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ON THE MILD ITÔ FORMULA IN BANACH SPACES

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Abstract. The mild Itô formula proposed in Theorem 1 in [Da Prato, G., Jentzen, A., & Röckner, M., A mild Ito formula for SPDEs, arXiv:1009.3526 (2012), To appear in the Trans. Amer. Math. Soc.] has turned out to be a useful instrument to study solutions and numerical approximations of stochastic partial differential equations (SPDEs) which are formulated as stochastic evolution equations (SEEs) on Hilbert spaces. In this article we generalize this mild Itô formula so that it is applicable to stopping times instead of deterministic time points and so that it is applicable to solutions and numerical approximations of SPDEs which are formulated as SEEs on UMD (unconditional martingale differences) Banach spaces. These generalizations are especially useful for proving essentially sharp weak convergence rates for numerical approximations of SPDEs such as stochastic heat equations with nonlinear diffusion coefficients.

1. Introduction.

The standard Itô formula for finite dimensional Itô processes has been generalized in the literature to infinite dimensions so that it is applicable to Itô processes with values in infinite dimensional Hilbert or Banach spaces; see Theorem 2.4 in Brzeźniak, Van Neerven, Veraar & Weiss [5]. This infinite dimensional generalization of the standard Itô formula is, however, typically not applicable to a solution (or a numerical approximation) of a stochastic partial differential equation (SPDE) as solutions of SPDEs are often only solutions in the mild or weak sense, which are not Itô processes on the considered state space of the SPDE. To overcome this lack of regularity of solutions of SPDEs, Da Prato et al. proposed in Theorem 1 in [8] (see also [14, Section 5]) an alternative formula which Da Prato et al. refer to as a mild Itô formula. The mild Itô formula in Theorem 1 in [8] is (even in finite dimensions) different to the standard Itô formula but it applies to the class of Hilbert

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space valued mild Itô processes which is a rather general class of Hilbert space valued stochastic processes that includes standard Itô processes as well as mild solutions and numerical approximations of semilinear SPDEs as special cases. In this work we generalize the mild Itô formula so that (i) it is applicable to stopping times instead of deterministic time points (see Theorem 3.5 below) and so that (ii) it is applicable to mild Itô processes which take values in UMD (unconditional martingale differences) Banach spaces with type 2; see Definition 3.1 in Subsection 3.2, see Theorem 3.5 in Subsection 3.4, and see Corollary 3 in Subsection 3.4 below. These generalizations of the mild Itô formula are especially useful for proving essentially sharp weak convergence rates for numerical approximations of SPDEs. In particular, Hefter et al. [13] employ the mild Itô formula in UMD Banach spaces established in this article to prove essentially sharp weak convergence rates for time-discrete exponential Euler approximations (cf., e.g., Lord & Rougemont [18, Section 3], Celledoni et al. [6, Section 2], Cohen & Gauckler [7, Section 2.2], Lord & Tambue [19, Section 2.3], and Wang [26, Section 1]) of a suitable class of semilinear SPDEs such as nonlinear stochastic heat equations as well as Cahn-Hilliard-Cook type equations. Semilinear SPDEs are a key ingredient in a number of models from economics and the natural sciences. They appear, for example, in models from neurobiology for the approximative description of the propagation of electrical impulses along nerve cells (cf., e.g., (1.1) in Sauer & Stannat [25]), in models from financial engineering for the approximative pricing of financial derivatives (cf., e.g., Theorem 2.5 in Harms et al. [12] and (1.2) in Filipović et al. [11]), in models from fluid mechanics for the approximative description of velocity fields in fully developed turbulent flows (cf., e.g., (7) in Birnir [2] and (1.5) in Birnir [3]), in models from quantum field theory for describing the temporal dynamics associated to Euclidean quantum field theories (cf., e.g., (1.1) in Mourrat & Weber [20]), and in models from chemistry for the approximative description of the temporal evolution of the concentration of an undesired chemical contaminant in the groundwater system (e.g., in a water basin, the groundwater system, or a river; cf., e.g., (1.1) in Kouritzin & Long [17] and also (2.2) in Kallianpur & Xiong [16]). In Section 2 below we also briefly review a few well-known results for Nemytskii and multiplication operators in Banach spaces (see Proposition 1, Proposition 2, and Corollary 1 in Section 2 below) which provide natural examples for the possibly nonlinear test function appearing in the mild Itô formula in Corollary 3 in Subsection 3.4 below.

1.1. **Notation.** Throughout this article the following notation is frequently used. Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of natural numbers. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ be the union of $\{0\}$ and the set of natural numbers. For all sets A and B let $\mathbb{M}(A, B)$ be the set of all functions from A to B. For all measurable spaces $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ let $\mathcal{M}(\mathcal{F}_1, \mathcal{F}_2)$ be the set of all $\mathcal{F}_1/\mathcal{F}_2$-measurable functions. For all separable $\mathbb{R}$-Hilbert spaces $(\check{H}, \langle\cdot,\cdot\rangle_{\check{H}}, \|\cdot\|_{\check{H}})$ and $(\hat{H}, \langle\cdot,\cdot\rangle_{\hat{H}}, \|\cdot\|_{\hat{H}})$ let $\mathcal{S}(\hat{H}, \check{H})$ be the sigma algebra on $L(\hat{H}, \check{H})$ given by $\mathcal{S}(\hat{H}, \check{H}) = \sigma_{L(\hat{H},\check{H})}(\cup_{v\in\hat{H}} \cup_{\mathcal{A}\in\mathcal{B}(\check{H})} \{A \in L(\hat{H}, \check{H}) \colon Av \in \mathcal{A}\})$ (see, e.g., [9, Section 1.2]). For every separable $\mathbb{R}$-Hilbert space $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ and every $\mathbb{R}$-Banach space $(V, \|\cdot\|_V)$ let $(\gamma(U, V), \|\cdot\|_{\gamma(U,V)})$ be the $\mathbb{R}$-Banach space of all γ-radonifying operators from U to V (cf., e.g, Van Neerven et al. [23, Section 3]) For every $d \in \mathbb{N}$ and every $A \in \mathcal{B}(\mathbb{R}^d)$ let $\lambda_A \colon \mathcal{B}(A) \to [0, \infty]$ be the Lebesgue-Borel measure on A. For every set X let $\#_X \in \mathbb{N}_0 \cup \{\infty\}$ be the number of elements of X. For every measure space $(\Omega, \mathcal{F}, \nu)$, every measurable space $(S, \mathcal{S})$, every set R, and every function $f \colon \Omega \to R$ let $[f]_{\nu,\mathcal{S}}$ be the set given by $[f]_{\nu,\mathcal{S}} = \{g \in \mathcal{M}(\mathcal{F}, \mathcal{S}) \colon (\exists\, A \in \mathcal{F} \colon \nu(A) = 0 \text{ and } \{\omega \in \Omega \colon f(\omega) \neq g(\omega)\} \subseteq A)\}$.

2. **Stochastic partial differential equations in Banach spaces.** In this section we recall a few well-known results for SPDEs on UMD Banach spaces. In particular, Proposition 1 below provides natural examples for the possibly nonlinear test function appearing in the mild Itô formula in Corollary 3 in Subsection 3.4 below.

2.1. **Preliminary results.** The following lemma and its proof can, e.g., be found in Van Neerven [21] (cf. [21, Theorem 6.2] and [21, Definition 3.7]).

Lemma 2.1 (An ideal property for γ-radonifying operators)**.** *Consider the notation in Subsection 1.1, let $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ and $(\mathcal{U}, \langle\cdot,\cdot\rangle_{\mathcal{U}}, \|\cdot\|_{\mathcal{U}})$ be $\mathbb{R}$-Hilbert spaces, let $(V, \|\cdot\|_V)$ and $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ be $\mathbb{R}$-Banach spaces, and let $A \in L(V, \mathcal{V})$, $B \in \gamma(U, V)$, $C \in L(\mathcal{U}, U)$. Then it holds that $ABC \in \gamma(\mathcal{U}, \mathcal{V})$ and*

$$\|ABC\|_{\gamma(\mathcal{U},\mathcal{V})} \leq \|A\|_{L(V,\mathcal{V})} \|B\|_{\gamma(U,V)} \|C\|_{L(\mathcal{U},U)}. \tag{1}$$

The next result is an elementary extension of Brzeźniak et al. [5, Lemma 2.3].

Lemma 2.2. *Consider the notation in Subsection 1.1, let $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be a separable $\mathbb{R}$-Hilbert space, let $(V, \|\cdot\|_V)$ and $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ be $\mathbb{R}$-Banach spaces, and let $\beta \in L^{(2)}(V, \mathcal{V})$ be a bounded bilinear operator from $V \times V$ to $\mathcal{V}$. Then*

(i) it holds for all $A_1, A_2 \in \gamma(U, V)$ and all orthonormal sets $\mathbb{U} \subseteq U$ of U that there exists a unique $v \in \mathcal{V}$ such that

$$\inf_{\substack{I \subseteq \mathbb{U}, \\ \#_I < \infty}} \sup_{\substack{I \subseteq J \subseteq \mathbb{U}, \\ \#_J < \infty}} \left\| v - \sum_{u \in J} \beta(A_1 u, A_2 u) \right\|_{\mathcal{V}} = 0, \tag{2}$$

(ii) it holds for all $A_1, A_2 \in \gamma(U, V)$ and all orthonormal bases $\mathbb{U}_1, \mathbb{U}_2 \subseteq U$ of U that

$$\sum_{u \in \mathbb{U}_1} \beta(A_1 u, A_2 u) = \sum_{u \in \mathbb{U}_2} \beta(A_1 u, A_2 u), \tag{3}$$

(iii) it holds for all $A_1, A_2 \in \gamma(U, V)$ and all orthonormal sets $\mathbb{U} \subseteq U$ of U that

$$\left\| \sum_{u \in \mathbb{U}} \beta(A_1 u, A_2 u) \right\|_{\mathcal{V}} \leq \|\beta\|_{L^{(2)}(V,\mathcal{V})} \|A_1\|_{\gamma(U,V)} \|A_2\|_{\gamma(U,V)}, \tag{4}$$

and

(iv) it holds for all orthonormal sets $\mathbb{U} \subseteq U$ of U that

$$\left(\gamma(U, V) \times \gamma(U, V) \ni (A_1, A_2) \mapsto \sum_{u \in \mathbb{U}} \beta(A_1 u, A_2 u) \in \mathcal{V} \right) \in L^{(2)}(\gamma(U, V), \mathcal{V}). \tag{5}$$

2.2. **Convergence properties of measurable functions.** Proposition 1 below provides natural examples for the possibly nonlinear test function appearing in the mild Itô formula in Corollary 3 in Subsection 3.4 below. Our proof of Proposition 1 employs the following elementary convergence statement for measurable functions.

Lemma 2.3. *Let $(\Omega, \mathcal{F}, \nu)$ be a finite measure space, let (E, d) and $(\mathcal{E}, \delta)$ be separable pseudometric spaces, let $p, q \in (0, \infty)$, let $\phi \colon E \to \mathcal{E}$ be a continuous and globally bounded function, and let $f_n \colon \Omega \to E$, $n \in \{0, 1, 2, \ldots\}$, be $\mathcal{F}/\mathcal{B}(E)$-measurable functions which satisfy $\limsup_{n\to\infty} \int_\Omega |d(f_n, f_0)|^p \, d\nu = 0$. Then*

$$\limsup_{n\to\infty} \int_\Omega |\delta(\phi \circ f_n, \phi \circ f_0)|^q \, d\nu = 0. \tag{6}$$

2.3. Regular test functions.

Proposition 1. *Consider the notation in Subsection 1.1, let* $k, l, d, n \in \mathbb{N}$, $p \in [1, \infty)$, $q \in (np, \infty)$, *let* $\mathcal{O} \in \mathcal{B}(\mathbb{R}^d)$ *be a bounded set, let* $f\colon \mathbb{R}^k \to \mathbb{R}^l$ *be an* n*-times continuously differentiable function with globally bounded derivatives, and let* $F\colon L^q(\lambda_{\mathcal{O}}; \mathbb{R}^k) \to L^p(\lambda_{\mathcal{O}}; \mathbb{R}^l)$ *be the function which satisfies for all* $v \in \mathcal{L}^q(\lambda_{\mathcal{O}}; \mathbb{R}^k)$ *that*

$$F([v]_{\lambda_{\mathcal{O}}, \mathcal{B}(\mathbb{R}^l)}) = [\{f(v(x))\}_{x \in \mathcal{O}}]_{\lambda_{\mathcal{O}}, \mathcal{B}(\mathbb{R}^l)} = [f \circ v]_{\lambda_{\mathcal{O}}, \mathcal{B}(\mathbb{R}^l)}. \tag{7}$$

Then

(i) it holds that F *is* n*-times continuously Fréchet differentiable with globally bounded derivatives,*

(ii) it holds for all $m \in \{1, 2, \ldots, n\}$, $v, u_1, \ldots, u_m \in \mathcal{L}^q(\lambda_{\mathcal{O}}; \mathbb{R}^k)$ *that*

$$\begin{aligned} &F^{(m)}([v]_{\lambda_{\mathcal{O}}, \mathcal{B}(\mathbb{R}^k)})([u_1]_{\lambda_{\mathcal{O}}, \mathcal{B}(\mathbb{R}^k)}, \ldots, [u_m]_{\lambda_{\mathcal{O}}, \mathcal{B}(\mathbb{R}^k)}) \\ &= [\{f^{(m)}(v(x))(u_1(x), \ldots, u_m(x))\}_{x \in \mathcal{O}}]_{\lambda_{\mathcal{O}}, \mathcal{B}(\mathbb{R}^l)}, \end{aligned} \tag{8}$$

(iii) it holds for all $m \in \{1, 2, \ldots, n\}$, $r \in [mp, \infty)$ *that*

$$\begin{aligned} &\sup_{v \in L^q(\lambda_{\mathcal{O}}; \mathbb{R}^k)} \sup_{u_1, \ldots, u_m \in L^{\max\{r,q\}}(\lambda_{\mathcal{O}}; \mathbb{R}^k) \setminus \{0\}} \left[\frac{\|F^{(m)}(v)(u_1, \ldots, u_m)\|_{L^p(\lambda_{\mathcal{O}}; \mathbb{R}^l)}}{\|u_1\|_{L^r(\lambda_{\mathcal{O}}; \mathbb{R}^k)} \cdot \ldots \cdot \|u_m\|_{L^r(\lambda_{\mathcal{O}}; \mathbb{R}^k)}} \right] \\ &\leq \left[\sup_{x \in \mathbb{R}^k} \|f^{(m)}(x)\|_{L^{(m)}(\mathbb{R}^k, \mathbb{R}^l)} \right] [\lambda_{\mathbb{R}^d}(\mathcal{O})]^{[\frac{1}{p} - \frac{m}{r}]} < \infty, \end{aligned} \tag{9}$$

(iv) it holds for all $m \in \{1, 2, \ldots, n\}$, $r, s \in (p, \infty)$, $v, w \in L^{\max\{r,q\}}(\lambda_{\mathcal{O}}; \mathbb{R}^k)$ *with* $\frac{1}{r} + \frac{m}{s} \leq \frac{1}{p}$ *that*

$$\begin{aligned} &\sup_{u_1, \ldots, u_m \in L^{\max\{s,q\}}(\lambda_{\mathcal{O}}; \mathbb{R}^k) \setminus \{0\}} \left[\frac{\|(F^{(m)}(v) - F^{(m)}(w))(u_1, \ldots, u_m)\|_{L^p(\lambda_{\mathcal{O}}; \mathbb{R}^l)}}{\|u_1\|_{L^s(\lambda_{\mathcal{O}}; \mathbb{R}^k)} \cdot \ldots \cdot \|u_m\|_{L^s(\lambda_{\mathcal{O}}; \mathbb{R}^k)}} \right] \\ &\leq \left[\sup_{\substack{x, y \in \mathbb{R}^k, \\ x \neq y}} \frac{\|f^{(m)}(x) - f^{(m)}(y)\|_{L^{(m)}(\mathbb{R}^k, \mathbb{R}^l)}}{\|x - y\|_{\mathbb{R}^k}} \right] [\lambda_{\mathbb{R}^d}(\mathcal{O})]^{[\frac{1}{p} - \frac{1}{r} - \frac{m}{s}]} \|v - w\|_{L^r(\lambda_{\mathcal{O}}; \mathbb{R}^k)}, \end{aligned} \tag{10}$$

and

(v) it holds for all $m \in \{1, 2, \ldots, n\}$, $r \in [(m+1)p, \infty)$, $v, w \in L^{\max\{r,q\}}(\lambda_{\mathcal{O}}; \mathbb{R}^k)$ *that*

$$\begin{aligned} &\sup_{u_1, \ldots, u_m \in L^{\max\{r,q\}}(\lambda_{\mathcal{O}}; \mathbb{R}^k) \setminus \{0\}} \left[\frac{\|(F^{(m)}(v) - F^{(m)}(w))(u_1, \ldots, u_m)\|_{L^p(\lambda_{\mathcal{O}}; \mathbb{R}^l)}}{\|u_1\|_{L^r(\lambda_{\mathcal{O}}; \mathbb{R}^k)} \cdot \ldots \cdot \|u_m\|_{L^r(\lambda_{\mathcal{O}}; \mathbb{R}^k)}} \right] \\ &\leq \left[\sup_{\substack{x, y \in \mathbb{R}^k, \\ x \neq y}} \frac{\|f^{(m)}(x) - f^{(m)}(y)\|_{L^{(m)}(\mathbb{R}^k, \mathbb{R}^l)}}{\|x - y\|_{\mathbb{R}^k}} \right] [\lambda_{\mathbb{R}^d}(\mathcal{O})]^{[\frac{1}{p} - \frac{m+1}{r}]} \|v - w\|_{L^r(\lambda_{\mathcal{O}}; \mathbb{R}^k)}. \end{aligned} \tag{11}$$

Proof of Proposition 1. Throughout this proof we assume w.l.o.g. that $\lambda_{\mathbb{R}^d}(\mathcal{O}) > 0$. We claim that for all $m \in \{1, 2, \ldots, n\}$ it holds

