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THE GEOMETRY OF THE UNKNOWN: BOMBELLI'S *ALGEBRA LINEARIA*

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Abstract. This paper studies the ways algebra and geometry are related in Bombelli's *L'algebra*. I show that despite Bombelli's careful adherence to a from of homogeneity, he constructs several different forms of relationship between algebra and geometry, building on Greek, Arabic, abbacist and original approaches. I further show how Bombelli's technique of reading diagrams, especially when representing algebraic unknowns, requires a multiple view that makes lines stand for much more than the diagrams present to an untrained eye. This multiplicity reflects an exploratory approach that seeks to integrate the algebraic and geometric strata without reducing one to the other and without suppressing the idiosyncrasies of either stratum.

1. Introduction

1.1. **Scope, purpose and methodology.** In the year 1572 Rafael Bombelli's *L'algebra* came out in print. This book, which is well known for its contribution to solving cubic and quartic equations by means of roots of negative numbers, contains significant work on the relation between algebra and geometry. The manuscript included two additional geometry books, of which only a few fragments were incorporated into the print edition.

This paper will analyse what Bombelli called "*algebra linearia*":[1] the inter-representation of geometry and algebra in Bombelli's *L'algebra*. I will map out the various ways that geometry and algebra justify, instantiate, translate and accompany each other. The purpose of the analysis is to point out the multiple relations between the two mathematical domains as rigorously set out in Bombelli's text, and the multiple vision that's required for understanding Bombelli's geometric diagrams.

This paper runs as follows. After a historical introduction and a review of Bombelli's notation, the second section of this paper will conduct a survey of Bombelli's geometric representation of algebra and its original regimentation of homogeneity considerations. I will then show how, despite the

Parts of the research for this paper have been conducted while visiting Boston University's Center for the Philosophy of Science, the Max Planck Institute for the History of Science and the Edelstein Center for the History and Philosophy of Science, Medicine and Technology in the Hebrew University in Jerusalem.

[1]This term is best translated as 'algebra in lines', to avoid confusion with contemporary linear algebra.

rigour of Bombelli's geometric representation of algebra, this representation does not dictate a one-to-one correspondence between algebraic and geometric elements, but rather allows different translations and interpretations to coexist.

The third section will then study the various functional relations between algebra and geometry. I will show that, building on Greek, Arabic, abbacist[2] and original practices, Bombelli's geometric representations sometimes serve to justify algebraic manoeuvres, sometimes to instantiate algebra in a different medium of representation, sometimes to connect different algebraic expressions through a common geometric translation, and sometimes simply to provide an independent accompaniment for algebra (this classification may serve as a framework to understand Renaissance juxtapositions of algebra and geometry).

The fourth section will go on to show how Bombelli's geometric representations of algebra confront negative magnitudes, expressions involving roots of negatives, and algebraic unknowns and their powers. This section will demonstrate that Bombelli implicitly hypothesised a co-expressivity of algebra and geometry, and that this hypothesis helped endorse questionable algebraic entities, generated new hybrid geometrico-algebraic entities and practices, and rendered geometric and algebraic signs polysemic and multi-layered. The overall result of this study will be a complex picture of multiple relations between algebra and geometry, which does not reduce or subject the one to the other, but assumes that they are deeply related, and builds on this relation to produce new mathematical practices.

My approach is semiotic and mostly intrinsic. I explore Bombelli's work as an inspiring example of practicing mathematics rigorously, but without a confined, pre-charted ontology. Bombelli's hybridisation of algebra and geometry, expanding the boundaries of each without reducing one to the role of a servant to the other, is an instructive example of the open horizons of mathematical constructions of meaning.

1.2. **Bombelli and *L'algebra*.** Not much is known about Bombelli's life. According to Jayawardene (1965, 1963) he was born around 1526 and died no later than 1573. He was an engineer and architect involved in reclaiming marshland and building bridges. There is no record that he studied in a university, but he was obviously a learned man, so much so that a scholar from the university of Rome invited him to cooperate on a translation of the works of Diophantus.

The writing of the manuscript draft of *L'algebra*, Bombelli's only known publication, took place during a long pause in the Val di Chiana marsh reclaiming, which Bortolotti (1929) dates to the early 1550s and Jayawardene (1965) to the late 1550s.

[2]As the abbacists — the arithmetic teachers for Italian traders youth, had very little to do with the abacus as instrument — I follow other researchers in retaining the 'bb' spelling.

The manuscript (Bombelli, 155?), which was uncovered by Ettore Bortolotti, is divided into five books. The first three books of the manuscript appeared with revisions in the 1572 print edition (Bombelli, 1572). Bortolotti published a modern edition of the remaining two books with an introduction and comments (Bombelli, 1929), which was later combined with the 1572 edition (Bombelli, 1966).[3]

The first book of *L'algebra* is a treatise on arithmetic, including root extraction and operating on sums of numbers and roots (mostly binomials, but some longer sums as well). The second book introduces the unknown (*Tanto*), presents an elementary algebra of polynomials up to and including division, and goes on to systematically present techniques for solving quadratic, cubic and quartic equations, following the discoveries of dal Ferro, Tartaglia, Cardano and Ferrari. The third book is a collection of recreational problems in the abbacist tradition together with problems borrowed directly from Diophantus. The problems are solved using the algebraic techniques taught in the second book.