(a) that F is m-times Fréchet differentiable and

(b) that for all $v, u_1, \ldots, u_m \in \mathcal{L}^q(\lambda_{\mathcal{O}}; \mathbb{R}^k)$ it holds that

$$\begin{aligned} &F^{(m)}([v]_{\lambda_{\mathcal{O}}, \mathcal{B}(\mathbb{R}^k)})([u_1]_{\lambda_{\mathcal{O}}, \mathcal{B}(\mathbb{R}^k)}, \ldots, [u_m]_{\lambda_{\mathcal{O}}, \mathcal{B}(\mathbb{R}^k)}) \\ &= [\{f^{(m)}(v(x))(u_1(x), \ldots, u_m(x))\}_{x \in \mathcal{O}}]_{\lambda_{\mathcal{O}}, \mathcal{B}(\mathbb{R}^l)}. \end{aligned} \tag{12}$$

We now prove item (a) and item (b) by induction on $m \in \{1, 2, \ldots, n\}$. For the base case $m = 1$ we note that Minkowski's integral inequality and Hölder's inequality show that for all $v, h \in \mathcal{L}^q(\lambda_{\mathcal{O}}; \mathbb{R}^k)$, $\varepsilon \in (0, \infty)$ it holds that

$$
\begin{aligned}
&\|f \circ (v + h) - f \circ v - (f' \circ v)h\|_{\mathcal{L}^p(\lambda_{\mathcal{O}};\mathbb{R}^l)} \\
&\leq \int_0^1 \|[f' \circ (v + rh) - f' \circ v]h\|_{\mathcal{L}^p(\lambda_{\mathcal{O}};\mathbb{R}^l)}\, dr \\
&\leq \int_0^1 \|f' \circ (v + rh) - f' \circ v\|_{\mathcal{L}^{p(1+1/\varepsilon)}(\lambda_{\mathcal{O}};L(\mathbb{R}^k,\mathbb{R}^l))}\, dr\, \|h\|_{\mathcal{L}^{p(1+\varepsilon)}(\lambda_{\mathcal{O}};\mathbb{R}^k)}.
\end{aligned} \tag{13}
$$

Next observe that Lemma 2.3 (with $(\Omega, \mathcal{F}, \nu) = (\mathcal{O}, \mathcal{B}(\mathcal{O}), \lambda_{\mathcal{O}})$, $E = \mathbb{R}^k$, $\mathcal{E} = L(\mathbb{R}^k, \mathbb{R}^l)$, $p = p(1 + \varepsilon)$, $q = p(1 + 1/\varepsilon)$, $\phi = f'$, $f_0 = v$, $f_j = v + rh_j$ for $r \in [0, 1]$, $j \in \mathbb{N}$, $v \in \mathcal{L}^q(\lambda_{\mathcal{O}}; \mathbb{R}^k)$, $(h_j)_{j\in\mathbb{N}} \in \{(u_j)_{j\in\mathbb{N}} \subseteq \mathcal{L}^{p(1+\varepsilon)}(\lambda_{\mathcal{O}}; \mathbb{R}^k)\colon \limsup_{j\to\infty} \|u_j\|_{\mathcal{L}^{p(1+\varepsilon)}(\lambda_{\mathcal{O}};\mathbb{R}^k)} = 0\}$, $\varepsilon \in (0, \infty)$ in the notation of Lemma 2.3), the fact that $\sup_{x\in\mathbb{R}^k} \|f'(x)\|_{L(\mathbb{R}^k,\mathbb{R}^l)} < \infty$, and the fact that f' is continuous ensure that for all $r \in [0, 1]$, $v \in \mathcal{L}^q(\lambda_{\mathcal{O}}; \mathbb{R}^k)$, $\varepsilon \in (0, \infty)$, $(h_j)_{j\in\mathbb{N}} \subseteq \mathcal{L}^{p(1+\varepsilon)}(\lambda_{\mathcal{O}}; \mathbb{R}^k)$ with $\limsup_{j\to\infty} \|h_j\|_{\mathcal{L}^{p(1+\varepsilon)}(\lambda_{\mathcal{O}};\mathbb{R}^k)} = 0$ it holds that

$$
\limsup_{j\to\infty} \|f' \circ (v + rh_j) - f' \circ v\|_{\mathcal{L}^{p(1+1/\varepsilon)}(\lambda_{\mathcal{O}};L(\mathbb{R}^k,\mathbb{R}^l))} = 0. \tag{14}
$$

This, the fact that $\sup_{x\in\mathbb{R}^k} \|f'(x)\|_{L(\mathbb{R}^k,\mathbb{R}^l)} < \infty$, and Lebesgue's theorem of dominated convergence prove that for all $v \in \mathcal{L}^q(\lambda_{\mathcal{O}}; \mathbb{R}^k)$, $\varepsilon \in (0, \infty)$, $(h_j)_{j\in\mathbb{N}} \subseteq \mathcal{L}^{p(1+\varepsilon)}(\lambda_{\mathcal{O}}; \mathbb{R}^k)$ with $\limsup_{j\to\infty} \|h_j\|_{\mathcal{L}^{p(1+\varepsilon)}(\lambda_{\mathcal{O}};\mathbb{R}^k)} = 0$ it holds that

$$
\limsup_{j\to\infty} \left(\int_0^1 \|f' \circ (v + rh_j) - f' \circ v\|_{\mathcal{L}^{p(1+1/\varepsilon)}(\lambda_{\mathcal{O}};L(\mathbb{R}^k,\mathbb{R}^l))}\, dr \right) = 0. \tag{15}
$$

This together with Hölder's inequality and (13) implies that for all $v \in \mathcal{L}^q(\lambda_{\mathcal{O}}; \mathbb{R}^k)$, $\varepsilon \in (0, q/p - 1)$, $(h_j)_{j\in\mathbb{N}} \subseteq \mathcal{L}^{p(1+\varepsilon)}(\lambda_{\mathcal{O}}; \mathbb{R}^k)$ with $\limsup_{j\to\infty} \|h_j\|_{\mathcal{L}^{p(1+\varepsilon)}(\lambda_{\mathcal{O}};\mathbb{R}^k)} = 0$ and $\forall\, j \in \mathbb{N}\colon \|h_j\|_{\mathcal{L}^{p(1+\varepsilon)}(\lambda_{\mathcal{O}};\mathbb{R}^k)} > 0$ it holds that

$$
\begin{aligned}
&\limsup_{j\to\infty} \left(\frac{\|f \circ (v + h_j) - f \circ v - (f' \circ v)h_j\|_{\mathcal{L}^p(\lambda_{\mathcal{O}};\mathbb{R}^l)}}{\|h_j\|_{\mathcal{L}^{p(1+\varepsilon)}(\lambda_{\mathcal{O}};\mathbb{R}^k)}} \right) \\
&\leq \limsup_{j\to\infty} \left(\int_0^1 \|f' \circ (v + rh_j) - f' \circ v\|_{\mathcal{L}^{p(1+1/\varepsilon)}(\lambda_{\mathcal{O}};L(\mathbb{R}^k,\mathbb{R}^l))}\, dr \right) = 0.
\end{aligned} \tag{16}
$$

Hölder's inequality hence shows that for all $v \in \mathcal{L}^q(\lambda_{\mathcal{O}}; \mathbb{R}^k)$, $\varepsilon \in (0, q/p - 1)$, $(h_j)_{j\in\mathbb{N}} \subseteq \mathcal{L}^{p(1+\varepsilon)}(\lambda_{\mathcal{O}}; \mathbb{R}^k)$ with $\limsup_{j\to\infty} \|h_j\|_{\mathcal{L}^q(\lambda_{\mathcal{O}};\mathbb{R}^k)} = 0$ and $\forall\, j \in \mathbb{N}\colon \|h_j\|_{\mathcal{L}^q(\lambda_{\mathcal{O}};\mathbb{R}^k)} > 0$ it holds that

$$
\limsup_{j\to\infty} \left(\frac{\|f \circ (v + h_j) - f \circ v - (f' \circ v)h_j\|_{\mathcal{L}^p(\lambda_{\mathcal{O}};\mathbb{R}^l)}}{\|h_j\|_{\mathcal{L}^q(\lambda_{\mathcal{O}};\mathbb{R}^k)}} \right) = 0. \tag{17}
$$

This demonstrates that F is Fréchet differentiable and that for all $v, h \in \mathcal{L}^q(\lambda_{\mathcal{O}}; \mathbb{R}^k)$ it holds that

$$
F'([v]_{\lambda_{\mathcal{O}},\mathcal{B}(\mathbb{R}^k)})[h]_{\lambda_{\mathcal{O}},\mathcal{B}(\mathbb{R}^k)} = [\{f'(v(x))h(x)\}_{x\in\mathcal{O}}]_{\lambda_{\mathcal{O}},\mathcal{B}(\mathbb{R}^l)}. \tag{18}
$$

This proves item (a) and item (b) in the base case $m = 1$. For the induction step $\mathbb{N} \cap [0, n - 1] \ni m \to m + 1 \in \{1, 2, \ldots, n\}$ assume that there exists a natural number $m \in \mathbb{N} \cap [0, n - 1]$ such that item (a) and item (b) hold for $m = m$. Next observe that Minkowski's integral inequality and Hölder's inequality show that for all $v, h, u_1 \ldots, u_m \in \mathcal{L}^q(\lambda_{\mathcal{O}}; \mathbb{R}^k)$, $\varepsilon \in (0, \frac{q}{p(1+m)} - 1)$ it holds that

$$\begin{aligned} &\|[(f^{(m)} \circ (v+h)) - (f^{(m)} \circ v)](u_1, \ldots, u_m) - (f^{(m+1)} \circ v)(h, u_1, \ldots, u_m)\|_{\mathcal{L}^p(\lambda_{\mathcal{O}}; \mathbb{R}^l)} \\ &\le \int_0^1 \|[f^{(m+1)} \circ (v+rh) - f^{(m+1)} \circ v](h, u_1, \ldots, u_m)\|_{\mathcal{L}^p(\lambda_{\mathcal{O}}; \mathbb{R}^l)} \, dr \\ &\le \int_0^1 \|f^{(m+1)} \circ (v+rh) - f^{(m+1)} \circ v\|_{\mathcal{L}^{p(1+1/\varepsilon)}(\lambda_{\mathcal{O}}; L^{(m+1)}(\mathbb{R}^k, \mathbb{R}^l))} \, dr \\ &\quad \cdot \|h\|_{\mathcal{L}^{p(1+\varepsilon)(1+m)}(\lambda_{\mathcal{O}}; \mathbb{R}^k)} \prod_{i=1}^m \|u_i\|_{\mathcal{L}^{p(1+\varepsilon)(1+m)}(\lambda_{\mathcal{O}}; \mathbb{R}^k)}. \end{aligned} \tag{19}$$

Moreover, note that Lemma 2.3 (with $(\Omega, \mathcal{F}, \nu) = (\mathcal{O}, \mathcal{B}(\mathcal{O}), \lambda_{\mathcal{O}})$, $E = \mathbb{R}^k$, $\mathcal{E} = L^{(m+1)}(\mathbb{R}^k, \mathbb{R}^l)$, $p = p(1+\varepsilon)(1+m)$, $q = p(1+1/\varepsilon)$, $\phi = f^{(m+1)}$, $f_0 = v$, $f_j = v + rh_j$ for $r \in [0,1]$, $j \in \mathbb{N}$, $v \in \mathcal{L}^q(\lambda_{\mathcal{O}}; \mathbb{R}^k)$, $(h_j)_{j\in\mathbb{N}} \in \{(u_j)_{j\in\mathbb{N}} \subseteq \mathcal{L}^{p(1+\varepsilon)(1+m)}(\lambda_{\mathcal{O}}; \mathbb{R}^k)\colon \limsup_{j\to\infty} \|u_j\|_{\mathcal{L}^{p(1+\varepsilon)(1+m)}(\lambda_{\mathcal{O}}; \mathbb{R}^k)} = 0\}$, $\varepsilon \in (0, \infty)$ in the notation of Lemma 2.3), the fact that $\sup_{x\in\mathbb{R}^k} \|f^{(m+1)}(x)\|_{L^{(m+1)}(\mathbb{R}^k, \mathbb{R}^l)} < \infty$, and the fact that $f^{(m+1)}$ is continuous ensure that for all $r \in [0,1]$, $v \in \mathcal{L}^q(\lambda_{\mathcal{O}}; \mathbb{R}^k)$, $\varepsilon \in (0, \infty)$, $(h_j)_{j\in\mathbb{N}} \subseteq \mathcal{L}^{p(1+\varepsilon)(1+m)}(\lambda_{\mathcal{O}}; \mathbb{R}^k)$ with $\limsup_{j\to\infty} \|h_j\|_{\mathcal{L}^{p(1+\varepsilon)(1+m)}(\lambda_{\mathcal{O}}; \mathbb{R}^k)} = 0$ it holds that

$$\limsup_{j\to\infty} \|f^{(m+1)} \circ (v + rh_j) - f^{(m+1)} \circ v\|_{\mathcal{L}^{p(1+1/\varepsilon)}(\lambda_{\mathcal{O}}; L^{(m+1)}(\mathbb{R}^k, \mathbb{R}^l))} = 0. \tag{20}$$

This, the fact that $\sup_{x\in\mathbb{R}^k} \|f^{(m+1)}(x)\|_{L^{(m+1)}(\mathbb{R}^k, \mathbb{R}^l)} < \infty$, and Lebesgue's theorem of dominated convergence prove that for all $v \in \mathcal{L}^q(\lambda_{\mathcal{O}}; \mathbb{R}^k)$, $\varepsilon \in (0, \infty)$, $(h_j)_{j\in\mathbb{N}} \subseteq \mathcal{L}^{p(1+\varepsilon)(1+m)}(\lambda_{\mathcal{O}}; \mathbb{R}^k)$ with $\limsup_{j\to\infty} \|h_j\|_{\mathcal{L}^{p(1+\varepsilon)(1+m)}(\lambda_{\mathcal{O}}; \mathbb{R}^k)} = 0$ it holds that

$$\limsup_{j\to\infty} \left(\int_0^1 \|f^{(m+1)} \circ (v + rh_j) - f^{(m+1)} \circ v]\|_{\mathcal{L}^{p(1+1/\varepsilon)}(\lambda_{\mathcal{O}}; L^{(m+1)}(\mathbb{R}^k, \mathbb{R}^l))} \, dr \right) = 0. \tag{21}$$

The fact that $\forall \, \varepsilon \in (0, \frac{q}{p(1+m)} - 1)\colon \mathcal{L}^{p(1+\varepsilon)(1+m)}(\lambda_{\mathcal{O}}; \mathbb{R}^k) \subseteq \mathcal{L}^q(\lambda_{\mathcal{O}}; \mathbb{R}^k)$ and (19) hence imply that for all $v \in \mathcal{L}^q(\lambda_{\mathcal{O}}; \mathbb{R}^k)$, $\varepsilon \in (0, \frac{q}{p(1+m)} - 1)$, and all $(h_j)_{j\in\mathbb{N}} \subseteq \mathcal{L}^{p(1+\varepsilon)(1+m)}(\lambda_{\mathcal{O}}; \mathbb{R}^k)$ with $\limsup_{j\to\infty} \|h_j\|_{\mathcal{L}^{p(1+\varepsilon)(1+m)}(\lambda_{\mathcal{O}}; \mathbb{R}^k)} = 0$ and $\forall \, j \in \mathbb{N}\colon \|h_j\|_{\mathcal{L}^{p(1+\varepsilon)(1+m)}(\lambda_{\mathcal{O}}; \mathbb{R}^k)} > 0$ it holds that

$$\begin{aligned} &\limsup_{j\to\infty} \sup_{\substack{u_1, \ldots, u_m \in \mathcal{L}^q(\lambda_{\mathcal{O}}; \mathbb{R}^k), \\ \prod_{i=1}^m \|u_i\|_{\mathcal{L}^{p(1+\varepsilon)(1+m)}(\lambda_{\mathcal{O}}; \mathbb{R}^k)} > 0}} \Bigg(\\ &\frac{\|[(f^{(m)} \circ (v+h_j)) - (f^{(m)} \circ v)](u_1, \ldots, u_m) - (f^{(m+1)} \circ v)(h_j, u_1, \ldots, u_m)\|_{\mathcal{L}^p(\lambda_{\mathcal{O}}; \mathbb{R}^l)}}{\|h_j\|_{\mathcal{L}^{p(1+\varepsilon)(1+m)}(\lambda_{\mathcal{O}}; \mathbb{R}^k)} \prod_{i=1}^m \|u_i\|_{\mathcal{L}^{p(1+\varepsilon)(1+m)}(\lambda_{\mathcal{O}}; \mathbb{R}^k)}} \Bigg) \\ &\le \limsup_{j\to\infty} \left(\int_0^1 \|f^{(m+1)} \circ (v + rh_j) - f^{(m+1)} \circ v\|_{\mathcal{L}^{p(1+1/\varepsilon)}(\lambda_{\mathcal{O}}; L^{(m+1)}(\mathbb{R}^k, \mathbb{R}^l))} \, dr \right) \\ &= 0. \end{aligned} \tag{22}$$

Hölder's inequality therefore shows that for all $v \in \mathcal{L}^q(\lambda_{\mathcal{O}}; \mathbb{R}^k)$ and all $(h_j)_{j\in\mathbb{N}} \subseteq \mathcal{L}^q(\lambda_{\mathcal{O}}; \mathbb{R}^k)$ with $\limsup_{j\to\infty} \|h_j\|_{\mathcal{L}^q(\lambda_{\mathcal{O}}; \mathbb{R}^k)} = 0$ and $\forall \, j \in \mathbb{N}\colon \|h_j\|_{\mathcal{L}^q(\lambda_{\mathcal{O}}; \mathbb{R}^k)} > 0$ it holds that

$$\limsup_{j\to\infty} \sup_{\substack{u_1,\dots,u_m\in\mathcal{L}^q(\lambda_{\mathcal{O}};\mathbb{R}^k),\\ \prod_{i=1}^m \|u_i\|_{\mathcal{L}^q(\lambda_{\mathcal{O}};\mathbb{R}^k)}>0}} \left(\frac{\|[(f^{(m)}\circ(v+h_j))-(f^{(m)}\circ v)](u_1,\dots,u_m)-(f^{(m+1)}\circ v)(h_j,u_1,\dots,u_m)\|_{\mathcal{L}^p(\lambda_{\mathcal{O}};\mathbb{R}^l)}}{\|h_j\|_{\mathcal{L}^q(\lambda_{\mathcal{O}};\mathbb{R}^k)}\prod_{i=1}^m \|u_i\|_{\mathcal{L}^q(\lambda_{\mathcal{O}};\mathbb{R}^k)}} \right) = 0. \tag{23}$$

The induction hypothesis hence implies that $F^{(m)}$ is Fréchet differentiable and that for all $v, h, u_1, \dots, u_m \in \mathcal{L}^q(\lambda_{\mathcal{O}};\mathbb{R}^k)$ it holds that

$$\begin{aligned} &F^{(m+1)}([v]_{\lambda_{\mathcal{O}},\mathcal{B}(\mathbb{R}^k)})([h]_{\lambda_{\mathcal{O}},\mathcal{B}(\mathbb{R}^k)}, [u_1]_{\lambda_{\mathcal{O}},\mathcal{B}(\mathbb{R}^k)}, \dots, [u_m]_{\lambda_{\mathcal{O}},\mathcal{B}(\mathbb{R}^k)}) \\ &= [\{f^{(m+1)}(v(x))(h(x), u_1(x), \dots u_m(x))\}_{x\in\mathcal{O}}]_{\lambda_{\mathcal{O}},\mathcal{B}(\mathbb{R}^l)}. \end{aligned} \tag{24}$$

This establishes item (a) and item (b) in the case $m+1$. Induction thus completes the proof of item (a) and item (b).

In the next step we observe that Hölder's inequality ensures that for all $m \in \{1, 2, \dots, n\}$, $v, w \in \mathcal{L}^q(\lambda_{\mathcal{O}};\mathbb{R}^k)$, $r, s \in (p, \infty)$, $u_1, \dots, u_m \in \mathcal{L}^s(\lambda_{\mathcal{O}};\mathbb{R}^k)$ with $\frac{1}{r} + \frac{m}{s} \leq \frac{1}{p}$ it holds that

$$\begin{aligned} &\|[f^{(m)}\circ v - f^{(m)}\circ w](u_1,\dots,u_m)\|_{\mathcal{L}^p(\lambda_{\mathcal{O}};\mathbb{R}^l)} \\ &\leq [\lambda_{\mathbb{R}^d}(\mathcal{O})]^{[\frac{1}{p}-\frac{1}{r}-\frac{m}{s}]}\|f^{(m)}\circ v - f^{(m)}\circ w\|_{\mathcal{L}^r(\lambda_{\mathcal{O}};L^{(m)}(\mathbb{R}^k,\mathbb{R}^l))}\prod_{i=1}^m \|u_i\|_{\mathcal{L}^s(\lambda_{\mathcal{O}};\mathbb{R}^k)}. \end{aligned} \tag{25}$$

This implies that for all $m \in \{1, 2, \dots, n\}$, $v, w \in \mathcal{L}^q(\lambda_{\mathcal{O}};\mathbb{R}^k)$, $r, s \in (p, \infty)$ with $\frac{1}{r} + \frac{m}{s} \leq \frac{1}{p}$ it holds that

$$\begin{aligned} &\sup_{\substack{u_1,\dots,u_m\in\mathcal{L}^s(\lambda_{\mathcal{O}};\mathbb{R}^k),\\ \prod_{i=1}^m \|u_i\|_{\mathcal{L}^s(\lambda_{\mathcal{O}};\mathbb{R}^k)}>0}} \left(\frac{\|[f^{(m)}\circ v - f^{(m)}\circ w](u_1,\dots,u_m)\|_{\mathcal{L}^p(\lambda_{\mathcal{O}};\mathbb{R}^l)}}{\|u_1\|_{\mathcal{L}^s(\lambda_{\mathcal{O}};\mathbb{R}^k)}\cdot\ldots\cdot\|u_m\|_{\mathcal{L}^s(\lambda_{\mathcal{O}};\mathbb{R}^k)}} \right) \\ &\leq [\lambda_{\mathbb{R}^d}(\mathcal{O})]^{[\frac{1}{p}-\frac{1}{r}-\frac{m}{s}]}\|f^{(m)}\circ v - f^{(m)}\circ w\|_{\mathcal{L}^r(\lambda_{\mathcal{O}};L^{(m)}(\mathbb{R}^k,\mathbb{R}^l))}. \end{aligned} \tag{26}$$

Lemma 2.3 (with $(\Omega, \mathcal{F}, \nu) = (\mathcal{O}, \mathcal{B}(\mathcal{O}), \lambda_{\mathcal{O}})$, $E = \mathbb{R}^k$, $\mathcal{E} = L^{(n)}(\mathbb{R}^k, \mathbb{R}^l)$, $p = q$, $q = r$, $\phi = f^{(n)}$, $f_j = v_j$ for $r \in (0, \infty)$, $j \in \mathbb{N}_0$ in the notation of Lemma 2.3) and the fact that $\sup_{x\in\mathbb{R}^k} \|f^{(n)}(x)\|_{L^{(n)}(\mathbb{R}^k,\mathbb{R}^l)} < \infty$ hence show that for all $(v_j)_{j\in\mathbb{N}_0} \subseteq \mathcal{L}^q(\lambda_{\mathcal{O}};\mathbb{R}^k)$, $r, s \in (p, \infty)$ with $\limsup_{j\to\infty} \|v_j\|_{\mathcal{L}^q(\lambda_{\mathcal{O}};\mathbb{R}^k)} = 0$ and $\frac{1}{r} + \frac{n}{s} \leq \frac{1}{p}$ it holds that

$$\begin{aligned} &\limsup_{j\to\infty} \sup_{\substack{u_1,\dots,u_n\in\mathcal{L}^s(\lambda_{\mathcal{O}};\mathbb{R}^k),\\ \prod_{i=1}^m \|u_i\|_{\mathcal{L}^s(\lambda_{\mathcal{O}};\mathbb{R}^k)}>0}} \left(\frac{\|[f^{(n)}\circ v_j - f^{(n)}\circ v_0](u_1,\dots,u_n)\|_{\mathcal{L}^p(\lambda_{\mathcal{O}};\mathbb{R}^l)}}{\|u_1\|_{\mathcal{L}^s(\lambda_{\mathcal{O}};\mathbb{R}^k)}\cdot\ldots\cdot\|u_n\|_{\mathcal{L}^s(\lambda_{\mathcal{O}};\mathbb{R}^k)}} \right) \\ &\leq [\lambda_{\mathbb{R}^d}(\mathcal{O})]^{[\frac{1}{p}-\frac{1}{r}-\frac{m}{s}]}\left[\limsup_{j\to\infty} \|f^{(n)}\circ v_j - f^{(n)}\circ v_0\|_{\mathcal{L}^r(\lambda_{\mathcal{O}};L^{(n)}(\mathbb{R}^k,\mathbb{R}^l))} \right] = 0. \end{aligned} \tag{27}$$

This establishes that $F^{(n)}$ is continuous. Combining this with item (a) and item (b) proves item (i) and item (ii). Next note that Hölder's inequality shows that for all $m \in \{1, 2, \dots, n\}$, $r \in [mp, \infty)$, $v \in \mathcal{L}^q(\lambda_{\mathcal{O}};\mathbb{R}^k)$, $u_1, \dots, u_m \in \mathcal{L}^{\max\{r,q\}}(\lambda_{\mathcal{O}};\mathbb{R}^k)$ it holds that

$$\begin{aligned} &\|(f^{(m)} \circ v)(u_1, \dots, u_m)\|_{\mathcal{L}^p(\lambda_{\mathcal{O}};\mathbb{R}^l)} \\ &\leq \left[\sup_{x\in\mathbb{R}^k} \|f^{(m)}(x)\|_{L^{(m)}(\mathbb{R}^k,\mathbb{R}^l)}\right] [\lambda_{\mathbb{R}^d}(\mathcal{O})]^{[\frac{1}{p}-\frac{m}{r}]} \prod_{i=1}^{m} \|u_i\|_{\mathcal{L}^r(\lambda_{\mathcal{O}};\mathbb{R}^k)}. \end{aligned} \tag{28}$$

This and item (ii) imply that for all $m \in \{1, 2, \dots, n\}$, $r \in [mp, \infty)$ it holds that

$$\begin{aligned} &\sup_{v\in L^q(\lambda_{\mathcal{O}};\mathbb{R}^k)} \sup_{u_1,\dots,u_m\in L^{\max\{r,q\}}(\lambda_{\mathcal{O}};\mathbb{R}^k)\setminus\{0\}} \left(\frac{\|F^{(m)}(v)(u_1, \dots, u_m)\|_{L^p(\lambda_{\mathcal{O}};\mathbb{R}^l)}}{\|u_1\|_{L^r(\lambda_{\mathcal{O}};\mathbb{R}^k)} \cdot \ldots \cdot \|u_m\|_{L^r(\lambda_{\mathcal{O}};\mathbb{R}^k)}} \right) \\ &\leq \left[\sup_{x\in\mathbb{R}^k} \|f^{(m)}(x)\|_{L^{(m)}(\mathbb{R}^k,\mathbb{R}^l)}\right] [\lambda_{\mathbb{R}^d}(\mathcal{O})]^{[\frac{1}{p}-\frac{m}{r}]} < \infty. \end{aligned} \tag{29}$$

This proves item (iii). In the next step we observe that (26) assures that for all $m \in \{1, 2, \dots, n\}$, $r, s \in (p, \infty)$, $v, w \in \mathcal{L}^{\max\{r,q\}}(\lambda_{\mathcal{O}}; \mathbb{R}^k)$ with $\frac{1}{r} + \frac{m}{s} \leq \frac{1}{p}$ it holds that

$$\begin{aligned} &\sup_{\substack{u_1,\dots,u_m\in\mathcal{L}^s(\lambda_{\mathcal{O}};\mathbb{R}^k),\\ \prod_{i=1}^m \|u_i\|_{\mathcal{L}^s(\lambda_{\mathcal{O}};\mathbb{R}^k)}>0}} \left(\frac{\|(f^{(m)} \circ v - f^{(m)} \circ w)(u_1, \dots, u_m)\|_{\mathcal{L}^p(\lambda_{\mathcal{O}};\mathbb{R}^l)}}{\|u_1\|_{\mathcal{L}^s(\lambda_{\mathcal{O}};\mathbb{R}^k)} \cdot \ldots \cdot \|u_m\|_{\mathcal{L}^s(\lambda_{\mathcal{O}};\mathbb{R}^k)}} \right) \\ &\leq \left[\sup_{\substack{x,y\in\mathbb{R}^k,\\ x\neq y}} \frac{\|f^{(m)}(x) - f^{(m)}(y)\|_{L^{(m)}(\mathbb{R}^k,\mathbb{R}^l)}}{\|x - y\|_{\mathbb{R}^k}} \right] [\lambda_{\mathbb{R}^d}(\mathcal{O})]^{[\frac{1}{p}-\frac{1}{r}-\frac{m}{s}]} \|v - w\|_{\mathcal{L}^r(\lambda_{\mathcal{O}};\mathbb{R}^k)}. \end{aligned} \tag{30}$$

This and item (ii) establish that for all $m \in \{1, 2, \dots, n\}$, $r, s \in (p, \infty)$, $v, w \in L^{\max\{r,q\}}(\lambda_{\mathcal{O}}; \mathbb{R}^k)$ with $\frac{1}{r} + \frac{m}{s} \leq \frac{1}{p}$ it holds that

$$\begin{aligned} &\sup_{u_1,\dots,u_m\in L^{\max\{s,q\}}(\lambda_{\mathcal{O}};\mathbb{R}^k)\setminus\{0\}} \left[\frac{\|F^{(m)}(v)(u_1,\dots,u_m) - F^{(m)}(w)(u_1,\dots,u_m)\|_{L^p(\lambda_{\mathcal{O}};\mathbb{R}^l)}}{\|u_1\|_{L^s(\lambda_{\mathcal{O}};\mathbb{R}^k)} \cdot \ldots \cdot \|u_m\|_{L^s(\lambda_{\mathcal{O}};\mathbb{R}^k)}} \right] \\ &\leq \left[\sup_{\substack{x,y\in\mathbb{R}^k,\\ x\neq y}} \frac{\|f^{(m)}(x) - f^{(m)}(y)\|_{L^{(m)}(\mathbb{R}^k,\mathbb{R}^l)}}{\|x - y\|_{\mathbb{R}^k}} \right] [\lambda_{\mathbb{R}^d}(\mathcal{O})]^{[\frac{1}{p}-\frac{1}{r}-\frac{m}{s}]} \|v - w\|_{L^r(\lambda_{\mathcal{O}};\mathbb{R}^k)}. \end{aligned} \tag{31}$$

This proves item (iv). Item (v) is an immediate consequence of item (iv). The proof of Proposition 1 is thus completed. ☐

2.4. **Regular diffusion coefficients.** The following result is essentially contained in [4, Remark 6.1(c)].

Lemma 2.4. *Consider the notation in Subsection 1.1, let* $p \in [2, \infty)$, $r \in (1/4, \infty)$, *let* $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H) = (L^2(\lambda_{(0,1)}; \mathbb{R}), \langle\cdot,\cdot\rangle_{L^2(\lambda_{(0,1)};\mathbb{R})}, \|\cdot\|_{L^2(\lambda_{(0,1)};\mathbb{R})})$, *and let* $A\colon D(A) \subseteq H \to H$ *be the Laplacian with Dirichlet boundary conditions on* H. *Then*

(i) it holds for all $v \in H$ *that* $(-A)^{-r}v \in L^p(\lambda_{(0,1)}; \mathbb{R})$,
(ii) it holds that $(H \ni v \mapsto (-A)^{-r}v \in L^p(\lambda_{(0,1)}; \mathbb{R})) \in \gamma(H, L^p(\lambda_{(0,1)}; \mathbb{R}))$, *and*

(iii) it holds that

$$\begin{aligned} &\|H \ni v \mapsto (-A)^{-r} v \in L^p(\lambda_{(0,1)}; \mathbb{R})\|_{\gamma(H, L^p(\lambda_{(0,1)};\mathbb{R}))} \\ &\leq \left[\int_{\mathbb{R}} \tfrac{|x|^p}{\sqrt{2\pi}} \, e^{-x^2/2} \, dx \right]^{1/p} \left[\sum_{n=1}^{\infty} \tfrac{1}{n^{4r}} \right]^{1/2} < \infty. \end{aligned} \tag{32}$$

Proof of Lemma 2.4. Throughout this proof let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\gamma_n : \Omega \to \mathbb{R}$, $n \in \mathbb{N}$, be independent standard normal random variables, let $f_n : (0,1) \to \mathbb{R}$, $n \in \mathbb{N}$, satisfy for all $n \in \mathbb{N}$, $x \in (0,1)$ that $f_n(x) = \sqrt{2} \sin(n\pi x)$, let $(\rho_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}$ satisfy for all $n \in \mathbb{N}$ that $\rho_n = \pi^2 n^2$, and let $e_n \in H$, $n \in \mathbb{N}$, satisfy for all $n \in \mathbb{N}$ that $e_n = [f_n]_{\lambda_{(0,1)}, \mathcal{B}(\mathbb{R})}$. Note that item (i) is an immediate consequence from the Sobolev embedding theorem. It thus remains to prove item (ii) and (iii). For this observe that Jensen's inequality ensures that for every finite set $\mathcal{N} \subseteq N$ it holds that

$$\begin{aligned} &\mathbb{E}\left[\left\| \sum_{n \in \mathcal{N}} \gamma_n (-A)^{-r} e_n \right\|^2_{L^p(\lambda_{(0,1)};\mathbb{R})} \right] = \mathbb{E}\left[\left\| \sum_{n \in \mathcal{N}} \gamma_n (\rho_n)^{-r} e_n \right\|^2_{L^p(\lambda_{(0,1)};\mathbb{R})} \right] \\ &\leq \left(\mathbb{E}\left[\left\| \sum_{n \in \mathcal{N}} \gamma_n (\rho_n)^{-r} e_n \right\|^p_{L^p(\lambda_{(0,1)};\mathbb{R})} \right] \right)^{2/p} \end{aligned} \tag{33}$$

This implies that for every finite set $\mathcal{N} \subseteq N$ it holds that

$$\begin{aligned} &\mathbb{E}\left[\left\| \sum_{n \in \mathcal{N}} \gamma_n (-A)^{-r} e_n \right\|^2_{L^p(\lambda_{(0,1)};\mathbb{R})} \right] \\ &\leq \left[\int_0^1 \mathbb{E}\left[\left| \sum_{n \in \mathcal{N}} \gamma_n (\rho_n)^{-r} f_n(x) \right|^p \right] dx \right]^{2/p} \\ &= \left[\int_0^1 \left| \mathbb{E}\left[\left| \sum_{n \in \mathcal{N}} \gamma_n (\rho_n)^{-r} f_n(x) \right|^2 \right] \right|^{p/2} \mathbb{E}\big[|\gamma_1|^p\big] \, dx \right]^{2/p} \\ &= \left[\int_0^1 \left(\sum_{n \in \mathcal{N}} (\rho_n)^{-2r} |f_n(x)|^2 \right)^{p/2} \mathbb{E}\big[|\gamma_1|^p\big] \, dx \right]^{2/p} \\ &= \left[\int_0^1 \left(\sum_{n \in \mathcal{N}} (\rho_n)^{-2r} |f_n(x)|^2 \right)^{p/2} dx \right]^{2/p} \|\gamma_1\|^2_{\mathcal{L}^p(\mathbb{P};\mathbb{R})}. \end{aligned} \tag{34}$$