The fourth book, which did not make it to print at the time, concerns what Bombelli calls *algebra linearia*, the reconstruction of algebra in geometric terms. It opens with some elementary Euclidean constructions, and then builds on them to geometrically reproduce the main techniques of Book II and some problems of Book III. Book V treats some more traditional geometric problems in both geometric and algebraic manner, goes on to teach some basic practical triangulation techniques, and concludes with a treatise on regular and semi-regular polyhedra. Book IV is not entirely complete. Many of the spaces left for diagrams remain empty. Book V is even less complete, and its sections do not appear in the manuscript table of contents.

There are some substantial differences between the manuscript and the print edition. Several sections that appear as marginalia in the manuscript were incorporated as text into the print version of books I and II. Some of the geometric reconstructions of the unpublished Book IV were incorporated into the first two books. The print edition also has a much more developed discussion of roots of negative numbers, and introduces some new terminology and notation that will be addressed below. Book III went through some major changes. Problems stated in terms of commerce in the manuscript were removed, and many Diophantine problems were incorporated (for a full survey of these changes see Jayawardene (1973)). The introduction to the print edition states an intention to produce a book that appears to be based on the manuscript Book IV, but this intention was never actualised (Bombelli died within a year of the print publication).

[3]In referring to Books III, IV and V, I use problem number and section number (the 1929 version is available online, so it makes more sense to use section numbers than the page numbers of the out-of-print 1966 edition). In references to Books I and II the page numbers of the 1966 edition are used, as there is no numerical sectioning. The translations from the vernacular Italian are my own.

According to Bortolotti's introduction, *L'algebra* seems to have been well received in early modern mathematical circles. Bortolotti quotes Leibniz as stating that Bombelli was an "excellent master of the analytical art", and brings evidence of Huygens' high esteem for Bombelli as well (Bombelli, 1929, 7–8). Jean Dieudonné, however, seems less impressed with Bombelli's achievements and renown (Dieudonné, 1972). Note, however, that the geometry books (IV and V) were not available in print, and are unlikely to have enjoyed considerable circulation.

The vast majority of technical achievements included in Bombelli's printed work had already been expounded by Cardano. The exceptions include some clever tinkering with root extraction and the fine tuning of techniques for solving cubics and quartics. Bombelli's achievements in reconciling algebra and geometry, which are the subject of this paper, were not published in print at the time. But Bombelli's one undeniable major achievement is the first documented use of roots of negative numbers in order to derive a real solution of a polynomial equation with integer coefficients. He is not the first to work with roots of negative numbers, but he is the first to manipulate them extensively beyond a basic statement of their rules.

However, judging Bombelli's book through the prism of technical novelty does not do it justice. Indeed, Bombelli explicitly states in his introduction that he is presenting existing knowledge. He explains that "in order to remove finally all obstacles for the speculative theoreticians and the practitioners of this science" (algebra) ... "I was taken by a desire to bring it to perfect order".[4] In fact, the main text that Bombelli sought to clarify was Girolamo Cardano's *Ars Magna*.

Despite this strictly pedagogical aim, Bombelli does occupy a special place in the history of algebra. While Bombelli's explicitly mentioned sources (Leonardo Fibonacci, Oronce Finé, Heinrich Schreiber, Michael Stifel, Luca Pacioli, Niccoló Tartaglia and Girolamo Cardano) fail to include two centuries of vernacular Italian algebra developed in the context of abbacus schools, Bombelli is, in a sense, the last proponent of the abbacist tradition. He is the last important and innovative author to organise his work around rules for solving first a comprehensive list of kinds of equations, and then around a much looser collection of recreational problems borrowed from the abbacist reservoir. Bombelli, like his sixteenth century predecessors, adds much to the knowledge of past abbacists, but in terms of practices, terminology and problems he is a direct descendent of their tradition.[5]

1.3. **Notation.** The name of the unknown in Italian abbacus algebra is usually *cosa* (thing), and occasionally *quantità* (quantity). Bombelli writes

[4]"per levare finalmente ogni impedimento alli speculativi e vaghi di questa scientia e togliere ogni scusa a' vili et inetti, mi son posto nell'animo di volere a perfetto ordine ridurla" (Bombelli, 1966, 8).

[5]The authoritative survey of abbacus algebra is still that of Franci & Rigatelli (1985). A comprehensive catalogue of abbacist algebra was compiled by Van Egmond (1980).

in his manuscript that he prefers the latter, but uses the former, because that's the received practice. In the print version, following Diophantine inspiration, Bombelli changes the name of the unknown to *Tanto* (so much, such; I retain these Italian terms in this paper in order to maintain a distance from modern practice). The second power is called *Censo* in the manuscript (Bombelli prefers *quadrata*, but again follows received practice) and *potenza* in the print version. *Cubo* is the third power, and *Censo di Censo* or *potenza di potenza* is the fourth. Higher powers are treated and named as well, but are not relevant for this paper.