The Minkowski inequality hence shows that for every finite set $\mathcal{N} \subseteq N$ it holds that

$$\begin{aligned} &\mathbb{E}\left[\left\| \sum_{n \in \mathcal{N}} \gamma_n (-A)^{-r} e_n \right\|^2_{L^p(\lambda_{(0,1)};\mathbb{R})} \right] \leq \|\gamma_1\|^2_{\mathcal{L}^p(\mathbb{P};\mathbb{R})} \left\| \sum_{n \in \mathcal{N}} (\rho_n)^{-2r} |f_n|^2 \right\|_{\mathcal{L}^{p/2}(\mathbb{P};\mathbb{R})} \\ &\leq \|\gamma_1\|^2_{\mathcal{L}^p(\mathbb{P};\mathbb{R})} \left(\sum_{n \in \mathcal{N}} (\rho_n)^{-2r} \|f_n\|^2_{\mathcal{L}^p(\mathbb{P};\mathbb{R})} \right). \end{aligned} \tag{35}$$

This proves that for every finite set $\mathcal{N} \subseteq N$ it holds that

$$\left\| \sum_{n \in \mathcal{N}} \gamma_n (-A)^{-r} e_n \right\|_{L^2(\mathbb{P}; L^p(\lambda_{(0,1)};\mathbb{R}))} \leq \|\gamma_1\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})} \left[\sum_{n=\min(\mathcal{N})}^{\infty} (\rho_n)^{-2r} \|f_n\|^2_{\mathcal{L}^p(\mathbb{P};\mathbb{R})} \right]^{1/2}$$

$$\leq \|\gamma_1\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})}\left[2\sum_{n=\min(\mathcal{N})}^{\infty}(\rho_n)^{-2r}\right]^{1/2} = \|\gamma_1\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})}\tfrac{\sqrt{2}}{\pi^{2r}}\left[\sum_{n=\min(\mathcal{N})}^{\infty}n^{-4r}\right]^{1/2}$$
$$\leq \|\gamma_1\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})}\left[\sum_{n=\min(\mathcal{N})}^{\infty}n^{-4r}\right]^{1/2} = \left[\int_{\mathbb{R}}\tfrac{|x|^p}{\sqrt{2\pi}}\,e^{-x^2/2}\,dx\right]^{1/p}\left[\sum_{n=\min(\mathcal{N})}^{\infty}n^{-4r}\right]^{1/2} < \infty. \tag{36}$$

This, the Sobolev embedding theorem, and, e.g, [21, Theorem 3.20] complete the proof of Lemma 2.4. □

Lemma 2.5. *Consider the notation in Subsection 1.1, let* $d \in \mathbb{N}$, $p \in (2,\infty)$, $\beta \in (-\infty, -\frac{d}{2p}]$, *let* $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H) = (L^2(\lambda_{(0,1)^d};\mathbb{R}), \langle\cdot,\cdot\rangle_{L^2(\lambda_{(0,1)^d};\mathbb{R})}, \|\cdot\|_{L^2(\lambda_{(0,1)^d};\mathbb{R})})$, *let* $A\colon D(A)\subseteq H \to H$ *be the Laplacian with Dirichlet boundary conditions on* H, *and let* $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, *be a family of interpolation spaces associated to* $-A$. *Then*

(i) there exists a unique bounded linear operator $B \in L(L^p(\lambda_{(0,1)^d};\mathbb{R}), L(H, H_\beta))$ *which satisfies for all* $v \in L^{\max\{p,4\}}(\lambda_{(0,1)^d};\mathbb{R})$, $u \in L^4(\lambda_{(0,1)^d};\mathbb{R})$ *that*
$$(Bv)u = v\cdot u \tag{37}$$
and

(ii) it holds that
$$\|B\|_{L(L^p(\lambda_{(0,1)^d};\mathbb{R}),L(H,H_\beta))} \leq \sup_{w\in H_{-\beta}\setminus\{0\}}\left[\frac{\|w\|_{L^{2p/(p-2)}(\lambda_{(0,1)^d};\mathbb{R})}}{\|w\|_{H_{-\beta}}}\right] < \infty. \tag{38}$$

Proof of Lemma 2.5. Throughout this proof let
$$M\colon L^{\max\{p,4\}}(\lambda_{(0,1)^d};\mathbb{R}) \to L(L^4(\lambda_{(0,1)^d};\mathbb{R}), H) \tag{39}$$
be the function which satisfies for all $v \in L^{\max\{p,4\}}(\lambda_{(0,1)^d};\mathbb{R})$, $u \in L^4(\lambda_{(0,1)^d};\mathbb{R})$ that $M(v)u = v\cdot u$. Observe that for all $v \in L^{\max\{p,4\}}(\lambda_{(0,1)^d};\mathbb{R})$, $u \in L^4(\lambda_{(0,1)^d};\mathbb{R})$ it holds that
$$\begin{aligned}\|(M(v))u\|_{H_\beta} &= \|v\cdot u\|_{H_\beta} = \|(-A)^\beta(u\cdot v)\|_H = \sup_{w\in H\setminus\{0\}}\left[\tfrac{|\langle w,(-A)^\beta(v\cdot u)\rangle_H|}{\|w\|_H}\right]\\ &= \sup_{w\in H\setminus\{0\}}\left[\frac{|\langle(-A)^\beta w, v\cdot u\rangle_H|}{\|(-A)^{-\beta}(-A)^\beta w\|_H}\right].\end{aligned} \tag{40}$$

This and the Hölder inequality demonstrates that for all $v \in L^{\max\{p,4\}}(\lambda_{(0,1)^d};\mathbb{R})$, $u \in L^4(\lambda_{(0,1)^d};\mathbb{R})$ it holds that
$$\begin{aligned}\|(M(v))u\|_{H_\beta} &= \sup_{w\in H_{-\beta}\setminus\{0\}}\left[\tfrac{|\langle w,v\cdot u\rangle_H|}{\|(-A)^{-\beta}w\|_H}\right]\\ &\leq \sup_{w\in H_{-\beta}\setminus\{0\}}\left[\frac{\|w\|_{L^{1/(1/2-1/p)}(\lambda_{(0,1)^d};\mathbb{R})}\|v\|_{L^p(\lambda_{(0,1)^d};\mathbb{R})}\|u\|_H}{\|(-A)^{-\beta}w\|_H}\right]\\ &= \left[\sup_{w\in H_{-\beta}\setminus\{0\}}\frac{\|w\|_{L^{2p/(p-2)}(\lambda_{(0,1)^d};\mathbb{R})}}{\|(-A)^{-\beta}w\|_H}\right]\|v\|_{L^p(\lambda_{(0,1)^d};\mathbb{R})}\|u\|_H.\end{aligned} \tag{41}$$

Combining this and the Sobolev embedding theorem with the fact that
$$(-2\beta) - 0 = -2\beta \geq \tfrac{d}{p} = d\big[\tfrac{1}{2} - [\tfrac{1}{2} - \tfrac{1}{p}]\big] = d\big[\tfrac{1}{2} - \tfrac{1}{(2p/(p-2))}\big] \tag{42}$$

proves that for all $v \in L^{\max\{p,4\}}(\lambda_{(0,1)^d};\mathbb{R})$ it holds that

$$\begin{aligned} &\sup_{u \in L^4(\lambda_{(0,1)^d};\mathbb{R})\setminus\{0\}} \left[\frac{\|(M(v))u\|_{H_\beta}}{\|u\|_H}\right] \\ &\leq \underbrace{\left[\sup_{w \in H_{-\beta}\setminus\{0\}} \frac{\|w\|_{L^{2p/(p-2)}(\lambda_{(0,1)^d};\mathbb{R})}}{\|(-A)^{-\beta} w\|_H}\right]}_{<\infty} \|v\|_{L^p(\lambda_{(0,1)^d};\mathbb{R})}. \end{aligned} \tag{43}$$

This implies that there exists a unique function $\mathcal{M}\colon L^{\max\{p,4\}}(\lambda_{(0,1)^d};\mathbb{R}) \to L(H, H_\beta)$ which satisfies for all $v \in L^{\max\{p,4\}}(\lambda_{(0,1)^d};\mathbb{R})$, $u \in L^4(\lambda_{(0,1)^d};\mathbb{R})$ that

$$(\mathcal{M}(v))u = (M(v))(u) = v \cdot u \tag{44}$$

and

$$\|\mathcal{M}(v)\|_{L(H,H_\beta)} \leq \left[\sup_{w \in H_{-\beta}\setminus\{0\}} \frac{\|w\|_{L^{2p/(p-2)}(\lambda_{(0,1)^d};\mathbb{R})}}{\|(-A)^{-\beta} w\|_H}\right] \|v\|_{L^p(\lambda_{(0,1)^d};\mathbb{R})} < \infty. \tag{45}$$

This, in turn, assures that there exists a unique bounded linear operator

$$B \in L(L^p(\lambda_{(0,1)};\mathbb{R}), L(H, H_\beta)) \tag{46}$$

which satisfies for all $v \in L^{\max\{p,4\}}(\lambda_{(0,1)^d};\mathbb{R})$ that

$$B(v) = \mathcal{M}(v) \tag{47}$$

and

$$\|B\|_{L(L^p(\lambda_{(0,1)};\mathbb{R}),L(H,H_\beta))} \leq \sup_{w \in H_{-\beta}\setminus\{0\}} \left[\frac{\|w\|_{L^{2p/(p-2)}(\lambda_{(0,1)^d};\mathbb{R})}}{\|(-A)^{-\beta} w\|_H}\right] < \infty. \tag{48}$$

Combining (44), (47), and (48) completes the proof of Lemma 2.5. □

Lemma 2.6. *Let $\lambda_{(0,1)}\colon \mathcal{B}((0,1)) \to [0,\infty]$ be the Lebesgue-Borel measure on $(0,1)$, let $p \in [2,\infty)$, $\varepsilon \in [0,\infty)$, $\beta \in (-\infty, -1/4 - \varepsilon)$, let*

$$(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H) = (L^2(\lambda_{(0,1)};\mathbb{R}), \langle\cdot,\cdot\rangle_{L^2(\lambda_{(0,1)};\mathbb{R})}, \|\cdot\|_{L^2(\lambda_{(0,1)};\mathbb{R})}), \tag{49}$$

let $(V, \|\cdot\|_V) = (L^p(\lambda_{(0,1)};\mathbb{R}), \|\cdot\|_{L^p(\lambda_{(0,1)};\mathbb{R})})$, let $A\colon D(A) \subseteq H \to H$ be the Laplacian with Dirichlet boundary conditions on H, let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$, let $\mathcal{A}\colon D(\mathcal{A}) \subseteq V \to V$ be the Laplacian with Dirichlet boundary conditions on V, and let $(V_r, \|\cdot\|_{V_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-\mathcal{A}$. Then

(i) there exists a unique continuous function $\iota\colon H_{-\varepsilon} \to V_\beta$ which satisfies for all $v \in V$ that $\iota(v) = v$,
(ii) it holds that $\iota \in \gamma(H_{-\varepsilon}, V_\beta)$, and
(iii) it holds that

$$\|\iota\|_{\gamma(H_{-\varepsilon},V_\beta)} \leq \left[\int_{\mathbb{R}} \frac{|x|^p}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx\right]^{1/p} \left[\sum_{n=1}^{\infty} n^{4(\beta+\varepsilon)}\right]^{1/2} < \infty. \tag{50}$$

Proof of Lemma 2.6. Throughout this proof let $\varphi \in L(H_{-\varepsilon}, H)$ be the unique bounded linear operator which satisfies for all $v \in H$ that

$$\varphi(v) = (-A)^{-\varepsilon} v \tag{51}$$

and let $\phi \in L(V, V_\beta)$ be the unique bounded linear operator which satisfies for all $v \in V_{-\beta}$ that

$$\phi(v) = (-\mathcal{A})^{-\beta} v. \tag{52}$$

Observe that Lemma 2.4 and the assumption that $\beta + \varepsilon < -1/4$ prove
(a) that $\forall\, v \in H\colon (-A)^{\beta+\varepsilon} v \in V$,
(b) that $(H \ni v \mapsto (-A)^{\beta+\varepsilon} v \in V) \in \gamma(H, V)$, and
(c) that

$$\|H \ni v \mapsto (-A)^{\beta+\varepsilon} v \in V\|_{\gamma(H,U)} \leq \left[\int_{\mathbb{R}} \frac{|x|^p}{\sqrt{2\pi}} e^{-x^2/2}\, dx\right]^{1/2} \left[\sum_{n=1}^{\infty} n^{4(\beta+\varepsilon)}\right]^{1/2} < \infty. \tag{53}$$

Note that item (a) assures that there exist functions $\Phi\colon H \to V$ and $\iota\colon H_{-\varepsilon} \to V_\beta$ which satisfy for all $v \in H$ that

$$\Phi(v) = (-A)^{\beta+\varepsilon} v \tag{54}$$

and

$$\iota = \phi \circ \Phi \circ \varphi. \tag{55}$$

Observe that item (b) and item (c) establish that $\Phi \in \gamma(H, V)$ and

$$\|\Phi\|_{\gamma(H,V)} \leq \left[\int_{\mathbb{R}} \frac{|x|^p}{\sqrt{2\pi}} e^{-x^2/2}\, dx\right]^{1/2} \left[\sum_{n=1}^{\infty} n^{4(\beta+\varepsilon)}\right]^{1/2} < \infty. \tag{56}$$

Combining this, the fact that $\varphi \in L(H_{-\varepsilon}, H)$, and the fact that $\phi \in L(V, V_\beta)$ with Lemma 2.1 ensures that $\iota \in \gamma(H_{-\varepsilon}, V_\beta)$ and

$$\begin{aligned} \|\iota\|_{\gamma(H_{-\varepsilon},V_\beta)} &\leq \|\phi\|_{L(V,V_\beta)} \|\Phi\|_{\gamma(H,V)} \|\varphi\|_{L(H_{-\varepsilon},H)} \\ &= \|\Phi\|_{\gamma(H,V)} \leq \left[\int_{\mathbb{R}} \frac{|x|^p}{\sqrt{2\pi}} e^{-x^2/2}\, dx\right]^{1/2} \left[\sum_{n=1}^{\infty} n^{4(\beta+\varepsilon)}\right]^{1/2} < \infty. \end{aligned} \tag{57}$$

Next note that the fact that $\forall\, v \in V, t \in [0,\infty)\colon e^{tA} v = e^{t\mathcal{A}} v$, e.g., [10, item (ii) of Theorem 1.10 in Chapter II] and, e.g., [10, Definition 5.25 in Chapter II] ensure that for all $v \in V$ it holds that

$$(-A)^\beta v = (-\mathcal{A})^\beta v. \tag{58}$$

Hence, we obtain for all $v \in V$ that

$$\begin{aligned} \iota(v) &= \phi(\Phi(\varphi(v))) = \phi((-A)^{\beta+\varepsilon}(-A)^{-\varepsilon} v) = \phi((-A)^\beta v) \\ &= \phi((-\mathcal{A})^\beta v) = (-\mathcal{A})^{-\beta}(-\mathcal{A})^\beta v = v. \end{aligned} \tag{59}$$

This and (57) complete the proof of Lemma 2.6. □

The following result, Proposition 2, is closely related to some arguments given in the proof of Theorem 10.2 in Van Neerven et al. [22].

Proposition 2. *Consider the notation in Subsection 1.1, let* $n \in \mathbb{N}$, $\beta \in (-\infty, -1/4)$, $p \in (\max\{\frac{n}{2(|\beta|-1/4)}, 2n\}, \infty)$, *let*

$$(H, \langle \cdot,\cdot \rangle_H, \|\cdot\|_H) = (L^2(\lambda_{(0,1)};\mathbb{R}), \langle \cdot,\cdot \rangle_{L^2(\lambda_{(0,1)};\mathbb{R})}, \|\cdot\|_{L^2(\lambda_{(0,1)};\mathbb{R})}), \tag{60}$$

let $(V, \|\cdot\|_V) = (L^p(\lambda_{(0,1)};\mathbb{R}), \|\cdot\|_{L^p(\lambda_{(0,1)};\mathbb{R})})$, *let* $b\colon \mathbb{R} \to \mathbb{R}$ *be an* n*-times continuously differentiable function with globally bounded derivatives, let* $A\colon D(A) \subseteq H \to H$ *be the Laplacian with Dirichlet boundary conditions on* H, *let* $(H_r, \langle \cdot,\cdot \rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, *be a family of interpolation spaces associated to* $-A$, *let* $\mathcal{A}\colon D(\mathcal{A}) \subseteq V \to V$

be the Laplacian with Dirichlet boundary conditions on V, and let $(V_r, \|\cdot\|_{V_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-\mathcal{A}$. Then

(i) *there exists a unique continuous function $B\colon V \to \gamma(L^2(\lambda_{(0,1)};\mathbb{R}), V_\beta)$ which satisfies for all $u, v \in \mathcal{L}^{2p}(\lambda_{(0,1)};\mathbb{R})$ that*

$$B\big([v]_{\lambda_{(0,1)},\mathcal{B}(\mathbb{R})}\big)[u]_{\lambda_{(0,1)},\mathcal{B}(\mathbb{R})} = \big[\{b(v(x)) \cdot u(x)\}_{x\in(0,1)}\big]_{\lambda_{(0,1)},\mathcal{B}(\mathbb{R})}, \tag{61}$$

(ii) *it holds that B is n-times continuously Fréchet differentiable with globally bounded derivatives,*

(iii) *it holds for all $\delta \in (\frac{1}{p}\max\{\frac{n}{2(|\beta|-1/4)}, 2n\}, 1)$ that*

$$\sup_{w\in H_{n/(2p\delta)}\setminus\{0\}} \left[\frac{\|w\|_{L^{2p\delta/(p\delta-2n)}(\lambda_{(0,1)};\mathbb{R})}}{\|w\|_{H_{n/(2p\delta)}}}\right] + \sum_{l=1}^{\infty} l^{4(\beta+n/(2p\delta))} < \infty, \tag{62}$$

(iv) *it holds for all $k \in \{1, \ldots, n\}$, $\delta \in (\frac{1}{p}\max\{\frac{n}{2(|\beta|-1/4)}, 2n\}, 1)$ that*

$$\begin{aligned}\sup_{v\in V}\|B^{(k)}(v)\|_{L^{(k)}(V,\gamma(H,V_\beta))} &\leq \left[\int_{\mathbb{R}} \frac{|x|^p}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx\right]^{1/p} \left[\sum_{l=1}^{\infty} l^{4(\beta+n/(2p\delta))}\right]^{1/2} \\ &\quad \cdot \left[\sup_{w\in H_{n/(2p\delta)}\setminus\{0\}} \frac{\|w\|_{L^{2p\delta/(p\delta-2n)}(\lambda_{(0,1)};\mathbb{R})}}{\|w\|_{H_{n/(2p\delta)}}}\right]\left[\sup_{x\in\mathbb{R}}|b^{(k)}(x)|\right] < \infty, \end{aligned}\tag{63}$$

and

(v) *it holds for all $k \in \{1, \ldots, n\}$, $\delta \in (\frac{1}{p}\max\{\frac{n}{2(|\beta|-1/4)}, 2n\}, 1)$, $r \in [\frac{p\delta}{n-k\delta}, \infty)$ that*

$$\begin{aligned}&\sup_{\substack{v,w\in L^{\max\{r,p\}}(\lambda_{(0,1)};\mathbb{R}),\\ v\neq w}} \left[\frac{\|B^{(k)}(v) - B^{(k)}(w)\|_{L^{(k)}(V,\gamma(H,V_\beta))}}{\|v-w\|_{L^r(\lambda_{(0,1)};\mathbb{R})}}\right] \\ &\leq \left[\int_{\mathbb{R}} \frac{|x|^p}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx\right]^{1/p} \left[\sum_{l=1}^{\infty} l^{4(\beta+n/(2p\delta))}\right]^{1/2} \\ &\quad\cdot \left[\sup_{w\in H_{n/(2p\delta)}\setminus\{0\}} \frac{\|w\|_{L^{2p\delta/(p\delta-2n)}(\lambda_{(0,1)};\mathbb{R})}}{\|w\|_{H_{n/(2p\delta)}}}\right]\left[\sup_{\substack{x,y\in\mathbb{R},\\ x\neq y}} \frac{|b^{(k)}(x)-b^{(k)}(y)|}{|x-y|}\right].\end{aligned}\tag{64}$$

Proof of Proposition 2. Throughout this proof let $\delta \in (\frac{1}{p}\max\{\frac{n}{2(|\beta|-1/4)}, 2n\}, 1)$ and let $\psi\colon V \to L^{p\delta/n}(\lambda_{(0,1)};\mathbb{R})$ be the function which satisfies for all $v \in \mathcal{L}^p(\lambda_{(0,1)};\mathbb{R})$ that

$$\psi([v]_{\lambda_{(0,1)},\mathcal{B}(\mathbb{R})}) = [\{b(v(x))\}_{x\in(0,1)}]_{\lambda_{(0,1)},\mathcal{B}(\mathbb{R})} = [b\circ v]_{\lambda_{(0,1)},\mathcal{B}(\mathbb{R})}. \tag{65}$$

Note that item (i) of Proposition 1 (with $k = 1$, $l = 1$, $d = 1$, $n = n$, $p = \frac{p\delta}{n}$, $q = p$, $\mathcal{O} = (0,1)$, $f = b$, $F = \psi$ in the notation of item (i) of Proposition 1) establishes that $\psi\colon V \to L^{p\delta/n}(\lambda_{(0,1)};\mathbb{R})$ is n-times continuously Fréchet differentiable with globally bounded derivatives. Moreover, observe that item (iii) of Proposition 1 (with $k = 1$, $l = 1$, $d = 1$, $n = n$, $p = \frac{p\delta}{n}$, $q = p$, $\mathcal{O} = (0,1)$, $f = b$, $F = \psi$, $m = k$, $r = p$ for $k \in \{1, \ldots, n\}$ in the notation of item (iii) of Proposition 1) proves that for all $k \in \{1, \ldots, n\}$ it holds that

$$\sup_{v\in V}\|\psi^{(k)}(v)\|_{L^{(k)}(V,L^{p\delta/n}(\lambda_{(0,1)};\mathbb{R}))} \leq \sup_{x\in\mathbb{R}}|b^{(k)}(x)| < \infty. \tag{66}$$

In addition, we apply item (iv) of Proposition 1 (with $k = 1$, $l = 1$, $d = 1$, $n = n$, $p = \frac{p\delta}{n}$, $q = p$, $\mathcal{O} = (0,1)$, $f = b$, $F = \psi$, $m = k$, $r = r$, $s = p$, $v = v$, $w = w$ for

$v, w \in L^{\max\{r,p\}}(\lambda_{(0,1)};\mathbb{R})$, $r \in [\frac{p\delta}{n-k\delta}, \infty)$, $k \in \{1, \dots, n\}$ in the notation of item (iv) of Proposition 1) to obtain that for all $k \in \{1, \dots, n\}$, $r \in [\frac{p\delta}{n-k\delta}, \infty)$ it holds that

$$\begin{aligned}
&\sup_{\substack{v,w \in L^{\max\{r,p\}}(\lambda_{(0,1)};\mathbb{R}),\\ v \neq w}} \left[\frac{\|\psi^{(k)}(v) - \psi^{(k)}(w)\|_{L^{(k)}(V, L^{p\delta/n}(\lambda_{(0,1)};\mathbb{R}))}}{\|v-w\|_{L^r(\lambda_{(0,1)};\mathbb{R})}} \right] \\
&= \sup_{\substack{v,w \in L^{\max\{r,p\}}(\lambda_{(0,1)};\mathbb{R}),\\ v \neq w}} \sup_{\substack{v_1,\dots,v_k\\ \in V \setminus \{0\}}} \left[\frac{\|[\psi^{(k)}(v) - \psi^{(k)}(w)](v_1, \dots, v_k)\|_{L^{p\delta/n}(\lambda_{(0,1)};\mathbb{R})}}{\|v-w\|_{L^r(\lambda_{(0,1)};\mathbb{R})} \cdot \|v_1\|_V \cdot \ldots \cdot \|v_k\|_V} \right] \\
&\leq \sup_{\substack{x,y \in \mathbb{R},\\ x \neq y}} \left[\frac{|b^{(k)}(x) - b^{(k)}(y)|}{|x-y|} \right].
\end{aligned} \tag{67}$$

Moreover, note that for all $q \in [p, \infty)$, $v \in \mathcal{L}^q(\lambda_{(0,1)};\mathbb{R})$ it holds that

$$\begin{aligned}
&\int_0^1 |b(v(x))|^q \, dx = \int_0^1 \left| b(0) + \int_0^1 b'(rv(x))v(x) \, dr \right|^q dx \\
&\leq \int_0^1 \left(|b(0)| + |v(x)| \sup_{y \in \mathbb{R}} |b'(y)| \right)^q dx \\
&\leq 2^{q-1} \int_0^1 \left(|b(0)|^q + |v(x)|^q \sup_{y \in \mathbb{R}} |b'(y)|^q \right) dx \\
&= 2^{q-1} \left(|b(0)|^q + \sup_{y \in \mathbb{R}} |b'(y)|^q \int_0^1 |v(x)|^q \, dx \right) < \infty.
\end{aligned} \tag{68}$$

This proves that for all $q \in [p, \infty)$, $v \in L^q(\lambda_{(0,1)};\mathbb{R})$ it holds that

$$\psi(v) \in L^q(\lambda_{(0,1)};\mathbb{R}). \tag{69}$$

In the next step we observe that Lemma 2.5 (with $d = 1$, $p = \frac{p\delta}{n}$, $\beta = -\frac{n}{2p\delta}$, $A = A$ in the notation of Lemma 2.5) assures that there exists a unique

$$M \in L(L^{p\delta/n}(\lambda_{(0,1)};\mathbb{R}), L(H, H_{-n/(2p\delta)})) \tag{70}$$

which satisfies for all $v \in L^{\max\{p\delta/n, 4\}}(\lambda_{(0,1)};\mathbb{R})$, $u \in L^4(\lambda_{(0,1)};\mathbb{R})$ that

$$(Mv)u = v \cdot u \tag{71}$$

and

$$\|M\|_{L(L^{p\delta/n}(\lambda_{(0,1)};\mathbb{R}), L(H, H_{-n/(2p\delta)}))} \leq \sup_{w \in H_{n/(2p\delta)} \setminus \{0\}} \left[\frac{\|w\|_{L^{2p\delta/(p\delta-2n)}(\lambda_{(0,1)};\mathbb{R})}}{\|w\|_{H_{n/(2p\delta)}}} \right] < \infty. \tag{72}$$

Moreover, we note that Hölder's inequality shows that for all $u, v \in L^{2p}(\lambda_{(0,1)};\mathbb{R})$ it holds that

$$(Mv)u \in V. \tag{73}$$

Furthermore, we observe that Lemma 2.6 (with $p = p$, $\varepsilon = \frac{n}{2p\delta}$, $\beta = \beta$, $A = A$, $\mathcal{A} = \mathcal{A}$ in the notation of Lemma 2.6) and the fact that

$$\beta + \tfrac{n}{(2p\delta)} = -|\beta| + \tfrac{n}{(2p\delta)} < -\tfrac{1}{4} \tag{74}$$

yield that there exists a unique

$$\iota \in \gamma(H_{-n/(2p\delta)}, V_\beta) \tag{75}$$

which satisfies for all $v \in V$ that

$$\iota(v) = v \tag{76}$$

and

$$\|\iota\|_{\gamma(H_{-n/(2p\delta)},V_\beta)} \leq \left[\int_{\mathbb{R}} \frac{|x|^p}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx\right]^{1/p} \left[\sum_{l=1}^{\infty} l^{4(\beta+n/(2p\delta))}\right]^{1/2} < \infty. \tag{77}$$

In addition, note that (65), (69), (71), (73), and (76) demonstrate that for all $u, v \in \mathcal{L}^{2p}(\lambda_{(0,1)}; \mathbb{R})$ it holds that

$$\iota\big(M(\psi([v]_{\lambda_{(0,1)},\mathcal{B}(\mathbb{R})}))[u]_{\lambda_{(0,1)},\mathcal{B}(\mathbb{R})}\big) = \big[\{b(v(x)) \cdot u(x)\}_{x\in(0,1)}\big]_{\lambda_{(0,1)},\mathcal{B}(\mathbb{R})}. \tag{78}$$

Next observe that Lemma 2.1, (70), (72), (75), and (77) establish that

(a) for all $v \in L^{p\delta/n}(\lambda_{(0,1)}; \mathbb{R})$ it holds that $\iota \circ [M(v)] = \iota M(v) \in \gamma(H, V_\beta)$ and

$$\|\iota M(v)\|_{\gamma(H,V_\beta)} \leq \|\iota\|_{\gamma(H_{-n/(2p\delta)},V_\beta)} \|M(v)\|_{L(H,H_{-n/(2p\delta)})} < \infty \tag{79}$$

and

(b) that

$$(L^{p\delta/n}(\lambda_{(0,1)}; \mathbb{R}) \ni w \mapsto \iota M(w) \in \gamma(H, V_\beta)) \in L(L^{p\delta/n}(\lambda_{(0,1)}; \mathbb{R}), \gamma(H, V_\beta)). \tag{80}$$

Combining this with the fact that $\psi\colon V \to L^{p\delta/n}(\lambda_{(0,1)}; \mathbb{R})$ is n-times continuously Fréchet differentiable with globally bounded derivatives and the chain rule for differentiation implies that there exists a unique function

$$B \in C^n_b(V, \gamma(H, V_\beta)) \tag{81}$$

which satisfies for all $v \in V$, $u \in H$ that

$$B(v)u = \iota\big(M(\psi(v))u\big). \tag{82}$$

This and (78) prove items (i) and (ii). Next observe that (72) and (77) establish item (iii). It thus remains to prove items (iv) and (v). For this note that (80), the fact that $\psi\colon V \to L^{p\delta/n}(\lambda_{(0,1)}; \mathbb{R})$ is n-times continuously Fréchet differentiable with globally bounded derivatives, and the chain rule for differentiation assure that for all $k \in \{1, \ldots, n\}$, $v, v_1, \ldots, v_k \in V$, $u \in H$ it holds that

$$B^{(k)}(v)(v_1, \ldots, v_k)(u) = \iota M(\psi^{(k)}(v)(v_1, \ldots, v_k))u. \tag{83}$$

Therefore, we obtain that for all $k \in \{1, \ldots, n\}$, $v \in V$ it holds that

$$\begin{aligned}
\|B^{(k)}(v)\|_{L^{(k)}(V,\gamma(H,V_\beta))} &= \sup_{v_1,\ldots,v_k\in V\setminus\{0\}} \frac{\|B^{(k)}(v)(v_1,\ldots,v_k)\|_{\gamma(H,V_\beta)}}{\|v_1\|_V \cdot \ldots \cdot \|v_k\|_V} \\
&= \sup_{v_1,\ldots,v_k\in V\setminus\{0\}} \frac{\|\iota M(\psi^{(k)}(v)(v_1,\ldots,v_k))\|_{\gamma(H,V_\beta)}}{\|v_1\|_V \cdot \ldots \cdot \|v_k\|_V} \\
&\leq \|\iota M\|_{L(L^{p\delta/n}(\lambda_{(0,1)};\mathbb{R}),\gamma(H,V_\beta))} \sup_{v_1,\ldots,v_k\in V\setminus\{0\}} \frac{\|\psi^{(k)}(v)(v_1,\ldots,v_k)\|_{L^{p\delta/n}(\lambda_{(0,1)};\mathbb{R})}}{\|v_1\|_V \cdot \ldots \cdot \|v_k\|_V} \\
&\leq \|\iota M\|_{L(L^{p\delta/n}(\lambda_{(0,1)};\mathbb{R}),\gamma(H,V_\beta))} \|\psi^{(k)}(v)\|_{L^{(k)}(V,L^{p\delta/n}(\lambda_{(0,1)};\mathbb{R}))}.
\end{aligned} \tag{84}$$

This and (66) ensure that for all $k \in \{1, \ldots, n\}$ it holds that

$$\sup_{v\in V} \|B^{(k)}(v)\|_{L^{(k)}(V,\gamma(H,V_\beta))} \leq \|\iota M\|_{L(L^{p\delta/n}(\lambda_{(0,1)};\mathbb{R}),\gamma(H,V_\beta))} \left[\sup_{x\in\mathbb{R}} |b^{(k)}(x)|\right]. \tag{85}$$

Combining this with (79), (72), and (77) shows that for all $k \in \{1, \ldots, n\}$ it holds that

$$\begin{aligned}
&\sup_{v \in V} \|B^{(k)}(v)\|_{L^{(k)}(V, \gamma(H, V_\beta))} \\
&\le \|\iota\|_{\gamma(H_{-n/(2p\delta)}, V_\beta)} \|M\|_{L(L^{p\delta/n}(\lambda_{(0,1)};\mathbb{R}), L(H, H_{-n/(2p\delta)}))} \left[\sup_{x \in \mathbb{R}} |b^{(k)}(x)|\right] \\
&\le \left[\int_{\mathbb{R}} \frac{|x|^p}{\sqrt{2\pi}} e^{-x^2/2} \, dx\right]^{1/p} \left[\sum_{l=1}^{\infty} l^{4(\beta + n/(2p\delta))}\right]^{1/2} \\
&\quad \cdot \left[\sup_{w \in H_{n/(2p\delta)} \setminus \{0\}} \frac{\|w\|_{L^{2p\delta/(p\delta - 2n)}(\lambda_{(0,1)};\mathbb{R})}}{\|w\|_{H_{n/(2p\delta)}}}\right] \left[\sup_{x \in \mathbb{R}} |b^{(k)}(x)|\right] < \infty.
\end{aligned} \tag{86}$$

This proves item (iv). Next note that (83) demonstrates that for all $k \in \{1, \ldots, n\}$, $r \in [\frac{p\delta}{n-k\delta}, \infty)$, $v, w \in L^{\max\{r,p\}}(\lambda_{(0,1)}; \mathbb{R})$, $v_1, \ldots, v_k \in V \setminus \{0\}$ it holds that

$$\begin{aligned}
&\frac{\|(B^{(k)}(v) - B^{(k)}(w))(v_1, \ldots, v_k)\|_{\gamma(H, V_\beta)}}{\|v_1\|_V \cdot \ldots \cdot \|v_k\|_V} \\
&= \frac{\|\iota M([\psi^{(k)}(v) - \psi^{(k)}(w)](v_1, \ldots, v_k))\|_{\gamma(H, V_\beta)}}{\|v_1\|_V \cdot \ldots \cdot \|v_k\|_V} \\
&\le \|\iota M\|_{L(L^{p\delta/n}(\lambda_{(0,1)};\mathbb{R}), \gamma(H, V_\beta))} \frac{\|[\psi^{(k)}(v) - \psi^{(k)}(w)](v_1, \ldots, v_k)\|_{L^{p\delta/n}(\lambda_{(0,1)};\mathbb{R})}}{\|v_1\|_V \cdot \ldots \cdot \|v_k\|_V} \\
&\le \|\iota M\|_{L(L^{p\delta/n}(\lambda_{(0,1)};\mathbb{R}), \gamma(H, V_\beta))} \|\psi^{(k)}(v) - \psi^{(k)}(w)\|_{L^{(k)}(V, L^{p\delta/n}(\lambda_{(0,1)};\mathbb{R}))}.
\end{aligned} \tag{87}$$

This, (67), and (79) assure that for all $k \in \{1, \ldots, n\}$, $r \in [\frac{p\delta}{n-k\delta}, \infty)$ it holds that

$$\begin{aligned}
&\sup_{\substack{v, w \in L^{\max\{r,p\}}(\lambda_{(0,1)};\mathbb{R}), \\ v \neq w}} \left[\frac{\|B^{(k)}(v) - B^{(k)}(w)\|_{L^{(k)}(V, \gamma(H, V_\beta))}}{\|v - w\|_{L^r(\lambda_{(0,1)};\mathbb{R})}}\right] \\
&\le \|\iota M\|_{L(L^{p\delta/n}(\lambda_{(0,1)};\mathbb{R}), \gamma(H, V_\beta))} \left[\sup_{\substack{x, y \in \mathbb{R}, \\ x \neq y}} \frac{|b^{(k)}(x) - b^{(k)}(y)|}{|x - y|}\right] \\
&\le \|\iota\|_{\gamma(H_{-n/(2p\delta)}, V_\beta)} \|M\|_{L(L^{p\delta/n}(\lambda_{(0,1)};\mathbb{R}), L(H, H_{-n/(2p\delta)}))} \left[\sup_{\substack{x, y \in \mathbb{R}, \\ x \neq y}} \frac{|b^{(k)}(x) - b^{(k)}(y)|}{|x - y|}\right].
\end{aligned} \tag{88}$$

Combining (88) with (77) and (72) establishes item (v). The proof of Proposition 2 is thus completed. □

Corollary 1. *Consider the notation in Subsection 1.1, let* $n \in \mathbb{N}$, $\beta \in (-\infty, -1/4)$, $p \in (\max\{\frac{n+1}{2(|\beta| - 1/4)}, 2(n+1)\}, \infty)$, $(V, \|\cdot\|_V) = (L^p(\lambda_{(0,1)}; \mathbb{R}), \|\cdot\|_{L^p(\lambda_{(0,1)};\mathbb{R})})$, *let* $b\colon \mathbb{R} \to \mathbb{R}$ *be an* n*-times continuously differentiable function with globally Lipschitz continuous and globally bounded derivatives, let* $A\colon D(A) \subseteq V \to V$ *be the Laplacian with Dirichlet boundary conditions on* V*, and let* $(V_r, \|\cdot\|_{V_r})$, $r \in \mathbb{R}$, *be a family of interpolation spaces associated to* $-A$*. Then*

(i) *there exists a unique continuous function* $B\colon V \to \gamma(L^2(\lambda_{(0,1)};\mathbb{R}), V_\beta)$ *which satisfies for all* $v, u \in \mathcal{L}^{2p}(\lambda_{(0,1)};\mathbb{R})$ *that*

$$B\big([v]_{\lambda_{(0,1)},\mathcal{B}(\mathbb{R})}\big)[u]_{\lambda_{(0,1)},\mathcal{B}(\mathbb{R})} = \big[\{b(v(x)) \cdot u(x)\}_{x\in(0,1)}\big]_{\lambda_{(0,1)},\mathcal{B}(\mathbb{R})} \tag{89}$$

and

(ii) *it holds that* B *is* n*-times continuously Fréchet differentiable with globally Lipschitz continuous and globally bounded derivatives.*

Proof of Corollary 1. First, note that for all $k \in \{1, 2, \ldots, n\}$ it holds that

$$\begin{aligned} &\tfrac{1}{p}\max\big\{\tfrac{n}{2(|\beta|-1/4)}, 2n\big\} = \tfrac{n}{p}\max\big\{\tfrac{1}{2(|\beta|-1/4)}, 2\big\} \\ &= \tfrac{n}{p(n+1)}\max\big\{\tfrac{n+1}{2(|\beta|-1/4)}, 2(n+1)\big\} < \tfrac{n}{(n+1)} < 1 \end{aligned} \tag{90}$$

and

$$\frac{p(\frac{n}{n+1})}{n-k(\frac{n}{n+1})} = \frac{p}{(n+1)-k} = \frac{p}{1+n-k} \le p. \tag{91}$$

Items (i), (ii), (iii), and (v) of Proposition 2 (with $n = n$, $\beta = \beta$, $p = p$, $b = b$, $\mathcal{A} = A$, $k = n$, $\delta = \frac{n}{(n+1)}$, $r = p$ in the notation of Proposition 2) therefore establish items (i) and (ii). The proof of Corollary 1 is thus completed. □

3. **Mild stochastic calculus in Banach spaces.** In this section we generalize the machinery in [14, Section 5] from separable Hilbert spaces to separable UMD Banach spaces with type 2. Throughout this section we frequently employ the well-known fact that for all separable $\mathbb{R}$-Banach spaces $(V_1, \|\cdot\|_{V_1})$ and $(V_2, \|\cdot\|_{V_2})$ with $V_1 \subseteq V_2$ continuously it holds that

$$\mathcal{B}(V_1) = \{B \in \mathcal{P}(V_1)\colon (\exists\, A \in \mathcal{B}(V_2)\colon B = A \cap V_1)\} \subseteq \mathcal{B}(V_2) \tag{92}$$

(cf., e.g., [1, Lemma 2.2] and [24, Theorem 2.4 in Chapter V]).

3.1. **Setting.** Throughout this section we frequently assume the following setting. Consider the notation in Subsection 1.1, let $t_0 \in [0,\infty)$, $T \in (t_0,\infty)$, $\angle = \big\{(t_1,t_2) \in [t_0,T]^2\colon t_1 < t_2\big\}$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with a normal filtration $\mathbb{F} = (\mathbb{F}_t)_{t\in[t_0,T]}$, let $(W_t)_{t\in[t_0,T]}$ be an Id_U-cylindrical $(\Omega,\mathcal{F},\mathbb{P},\mathbb{F})$-Wiener process, let $(\check{V}, \|\cdot\|_{\check{V}})$, $(V, \|\cdot\|_V)$, $(\hat{V}, \|\cdot\|_{\hat{V}})$, and $(\mathcal{V}, \|\cdot\|_V)$ be separable UMD $\mathbb{R}$-Banach spaces with type 2 which satisfy $\check{V} \subseteq V \subseteq \hat{V}$ continuously and densely, let $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be a separable $\mathbb{R}$-Hilbert space, let $\mathbb{U} \subseteq U$ be an orthonormal basis of U, and for every separable $\mathbb{R}$-Banach space $(E, \|\cdot\|_E)$, every convex set $A \subseteq \mathbb{R}$, and all $X \in \mathcal{M}(\mathcal{B}(A)\otimes\mathcal{F}, \mathcal{B}(E))$, $a, b \in \mathcal{M}(\mathcal{F}, \mathcal{B}(A))$ with $a \le b$ and $\mathbb{P}\big(\int_a^b \|X_s\|_E\, ds < \infty\big) = 1$ let $\int_a^b X_s\, \mathbf{ds} \in L^0(\mathbb{P};E)$ be given by $\int_a^b X_s\, \mathbf{ds} = \big[\int_a^b \mathbb{1}_{\{\int_a^b \|X_u\|_E\, du<\infty\}} X_s\, ds\big]_{\mathbb{P},\mathcal{B}(E)}$.

3.2. **Mild Itô processes.**

Definition 3.1 (Mild Itô process)**.** Consider the notation in Subsection 1.1, let $(\check{V}, \|\cdot\|_{\check{V}})$, $(V, \|\cdot\|_V)$, and $(\hat{V}, \|\cdot\|_{\hat{V}})$ be separable UMD $\mathbb{R}$-Banach spaces with type 2 which satisfy $\check{V} \subseteq V \subseteq \hat{V}$ continuously and densely, let $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be a separable $\mathbb{R}$-Hilbert space, let $t_0 \in [0,\infty)$, $T \in (t_0,\infty)$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with a normal filtration $\mathbb{F} = (\mathbb{F}_t)_{t\in[t_0,T]}$, and let $(W_t)_{t\in[t_0,T]}$ be an Id_U-cylindrical $(\Omega,\mathcal{F},\mathbb{P},\mathbb{F})$-Wiener process. Then we say that X is a mild Itô process on $(\Omega,\mathcal{F},\mathbb{P},\mathbb{F},W,(\check{V}, \|\cdot\|_{\check{V}}),(V, \|\cdot\|_V),(\hat{V}, \|\cdot\|_{\hat{V}}))$ with evolution family S, mild drift Y, and mild diffusion Z (we say that X is a mild Itô process with evolution family

S, mild drift Y, and mild diffusion Z, we say that X is a mild Itô process) if and only if it holds

(i) that $X \in \mathbb{M}([t_0,T] \times \Omega, V)$ is an $\mathbb{F}/\mathcal{B}(V)$-predictable stochastic process,
(ii) that $Y \in \mathbb{M}([t_0,T] \times \Omega, \hat{V})$ is an $\mathbb{F}/\mathcal{B}(\hat{V})$-predictable stochastic process,
(iii) that $Z \in \mathbb{M}([t_0,T] \times \Omega, \gamma(U,\hat{V}))$ is an $\mathbb{F}/\mathcal{B}(\gamma(U,\hat{V}))$-predictable stochastic process,
(iv) that $S \in \mathbb{M}(\{(t_1,t_2) \in [t_0,T]^2 \colon t_1 < t_2\}, L(\hat{V},\check{V}))$ is a $\mathcal{B}(\{(t_1,t_2) \in [t_0,T]^2 \colon t_1 < t_2\})/\, \mathcal{S}(\hat{V},\check{V})$-measurable function which satisfies for all $t_1,t_2,t_3 \in [t_0,T]$ with $t_1 < t_2 < t_3$ that $S_{t_2,t_3} S_{t_1,t_2} = S_{t_1,t_3}$,
(v) that $\forall\, t \in (t_0,T] \colon \mathbb{P}(\int_{t_0}^t \|S_{s,t} Y_s\|_{\check{V}} + \|S_{s,t} Z_s\|^2_{\gamma(U,\check{V})}\, ds < \infty) = 1$, and
(vi) that for all $t \in (t_0,T]$ it holds that

$$[X_t]_{\mathbb{P},\mathcal{B}(V)} = \left[S_{t_0,t} X_{t_0} + \int_{t_0}^t \mathbb{1}_{\{\int_{t_0}^t \|S_{u,t}\, Y_u\|_V\, du < \infty\}} S_{s,t} Y_s \, ds \right]_{\mathbb{P},\mathcal{B}(V)} + \int_{t_0}^t S_{s,t} Z_s \, dW_s. \tag{93}$$

Lemma 3.2 (Regularization of mild Itô processes)**.** *Assume the setting in Subsection 3.1 and let $X \colon [t_0,T] \times \Omega \to V$ be a mild Itô process with evolution family $S \colon \angle \to L(\hat{V},\check{V})$, mild drift $Y \colon [t_0,T] \times \Omega \to \hat{V}$, and mild diffusion $Z \colon [t_0,T] \times \Omega \to \gamma(U,\hat{V})$. Then there exists an up to indistinguishability unique stochastic process $\bar{X} \colon [t_0,T] \times \Omega \to \check{V}$ with continuous sample paths which satisfies $\forall\, t \in [t_0,T) \colon \mathbb{P}\big(\bar{X}_t = S_{t,T} X_t\big) = 1$.*

Proof of Lemma 3.2. The assumption that X is a mild Itô process, in particular, ensures that $\mathbb{P}(\int_{t_0}^T \|S_{s,T} Y_s\|_{\check{V}} + \|S_{s,T} Z_s\|^2_{\gamma(U,\check{V})}\, ds < \infty) = 1$. This implies that there exists a stochastic process $\bar{X} \colon [t_0,T] \times \Omega \to \check{V}$ with continuous sample paths which satisfies for all $t \in [t_0,T]$ that

$$[\bar{X}_t]_{\mathbb{P},\mathcal{B}(\check{V})} = [S_{t_0,T}\, X_{t_0}]_{\mathbb{P},\mathcal{B}(\check{V})} + \int_{t_0}^t S_{s,T}\, Y_s \, \mathbf{ds} + \int_{t_0}^t S_{s,T}\, Z_s \, dW_s. \tag{94}$$

Next observe that Definition 3.1 ensures for all $t \in (t_0,T)$ that

$$\begin{aligned} &[S_{t_0,T}\, X_{t_0}]_{\mathbb{P},\mathcal{B}(\check{V})} + \int_{t_0}^t S_{s,T}\, Y_s \, \mathbf{ds} + \int_{t_0}^t S_{s,T}\, Z_s \, dW_s \\ &= S_{t,T} \left([S_{t_0,t}\, X_{t_0}]_{\mathbb{P},\mathcal{B}(\check{V})} + \int_{t_0}^t S_{s,t}\, Y_s \, \mathbf{ds} + \int_{t_0}^t S_{s,t}\, Z_s \, dW_s \right) = S_{t,T}\, [X_t]_{\mathbb{P},\mathcal{B}(\check{V})}. \end{aligned} \tag{95}$$

Combining this and (94) shows that for all $t \in [t_0,T)$ it holds that

$$[\bar{X}_t]_{\mathbb{P},\mathcal{B}(\check{V})} = [S_{t,T} X_t]_{\mathbb{P},\mathcal{B}(\check{V})}. \tag{96}$$

Moreover, observe that for all stochastic processes $A, B \colon [0,T] \times \Omega \to \check{V}$ with continuous sample paths which satisfy $\forall\, t \in [t_0,T) \colon \mathbb{P}\big(A_t = B_t\big) = 1$ it holds that $\mathbb{P}\big(\forall\, t \in [t_0,T] \colon A_t = B_t\big) = 1$. Combining this with (96) completes the proof of Lemma 3.2. □

Lemma 3.3 (Regularization of mild Itô processes)**.** *Assume the setting in Subsection 3.1, let $X \colon [t_0,T] \times \Omega \to V$ be a mild Itô process with evolution family $S \colon \angle \to L(\hat{V},\check{V})$, mild drift $Y \colon [t_0,T] \times \Omega \to \hat{V}$, and mild diffusion $Z \colon [t_0,T] \times \Omega \to \gamma(U,\hat{V})$, and let $\bar{X} \colon [t_0,T] \times \Omega \to \check{V}$ be a stochastic process with continuous sample paths which satisfies $\forall\, t \in [t_0,T) \colon \mathbb{P}\big(\bar{X}_t = S_{t,T} X_t\big) = 1$. Then*

(i) *it holds that* $\bar{X}$ *is* $\mathbb{F}/\mathcal{B}(\check{V})$*-predictable,*
(ii) *it holds that* $\mathbb{P}(\bar{X}_T = X_T) = 1$*,*
(iii) *it holds that* $\mathbb{P}(\int_{t_0}^T \|S_{s,T}Y_s\|_{\check{V}} + \|S_{s,T}Z_s\|^2_{\gamma(U,\check{V})}\, ds < \infty) = 1$*, and*
(iv) *it holds that*

$$\forall\, t \in [t_0, T] \colon [\bar{X}_t]_{\mathbb{P},\mathcal{B}(\check{V})} = [S_{t_0,T}\, X_{t_0}]_{\mathbb{P},\mathcal{B}(\check{V})} + \int_{t_0}^t S_{s,T}\, Y_s\, \mathbf{ds} + \int_{t_0}^t S_{s,T}\, Z_s\, dW_s. \quad (97)$$

Proof of Lemma 3.3. The assumption that $\bar{X}$ has continuous sample paths, the fact that X is $\mathbb{F}/\mathcal{B}(V)$-adapted, and the fact that $\forall\, t \in [t_0, T) \colon \mathbb{P}(\bar{X}_t = S_{t,T}X_t) = 1$ establish item (i). Moreover, note that the assumption that X is a mild Itô process proves item (iii). In addition, observe that the assumption that $\forall\, t \in [t_0, T) \colon \mathbb{P}\big(\bar{X}_t = S_{t,T}X_t\big) = 1$ implies that for all $t \in [t_0, T)$ it holds that

$$[\bar{X}_t]_{\mathbb{P},\mathcal{B}(\check{V})} = [S_{t,T}X_t]_{\mathbb{P},\mathcal{B}(\check{V})} = [S_{t_0,T}\, X_{t_0}]_{\mathbb{P},\mathcal{B}(\check{V})} + \int_{t_0}^t S_{s,T}\, Y_s\, \mathbf{ds} + \int_{t_0}^t S_{s,T}\, Z_s\, dW_s. \quad (98)$$

Combining this with the assumption that $\bar{X}$ has continuous sample paths shows items (iv) and (ii). The proof of Lemma 3.3 is thus completed. □

3.3. **Standard Itô formula.** Theorem 3.4 is an elementary modification of Theorem 2.4 in Brzeźniak et al. [5] (cf. Lemma 2.2 in Subsection 2.1 above).

Theorem 3.4. *Assume the setting in Subsection 3.1, let* $\varphi = (\varphi(t,x))_{t\in[t_0,T],x\in V} \in C^{1,2}([t_0,T] \times V, \mathcal{V})$*,* $\xi \in \mathcal{M}(\mathbb{F}_{t_0}, \mathcal{B}(V))$*, let* $Z \colon [t_0,T] \times \Omega \to \gamma(U,V)$ *be an* $\mathbb{F}/\mathcal{B}(\gamma(U,V))$*-predictable stochastic process which satisfies* $\mathbb{P}(\int_{t_0}^T \|Z_t\|^2_{\gamma(U,V)}\, dt < \infty) = 1$*, let* $Y \colon [t_0,T] \times \Omega \to V$ *be an* $\mathbb{F}/\mathcal{B}(V)$*-predictable stochastic process which satisfies* $\mathbb{P}(\int_{t_0}^T \|Y_t\|_V\, dt < \infty) = 1$*, and let* $X \colon [t_0,T] \times \Omega \to V$ *be an* $\mathbb{F}/\mathcal{B}(V)$*-predictable stochastic process which satisfies for all* $t \in [t_0,T]$ *that*

$$[X_t]_{\mathbb{P},\mathcal{B}(V)} = [\xi]_{\mathbb{P},\mathcal{B}(V)} + \int_{t_0}^t Y_s\, \mathbf{ds} + \int_{t_0}^t Z_s\, dW_s. \quad (99)$$

Then

(i) *it holds that* $\mathbb{P}(\int_{t_0}^T \|(\frac{\partial}{\partial t}\varphi)(s, X_s)\|_{\mathcal{V}}\, ds < \infty) = 1$*,*
(ii) *it holds that* $\mathbb{P}(\int_{t_0}^T \|(\frac{\partial}{\partial x}\varphi)(s, X_s)Y_s\|_{\mathcal{V}}\, ds < \infty) = 1$*,*
(iii) *it holds that* $\mathbb{P}(\int_{t_0}^T \|(\frac{\partial}{\partial x}\varphi)(s, X_s)Z_s\|^2_{\gamma(U,\mathcal{V})}\, ds < \infty) = 1$*,*
(iv) *it holds for all* $\omega \in \Omega$*,* $s \in [t_0, T]$ *that there exists a unique* $v \in \mathcal{V}$ *such that*

$$\inf_{\substack{I \subseteq \mathbb{U}, \\ \#_I < \infty}} \sup_{\substack{I \subseteq J \subseteq \mathbb{U}, \\ \#_J < \infty}} \left\| v - \sum_{u\in J} (\tfrac{\partial^2}{\partial x^2}\varphi)(s, X_s(\omega))(Z_s(\omega)u, Z_s(\omega)u) \right\|_{\mathcal{V}} = 0, \quad (100)$$

(v) *it holds that*

$$\mathbb{P}\left(\int_{t_0}^T \left\| \sum_{u\in\mathbb{U}} (\tfrac{\partial^2}{\partial x^2}\varphi)(s, X_s)(Z_s u, Z_s u) \right\|_{\mathcal{V}} ds < \infty \right) = 1, \quad (101)$$

and

(vi) *it holds for all* $t_1 \in [t_0, T]$ *that*

$$[\varphi(t_1, X_{t_1}) - \varphi(t_0, X_{t_0})]_{\mathbb{P},\mathcal{B}(\mathcal{V})} = \int_{t_0}^{t_1} \big[(\tfrac{\partial}{\partial t}\varphi)(s, X_s) + (\tfrac{\partial}{\partial x}\varphi)(s, X_s)Y_s\big]\, \mathbf{ds} \quad (102)$$

$$+ \tfrac{1}{2} \int_{t_0}^{t_1} \sum_{u \in \mathbb{U}} (\tfrac{\partial^2}{\partial x^2}\varphi)(s, X_s)(Z_s u, Z_s u) \, \mathbf{ds} + \int_{t_0}^{t_1} (\tfrac{\partial}{\partial x}\varphi)(s, X_s) Z_s \, dW_s.$$

3.4. **Mild Itô formula for stopping times.**

Theorem 3.5 (Mild Itô formula)**.** *Assume the setting in Subsection 3.1, let $X\colon [t_0, T] \times \Omega \to V$ be a mild Itô process with evolution family $S\colon \angle \to L(\hat{V}, \check{V})$, mild drift $Y\colon [t_0, T] \times \Omega \to \hat{V}$, and mild diffusion $Z\colon [t_0, T] \times \Omega \to \gamma(U, \hat{V})$, let $\bar{X}\colon [t_0, T] \times \Omega \to \check{V}$ be a stochastic process with continuous sample paths which satisfies $\forall\, t \in [t_0, T)\colon \mathbb{P}(\bar{X}_t = S_{t,T} X_t) = 1$ (see Lemma 3.2), let $r \in [t_0, T)$, $\varphi = (\varphi(t,x))_{t \in [r,T],\, x \in \check{V}} \in C^{1,2}([r,T] \times \check{V}, \mathcal{V})$, and let $\tau\colon \Omega \to [r, T]$ be an $\mathbb{F}$-stopping time. Then*

(i) it holds that $\mathbb{P}(\int_r^T \|(\tfrac{\partial}{\partial x}\varphi)(s, S_{s,T}X_s) S_{s,T} Y_s\|_{\mathcal{V}} \, ds < \infty) = 1$,
(ii) it holds that $\mathbb{P}(\int_r^T \|(\tfrac{\partial}{\partial x}\varphi)(s, S_{s,T}X_s) S_{s,T} Z_s\|^2_{\gamma(U,\mathcal{V})} \, ds < \infty) = 1$,
(iii) it holds that $\mathbb{P}(\int_r^T \|(\tfrac{\partial}{\partial t}\varphi)(s, S_{s,T}X_s)\|_{\mathcal{V}} \, ds < \infty) = 1$,
(iv) it holds that $\mathbb{P}(\int_r^T \|(\tfrac{\partial^2}{\partial x^2}\varphi)(s, S_{s,T}X_s)\|_{L^{(2)}(\check{V},\mathcal{V})} \|S_{s,T} Z_s\|^2_{\gamma(U,\check{V})} \, ds < \infty) = 1$,
(v) it holds for all $\omega \in \Omega$, $s \in [r,T]$ that there exists a unique $v \in \mathcal{V}$ such that

$$\inf_{\substack{I \subseteq \mathbb{U},\\ \#_I < \infty}} \sup_{\substack{I \subseteq J \subseteq \mathbb{U},\\ \#_J < \infty}} \left\| v - \sum_{u \in J} (\tfrac{\partial^2}{\partial x^2}\varphi)(s, S_{s,T}X_s(\omega))(S_{s,T}Z_s(\omega)u, S_{s,T}Z_s(\omega)u) \right\|_{\mathcal{V}} = 0, \tag{103}$$

(vi) it holds that

$$\mathbb{P}\biggl(\int_r^T \left\| \sum_{u \in \mathbb{U}} (\tfrac{\partial^2}{\partial x^2}\varphi)(s, S_{s,T}X_s)(S_{s,T}Z_s u, S_{s,T}Z_s u) \right\|_{\mathcal{V}} ds < \infty \biggr) = 1, \tag{104}$$

and

(vii) it holds that

$$\begin{aligned} [\varphi(\tau, \bar{X}_\tau)]_{\mathbb{P},\mathcal{B}(\mathcal{V})} &= [\varphi(r, S_{r,T}X_r)]_{\mathbb{P},\mathcal{B}(\mathcal{V})} + \int_r^\tau (\tfrac{\partial}{\partial x}\varphi)(s, S_{s,T}X_s)\, S_{s,T}\, Z_s \, dW_s \\ &\quad + \int_r^\tau \bigl[(\tfrac{\partial}{\partial t}\varphi)(s, S_{s,T}X_s) + (\tfrac{\partial}{\partial x}\varphi)(s, S_{s,T}X_s)\, S_{s,T}\, Y_s \bigr] \, \mathbf{ds} \\ &\quad + \tfrac{1}{2} \int_r^\tau \sum_{u \in \mathbb{U}} (\tfrac{\partial^2}{\partial x^2}\varphi)(s, S_{s,T}X_s)(S_{s,T}Z_s u, S_{s,T}Z_s u) \, \mathbf{ds}. \end{aligned} \tag{105}$$

Proof of Theorem 3.5. Throughout this proof let $\varphi_{1,0}\colon [r,T] \times \check{V} \to \mathcal{V}$, $\varphi_{0,1}\colon [r,T] \times \check{V} \to L(\check{V}, \mathcal{V})$, and $\varphi_{0,2}\colon [r,T] \times \check{V} \to L^{(2)}(\check{V}, \mathcal{V})$ be the functions which satisfy for all $t \in [r,T]$, $x, v_1, v_2 \in \check{V}$ that $\varphi_{1,0}(t,x) = \bigl(\tfrac{\partial}{\partial t}\varphi\bigr)(t,x)$, $\varphi_{0,1}(t,x)\, v_1 = \bigl(\tfrac{\partial}{\partial x}\varphi\bigr)(t,x)\, v_1$, and $\varphi_{0,2}(t,x)(v_1,v_2) = \bigl(\tfrac{\partial^2}{\partial x^2}\varphi\bigr)(t,x)(v_1,v_2)$. Note that Lemma 3.3 ensures that $\bar{X}$ is an $\mathbb{F}/\mathcal{B}(\check{V})$-adapted stochastic process with continuous sample paths which satisfies for all $t \in [t_0, T]$ that

$$[\bar{X}_t]_{\mathbb{P},\mathcal{B}(\check{V})} = [S_{t_0,T}\, X_{t_0}]_{\mathbb{P},\mathcal{B}(\check{V})} + \int_{t_0}^t S_{s,T}\, Y_s \, \mathbf{ds} + \int_{t_0}^t S_{s,T}\, Z_s \, dW_s. \tag{106}$$

Moreover, the assumption that $\varphi \in C^{1,2}([r,T] \times \check{V}, \mathcal{V})$, the assumption that $\bar{X}\colon [t_0, T] \times \Omega \to \check{V}$ has continuous sample paths, and the fact that $\forall\, t \in$

$[t_0, T]\colon \mathbb{P}(\int_{t_0}^t \|S_{s,t} Y_s\|_{\check{V}} + \|S_{s,t} Z_s\|^2_{\gamma(U,\check{V})}\, ds < \infty) = 1$ imply that

$$\mathbb{P}\left(\int_r^T \|\varphi_{0,1}(s, \bar{X}_s) S_{s,T} Y_s\|_{\mathcal{V}} + \|\varphi_{0,1}(s, \bar{X}_s) S_{s,T} Z_s\|^2_{\gamma(U,\mathcal{V})}\, ds < \infty\right) = 1 \tag{107}$$

and

$$\mathbb{P}\left(\int_r^T \|\varphi_{1,0}(s, \bar{X}_s)\|_{\mathcal{V}} + \|\varphi_{0,2}(s, \bar{X}_s)\|_{L^{(2)}(\check{V},\mathcal{V})} \|S_{s,T} Z_s\|^2_{\gamma(U,\check{V})}\, ds < \infty\right) = 1. \tag{108}$$

Combining this with, e.g., Lemma 3.1 in [15] proves items (i)–(iv). Then note that Lemma 3.3 and Theorem 3.4 show

(a) that for all $\omega \in \Omega$, $s \in [r, T]$ there exists a unique $v \in \mathcal{V}$ such that

$$\inf_{\substack{I \subseteq \mathbb{U},\\ \#_I < \infty}} \sup_{\substack{I \subseteq J \subseteq \mathbb{U},\\ \#_J < \infty}} \left\| v - \sum_{h \in J} \varphi_{0,2}(s, \bar{X}_s(\omega))(S_{s,T} Z_s(\omega) u, S_{s,T} Z_s(\omega) u) \right\|_{\mathcal{V}} = 0, \tag{109}$$

(b) that

$$\mathbb{P}\left(\int_r^T \left\| \sum_{u \in \mathbb{U}} \varphi_{0,2}(s, \bar{X}_s)(S_{s,T} Z_s u, S_{s,T} Z_s u) \right\|_{\mathcal{V}} ds < \infty\right) = 1, \tag{110}$$

and

(c) that

$$\begin{aligned} [\varphi(\tau, \bar{X}_\tau)]_{\mathbb{P},\mathcal{B}(\mathcal{V})} &= [\varphi(r, \bar{X}_r)]_{\mathbb{P},\mathcal{B}(\mathcal{V})} + \int_r^\tau \varphi_{1,0}(s, \bar{X}_s) + \varphi_{0,1}(s, \bar{X}_s) S_{s,T} Y_s \,\mathbf{ds} \\ &+ \int_r^\tau \varphi_{0,1}(s, \bar{X}_s) S_{s,T} Z_s \, dW_s + \tfrac{1}{2} \int_r^\tau \sum_{u \in \mathbb{U}} \varphi_{0,2}(s, \bar{X}_s)\, (S_{s,T} Z_s u, S_{s,T} Z_s u) \,\mathbf{ds}. \end{aligned} \tag{111}$$

Combining this with, e.g., Lemma 3.1 in [15], the fact that $\forall\, t \in [t_0, T)\colon \mathbb{P}\big(\bar{X}_t = S_{t,T}\, X_t\big) = 1$, and the fact that $\forall\, t \in [t_0, T]\colon \mathbb{P}(\sum_{u \in \mathbb{U}} \varphi_{0,2}(s, \bar{X}_s)(S_{s,T} Z_s u, S_{s,T} Z_s u) = \sum_{u \in \mathbb{U}} \varphi_{0,2}(s, S_{s,T} X_s)(S_{s,T} Z_s u, S_{s,T} Z_s u)) = 1$ demonstrates that item (v) holds, that item (vi) holds, and that for all $t \in [r, T]$ it holds that

$$\begin{aligned} [\varphi(t, \bar{X}_t)]_{\mathbb{P},\mathcal{B}(\mathcal{V})} = [\varphi(r, S_{r,T} X_r)]_{\mathbb{P},\mathcal{B}(\mathcal{V})} &+ \int_r^t \varphi_{0,1}(s, S_{s,T} X_s)\, S_{s,T}\, Z_s \, dW_s \\ &+ \int_r^t \varphi_{1,0}(s, S_{s,T} X_s) + \varphi_{0,1}(s, S_{s,T} X_s)\, S_{s,T}\, Y_s \,\mathbf{ds} \\ &+ \tfrac{1}{2} \int_r^t \sum_{u \in \mathbb{U}} \varphi_{0,2}(s, S_{s,T} X_s)\, (S_{s,T} Z_s u, S_{s,T} Z_s u) \,\mathbf{ds}. \end{aligned} \tag{112}$$

This implies item (vii). The proof of Theorem 3.5 is thus completed. □

Definition 3.6 (Extended mild Kolmogorov operators)**.** Assume the setting in Subsection 3.1, let $S\colon \angle \to L(\hat{V}, \check{V})$ be a $\mathcal{B}(\angle)/\mathcal{S}(\hat{V}, \check{V})$-measurable function which satisfies for all $t_1, t_2, t_3 \in [t_0, T]$ with $t_1 < t_2 < t_3$ that $S_{t_2,t_3} S_{t_1,t_2} = S_{t_1,t_3}$, and let $(t_1, t_2) \in \angle$. Then we denote by $\mathcal{L}^S_{t_1,t_2}\colon C^2(\check{V}, \mathcal{V}) \to C(V \times \hat{V} \times \gamma(U, \hat{V}), \mathcal{V})$ the function which satisfies for all $\varphi \in C^2(\check{V}, \mathcal{V})$, $x \in V$, $y \in \hat{V}$, $z \in \gamma(U, \hat{V})$ that

$$\big(\mathcal{L}^S_{t_1,t_2} \varphi\big)(x, y, z) = \varphi'(S_{t_1,t_2}\, x)\, S_{t_1,t_2}\, y + \tfrac{1}{2} \sum_{u \in \mathbb{U}} \varphi''(S_{t_1,t_2}\, x)(S_{t_1,t_2} z u, S_{t_1,t_2} z u). \tag{113}$$

The next corollary of Theorem 3.5 specialises Theorem 3.5 to the case where $r = t_0$ and where the test function $(\varphi(t,x))_{t\in[t_0,T],\, x\in\check{V}} \in C^{1,2}([t_0,T]\times\check{V},\mathcal{V})$ depends on $x \in \check{V}$ only.

Corollary 2. *Assume the setting in Subsection 3.1, let $X\colon [t_0,T]\times\Omega \to V$ be a mild Itô process with evolution family $S\colon \angle \to L(\hat{V},\check{V})$, mild drift $Y\colon [t_0,T]\times\Omega \to \hat{V}$, and mild diffusion $Z\colon [t_0,T]\times\Omega \to \gamma(U,\hat{V})$, let $\bar{X}\colon [t_0,T]\times\Omega \to \check{V}$ be a stochastic process with continuous sample paths which satisfies $\forall\, t \in [t_0,T)\colon \mathbb{P}\big(\bar{X}_t = S_{t,T}X_t\big) = 1$ (see Lemma 3.2), let $\varphi \in C^2(\check{V},\mathcal{V})$, and let $\tau\colon \Omega \to [t_0,T]$ be an $\mathbb{F}$-stopping time. Then*

(i) it holds that $\mathbb{P}(\int_{t_0}^T \|(\mathcal{L}^S_{s,T}\varphi)(X_s,Y_s,Z_s)\|_{\mathcal{V}}\,ds < \infty) = 1$,

(ii) it holds that $\mathbb{P}(\int_{t_0}^T \|\varphi'(S_{s,T}X_s)S_{s,T}Z_s\|^2_{\gamma(U,\mathcal{V})}\,ds < \infty) = 1$, and

(iii) it holds that

$$[\varphi(\bar{X}_\tau)]_{\mathbb{P},\mathcal{B}(\mathcal{V})} = [\varphi(S_{t_0,T}X_{t_0})]_{\mathbb{P},\mathcal{B}(\mathcal{V})} + \int_{t_0}^{\tau} (\mathcal{L}^S_{s,T}\varphi)(X_s,Y_s,Z_s)\,\mathbf{ds} + \int_{t_0}^{\tau} \varphi'(S_{s,T}X_s)\,S_{s,T}\,Z_s\,dW_s. \tag{114}$$

The next result, Corollary 3, specializes Corollary 2 to the case where $\forall\,\omega \in \Omega\colon \tau(\omega) = T$. Corollary 3 is an immediate consequence of Corollary 2, Lemma 3.2, and Lemma 3.3.

Corollary 3. *Assume the setting in Subsection 3.1, let $X\colon [t_0,T]\times\Omega \to V$ be a mild Itô process with evolution family $S\colon \angle \to L(\hat{V},\check{V})$, mild drift $Y\colon [t_0,T]\times\Omega \to \hat{V}$, and mild diffusion $Z\colon [t_0,T]\times\Omega \to \gamma(U,\hat{V})$, and let $\varphi \in C^2(\check{V},\mathcal{V})$. Then*

(i) it holds that $\mathbb{P}(\int_{t_0}^T \|(\mathcal{L}^S_{s,T}\varphi)(X_s,Y_s,Z_s)\|_{\mathcal{V}}\,ds < \infty) = 1$,

(ii) it holds that $\mathbb{P}(\int_{t_0}^T \|\varphi'(S_{s,T}X_s)S_{s,T}Z_s\|^2_{\gamma(U,\mathcal{V})}\,ds < \infty) = 1$,

(iii) it holds that $\mathbb{P}(X_T \in \check{V}) = 1$, and

(iv) it holds that

$$\big[\varphi(X_T \mathbb{1}_{\{X_T\in\check{V}\}})\big]_{\mathbb{P},\mathcal{B}(\mathcal{V})} = [\varphi(S_{t_0,T}X_{t_0})]_{\mathbb{P},\mathcal{B}(\mathcal{V})} + \int_{t_0}^{T} (\mathcal{L}^S_{s,T}\varphi)(X_s,Y_s,Z_s)\,\mathbf{ds} + \int_{t_0}^{T} \varphi'(S_{s,T}X_s)S_{s,T}\,Z_s\,dW_s. \tag{115}$$

3.5. **Mild Dynkin-type formula.** Under suitable additional assumptions (see Corollary 4 below), the stochastic integral in item (iii) of Corollary 2 is integrable and centered. This is the subject of the following result.

Corollary 4 (Mild Dynkin-type formula)**.** *Assume the setting in Subsection 3.1, let $X\colon [t_0,T]\times\Omega \to V$ be a mild Itô process with evolution family $S\colon \angle \to L(\hat{V},\check{V})$, mild drift $Y\colon [t_0,T]\times\Omega \to \hat{V}$, and mild diffusion $Z\colon [t_0,T]\times\Omega \to \gamma(U,\hat{V})$, let $\bar{X}\colon [t_0,T]\times\Omega \to \check{V}$ be a stochastic process with continuous sample paths which satisfies $\forall\, t \in [t_0,T)\colon \mathbb{P}\big(\bar{X}_t = S_{t,T}X_t\big) = 1$ (see Lemma 3.2), let $\varphi \in C^2(\check{V},\mathcal{V})$, and let $\tau\colon \Omega \to [t_0,T]$ be an $\mathbb{F}$-stopping time which satisfies that*

$$\min\bigg\{\mathbb{E}\bigg[\bigg\|[\varphi(S_{t_0,T}X_{t_0})]_{\mathbb{P},\mathcal{B}(\mathcal{V})} + \int_{t_0}^{\tau} (\mathcal{L}^S_{s,T}\varphi)(X_s,Y_s,Z_s)\,\mathbf{ds}\bigg\|_{\mathcal{V}}\bigg], \mathbb{E}\big[\|\varphi(\bar{X}_\tau)\|_{\mathcal{V}}\big]\bigg\} + \mathbb{E}\bigg[\bigg|\int_{t_0}^{\tau} \|\varphi'(S_{s,T}X_s)\,S_{s,T}\,Z_s\|^2_{\gamma(U,\mathcal{V})}\,ds\bigg|^{1/2}\bigg] < \infty. \tag{116}$$

Then

(i) it holds that

$$\mathbb{E}\big[\|\varphi(\bar{X}_\tau)\|_{\mathcal{V}}\big] + \mathbb{E}\bigg[\bigg\|[\varphi(S_{t_0,T}X_{t_0})]_{\mathbb{P},\mathcal{B}(\mathcal{V})} + \int_{t_0}^{\tau}(\mathcal{L}^S_{s,T}\varphi)(X_s,Y_s,Z_s)\,\mathbf{ds}\bigg\|_{\mathcal{V}}\bigg] < \infty \quad (117)$$

and

(ii) it holds that

$$\mathbb{E}\big[\varphi(\bar{X}_\tau)\big] = \mathbb{E}\Big[[\varphi(S_{t_0,T}X_{t_0})]_{\mathbb{P},\mathcal{B}(\mathcal{V})} + \int_{t_0}^{\tau}(\mathcal{L}^S_{s,T}\varphi)(X_s,Y_s,Z_s)\,\mathbf{ds}\Big]. \quad (118)$$

Proof of Corollary 4. First, note that item (iii) of Corollary 2 proves that

$$\begin{aligned}[\varphi(\bar{X}_\tau)]_{\mathbb{P},\mathcal{B}(\mathcal{V})} &= [\varphi(S_{t_0,T}X_{t_0})]_{\mathbb{P},\mathcal{B}(\mathcal{V})} \\ &+ \int_{t_0}^{\tau}(\mathcal{L}^S_{s,T}\varphi)(X_s,Y_s,Z_s)\,\mathbf{ds} + \int_{t_0}^{\tau}\varphi'(S_{s,T}X_s)\,S_{s,T}\,Z_s\,dW_s.\end{aligned} \quad (119)$$

Moreover, the fact that $\int_{t_0}^{\min\{t,\tau\}}\varphi'(S_{s,T}X_s)S_{s,T}Z_s\,dW_s$, $t\in[t_0,T]$, is a local $\mathbb{F}$-martingale, the assumption that $\mathbb{E}\big[|\int_{t_0}^{\tau}\|\varphi'(S_{s,T}X_s)\,S_{s,T}Z_s\|^2_{\gamma(U,\mathcal{V})}\,ds|^{1/2}\big] < \infty$, and, e.g., the Burkholder-Davis-Gundy type inequality in Van Neerven et al. [23, Theorem 4.7] ensure that

$$\int_{t_0}^{\min\{t,\tau\}}\varphi'(S_{s,T}X_s)S_{s,T}Z_s\,dW_s, \quad t\in[t_0,T], \quad (120)$$

is an $\mathbb{F}$-martingale. This, the fact that

$$\min\big\{\mathbb{E}\big[\|[\varphi(S_{t_0,T}X_{t_0})]_{\mathbb{P},\mathcal{B}(\mathcal{V})} + \textstyle\int_{t_0}^{\tau}(\mathcal{L}^S_{s,T}\varphi)(X_s,Y_s,Z_s)\,\mathbf{ds}\|_{\mathcal{V}}\big], \mathbb{E}\big[\|\varphi(\bar{X}_\tau)\|_{\mathcal{V}}\big]\big\} < \infty, \quad (121)$$

and (119) prove that item (i) holds and that

$$\mathbb{E}[\varphi(\bar{X}_\tau)] = \mathbb{E}\bigg[[\varphi(S_{t_0,T}X_{t_0})]_{\mathbb{P},\mathcal{B}(\mathcal{V})} + \int_{t_0}^{\tau}(\mathcal{L}^S_{s,T}\varphi)(X_s,Y_s,Z_s)\,\mathbf{ds}\bigg]. \quad (122)$$

The proof of Corollary 4 is thus completed. □

3.6. Weak estimates for terminal values of mild Itô processes.

Proposition 3. *Assume the setting in Subsection 3.1, let* $X\colon[t_0,T]\times\Omega\to V$ *be a mild Itô process with evolution family* $S\colon\angle\to L(\hat{V},\check{V})$*, mild drift* $Y\colon[t_0,T]\times\Omega\to\hat{V}$*, and mild diffusion* $Z\colon[t_0,T]\times\Omega\to\gamma(U,\hat{V})$*, let* $\varphi\in C^2(\check{V},\mathcal{V})$*, and assume that*

$$\left\{\big\|\varphi\big([S_{t_0,T}\,X_{t_0}]_{\mathbb{P},\mathcal{B}(\check{V})} + \textstyle\int_{t_0}^{\tau}S_{s,T}\,Y_s\,\mathbf{ds} + \int_{t_0}^{\tau}S_{s,T}\,Z_s\,dW_s\big)\big\|_{\mathcal{V}}\colon \begin{array}{c}\mathbb{F}\text{-stopping time}\\ \tau\colon\Omega\to[t_0,T]\end{array}\right\} \quad (123)$$

is uniformly $\mathbb{P}$*-integrable. Then*

(i) it holds that $\mathbb{P}(X_T\in\check{V})=1$*,*

(ii) it holds that $\mathbb{E}\big[\big\|\varphi(X_T\mathbb{1}_{\{X_T\in\check{V}\}})\big\|_{\mathcal{V}} + \|\varphi(S_{t_0,T}X_{t_0})\|_{\mathcal{V}}\big] < \infty$*, and*

(iii) it holds that

$$\big\|\mathbb{E}\big[\varphi(X_T\mathbb{1}_{\{X_T\in\check{V}\}})\big]\big\|_{\mathcal{V}} \le \big\|\mathbb{E}\big[\varphi(S_{t_0,T}X_{t_0})\big]\big\|_{\mathcal{V}} + \int_{t_0}^{T}\mathbb{E}\big[\|(\mathcal{L}^S_{s,T}\varphi)(X_s,Y_s,Z_s)\|_{\mathcal{V}}\big]\,ds. \quad (124)$$

Proof of Proposition 3. Throughout this proof let $\tau_n \colon \Omega \to [t_0, T]$, $n \in \mathbb{N}$, be the functions which satisfy for all $n \in \mathbb{N}$ that

$$\tau_n = \inf\left(\{T\} \cup \left\{t \in [t_0, T] \colon \int_{t_0}^{t} \|\varphi'(S_{s,T} X_s)\, S_{s,T}\, Z_s\|^2_{\gamma(U,\mathcal{V})}\, ds \geq n\right\}\right) \tag{125}$$

and let $\bar{X} \colon [t_0, T] \times \Omega \to \check{V}$ be a stochastic process with continuous sample paths which satisfies

$$\forall\, t \in [t_0, T) \colon \mathbb{P}\big(\bar{X}_t = S_{t,T} X_t\big) = 1 \tag{126}$$

(cf. Lemma 3.2). Note that item (iii) of Corollary 3 establishes item (i). Moreover, observe that the assumption that the set $\big\{\|\varphi([S_{t_0,T}\, X_{t_0}]_{\mathbb{P},\mathcal{B}(\check{V})} + \int_{t_0}^{\tau} S_{s,T}\, Y_s\, \mathbf{ds} + \int_{t_0}^{\tau} S_{s,T}\, Z_s\, dW_s)\|_{\mathcal{V}} \colon \mathbb{F}\text{-stopping time } \tau \colon \Omega \to [t_0, T]\big\}$ is uniformly $\mathbb{P}$-integrable proves item (ii). Next note that item (ii) of Corollary 2 shows that

$$\mathbb{P}\left(\int_{t_0}^{T} \|\varphi'(S_{s,T} X_s)\, S_{s,T}\, Z_s\|^2_{\gamma(U,\mathcal{V})}\, ds < \infty\right) = 1. \tag{127}$$

This establishes that

$$\mathbb{P}\Big(\lim_{n \to \infty} \tau_n = T\Big) = 1. \tag{128}$$

Note that assumption (123) and Lemma 3.3 ensure that the set $\{\|\varphi(\bar{X}_{\tau_n})\|_{\mathcal{V}} \colon n \in \mathbb{N}\}$ is uniformly $\mathbb{P}$-integrable. Equation (125) hence shows that for all $n \in \mathbb{N}$ it holds that

$$\mathbb{E}\big[\|\varphi(\bar{X}_{\tau_n})\|_{\mathcal{V}}\big] + \mathbb{E}\left[\int_0^{\tau_n} \|\varphi'(S_{s,T} X_s) S_{s,T} Z_s)\|^2_{\gamma(U,\mathcal{V})}\, ds\right] < \infty. \tag{129}$$

The fact that for all $n \in \mathbb{N}$ it holds that τ_n is an $\mathbb{F}$-stopping time thus allows us to apply Corollary 4 to obtain that for all $n \in \mathbb{N}$ it holds that

$$\begin{aligned} \mathbb{E}\big[\varphi(\bar{X}_{\tau_n})\big] &= \mathbb{E}\left[[\varphi(S_{t_0,T} X_{t_0})]_{\mathbb{P},\mathcal{B}(\mathcal{V})} + \int_{t_0}^{\tau_n} (\mathcal{L}^S_{s,T}\varphi)(X_s, Y_s, Z_s)\, \mathbf{ds}\right] \\ &= \mathbb{E}[\varphi(S_{t_0,T} X_{t_0})] + \mathbb{E}\left[\int_{t_0}^{\tau_n} (\mathcal{L}^S_{s,T}\varphi)(X_s, Y_s, Z_s)\, \mathbf{ds}\right]. \end{aligned} \tag{130}$$

The triangle inequality hence proves that

$$\limsup_{n \to \infty} \big\|\mathbb{E}\big[\varphi(\bar{X}_{\tau_n})\big]\big\|_{\mathcal{V}} \leq \big\|\mathbb{E}\big[\varphi(S_{t_0,T} X_{t_0})\big]\|_{\mathcal{V}} + \textstyle\int_{t_0}^{T} \mathbb{E}\big[\|(\mathcal{L}^S_{s,T}\varphi)(X_s, Y_s, Z_s)\|_{\mathcal{V}}\big]\, ds. \tag{131}$$

This together with (128), item (ii) of Lemma 3.3, and the uniform $\mathbb{P}$-integrability of $\{\|\varphi(\bar{X}_{\tau_n})\|_{\mathcal{V}} \colon n \in \mathbb{N}\}$ assures (124). The proof of Proposition 3 is thus completed. □

Proposition 4 (Test functions with at most polynomial growth)**.** *Assume the setting in Subsection 3.1, let $X \colon [t_0, T] \times \Omega \to V$ be a mild Itô process with evolution family $S \colon \angle \to L(\hat{V}, \check{V})$, mild drift $Y \colon [t_0, T] \times \Omega \to \hat{V}$, and mild diffusion $Z \colon [t_0, T] \times \Omega \to \gamma(U, \hat{V})$, and let $p \in [0, \infty)$, $\varphi \in C^2(\check{V}, \mathcal{V})$ satisfy*

$$\sup_{x \in \check{V}} \big[\|\varphi(x)\|_{\mathcal{V}} (1 + \|x\|^p_{\check{V}})^{-1}\big] < \infty \tag{132}$$

and

$$\mathbb{E}\left[\left|\int_{t_0}^{T} \|S_{s,T} Z_s\|^2_{\gamma(U,\check{V})}\, ds\right|^{p/2} + \|S_{t_0,T} X_{t_0}\|^p_{\check{V}} + \left|\int_{t_0}^{T} \|S_{s,T} Y_s\|_{\check{V}}\, ds\right|^p\right] < \infty. \tag{133}$$

Then

(i) it holds that $\mathbb{P}(X_T \in \check{V}) = 1$,

(ii) *it holds that* $\mathbb{E}\big[\|\varphi(X_T \mathbb{1}_{\{X_T\in\check{V}\}})\|_{\mathcal{V}} + \|\varphi(S_{t_0,T}X_{t_0})\|_{\mathcal{V}}\big] < \infty$, *and*

(iii) *it holds that*

$$\big\|\mathbb{E}\big[\varphi(X_T \mathbb{1}_{\{X_T\in\check{V}\}})\big]\big\|_{\mathcal{V}} \leq \big\|\mathbb{E}\big[\varphi(S_{t_0,T}X_{t_0})\big]\big\|_{\mathcal{V}} + \int_{t_0}^{T} \mathbb{E}\big[\|(\mathcal{L}^S_{s,T}\varphi)(X_s,Y_s,Z_s)\|_{\mathcal{V}}\big]\,ds. \tag{134}$$

Proof of Proposition 4. Throughout this proof let $\bar{X}\colon [t_0,T]\times\Omega\to\check{V}$ be a stochastic process with continuous sample paths which satisfies $\forall\, t\in[t_0,T)\colon \mathbb{P}\big(\bar{X}_t = S_{t,T}X_t\big) = 1$ (cf. Lemma 3.2) and let $\mathcal{Z}\colon[t_0,T]\times\Omega\to\check{V}$ be a stochastic process with continuous sample paths which satisfies for all $t\in[t_0,T]$ that

$$[\mathcal{Z}_t]_{\mathbb{P},\mathcal{B}(\check{V})} = \int_{t_0}^{t} S_{s,T}Z_s\,dW_s. \tag{135}$$

Observe that Lemma 3.3 implies that for all $t\in[t_0,T]$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned}\|\varphi(\bar{X}_t)\|_{\mathcal{V}} &\leq \left[\sup_{x\in\check{V}} \frac{\|\varphi(x)\|_{\mathcal{V}}}{(1+\|x\|^p_{\check{V}})}\right](1+\|\bar{X}_t\|^p_{\check{V}}) \\ &\leq 3^p\left[\sup_{x\in\check{V}} \frac{\|\varphi(x)\|_{\mathcal{V}}}{(1+\|x\|^p_{\check{V}})}\right]\left(1+\|S_{t_0,T}X_{t_0}\|^p_{\check{V}} + \left|\int_{t_0}^{T}\|S_{s,T}Y_s\|_{\check{V}}\,ds\right|^p + \|\mathcal{Z}_t\|^p_{\check{V}}\right).\end{aligned} \tag{136}$$

Moreover, e.g., the Burkholder-Davis-Gundy type inequality in Van Neerven et al. [23, Theorem 4.7] shows that there exists a real number $C\in[1,\infty)$ such that

$$\mathbb{E}\left[\sup_{t\in[t_0,T]}\|\mathcal{Z}_t\|^p_{\check{V}}\right] \leq C\,\mathbb{E}\left[\left|\int_{t_0}^{T}\|S_{s,T}\,Z_s\|^2_{\gamma(U,\check{V})}\,ds\right|^{p/2}\right]. \tag{137}$$

Combining (136) and (137) yields that there exists a real number $C\in[1,\infty)$ such that

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[t_0,T]}\|\varphi(\bar{X}_t)\|_{\mathcal{V}}\right] &\leq C\Bigg(1+\mathbb{E}[\|S_{t_0,T}\,X_{t_0}\|^p_{\check{V}}] \\ &+\mathbb{E}\left[\left|\int_{t_0}^{T}\|S_{s,T}\,Y_s\|_{\check{V}}\,ds\right|^p\right] + \mathbb{E}\left[\left|\int_{t_0}^{T}\|S_{s,T}\,Z_s\|^2_{\gamma(U,\check{V})}\,ds\right|^{p/2}\right]\Bigg).\end{aligned} \tag{138}$$

Assumption (133) hence ensures that

$$\mathbb{E}\left[\sup_{t\in[t_0,T]}\|\varphi(\bar{X}_t)\|_{V}\right] < \infty. \tag{139}$$

Lemma 3.3 therefore proves that

$$\mathbb{E}\left[\sup_{t\in[t_0,T]}\left\|\varphi\left(S_{t_0,T}X_{t_0} + \int_{t_0}^{t} S_{s,T}Y_s\mathbb{1}_{\{\int_{t_0}^{T}\|S_{r,T}Y_r\|_{\check{V}}\,dr<\infty\}}\,ds + \mathcal{Z}_t\right)\right\|_{\mathcal{V}}\right] < \infty. \tag{140}$$

Combining this with Proposition 3 completes the proof of Proposition 4. □

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