Bombelli's manuscript notation for powers of the unknown is a semicircle with the ordinal number of the power over the coefficient. The print edition reproduces this notation in diagrams of calculations, but, due to the limitations of print, places the semicircle next to the coefficient in the running text. So a contemporary $5x^2$ would be rendered as $5\ \overset{2}{\smile}$ in print and as $\overset{\overset{2}{\smile}}{5}$ in the manuscript. The manuscript Book II accompanies plain numbers with a $\overset{0}{\smile}$ above them, but this is almost entirely discarded in the print edition and in the other manuscript books.

Bombelli uses the shorthand *m.* for *meno* (minus) and *p.* for *più* (plus). The modern edition replaces those signs with the contemporary $-$ and $+$, but otherwise respects the notations of the 1572 edition. I follow this practice here as well.

2. Elements of *Algebra Linearia*

In this section I will discuss Bombelli's inter-representation of algebra and geometry. The first subsection will raise the issue of relations between algebra and geometry in the abbacist tradition. The second subsection will consider the problem of homogeneity[6] in Cardano and Bombelli, and describe Bombelli's method of preserving homogeneity without committing himself to a fixed translation of algebraic powers to geometric dimensions. The third subsection will present Bombelli's algebraic-geometric translation system, but will also show that Bombelli's geometric representations of algebra were not restricted to this system. Along the way we will encounter a couple of preliminary examples of geometric signs that function on several levels at once. The point of this section is that Bombelli's practice is highly rigorous, but still allows for different algebraic-geometric interpretations to coexist.

2.1. **Letting representation run wild.**

The second book of *L'algebra*, which deals with the solution of polynomial equations up to the fourth degree, concludes with Bombelli's "reserving it for later, at my leisure and

[6] Homogeneity here means never adding or equating geometric entities of different dimensions. If a practice is homogeneous, the constructed geometrical object will be invariant with respect to the choice of units of measurement.

convenience, to give to the world all these Problems in geometric demonstrations".[7] The third book, a collection of algebraic problems solved by the techniques of Book II, concludes with a more elaborate statement: "I had in mind to verify with geometrical demonstrations the working out of all these Arithmetical problems, knowing that these two sciences (that is Arithmetic and Geometry) have between them such accord that the former is the verification of the latter and the latter is the demonstration of the former. Never could the mathematician be perfect, who is not versed in both, although in these our times many are those who let themselves believe otherwise; how they deceive themselves they will clearly recognise when they will have seen my former and latter work; but because it is not yet brought to such perfection as the excellence of this discipline requires, I decided I want to consider it better first, before I were to present it for the scrutiny of men".[8] Unfortunately, Bombelli died soon after, and never got to perfect this part of his work, which remained incomplete in manuscript.

Of course, the notion of relating algebra or arithmetic and geometry was hardly new. The tradition of representing numbers by lines goes back to Euclid's diagrams in his arithmetic books, if not earlier. But while Euclid made an effort to set apart the theories of ratios between general homogeneous magnitudes, of ratios between numbers, and of relations between lengths, areas and volumes (books V, VII and XI of the *Elements* respectively), a renaissance author such as Cardano (1968, 28) could refer simultaneously to V.19, VII.17 and XI.31 to explain why, if three cubes equal 24, then one cube equals 8.

But this conflation emerged much earlier. Abbacus algebra imported Arabic geometric diagrams that had already been interpreted as referring to numbers.[9] Among the abbacists, the most emblematic 'algebraic geometer' was the artist Piero della Francesca. Piero's geometry dealt with diagrams with some line lengths given numerically. It used *cosa* terms to model unknown lengths of lines, translate information concerning these lengths into polynomial equations, and derive their solutions using abbacist rules. His

[7] "riserbandomi poi con più mio agio e commodità di dare al mondo tutti questi Problemi in dimostrationi geometriche" (Bombelli, 1966, 314).

[8] "io fussi di animo di provare con dimostrationi Geometriche l'operatione di tutti questi problemi Arimetici, sapendo che queste due scientie (cioè l'Arimetica e Geometria) hanno intra di loro tanta convenientia che l'una è la prova dell'altra e l'altra è la dimostration dell'una, nè già puote il Matematico esser perfetto il quale in ambedue non sia versato, benchè a questi nostri tempi molti siano i quali si danno a credere altrimente; del che quanto si ingannino all'hor chiaramente lo conosceranno quando che l'una e l'altra mia opera havranno veduta; ma perchè non è per ancora ridutta a quella perfettione che la eccellentia di questa disciplina ricerca, mi son risoluto di volerla prima meglio considerare, avanti che la mandi nel conspetto de gli huomini" (Bombelli, 1966, 476).

[9] For the earlier roots of the arithmetisation of Greek geometry see Reviel Netz' highly remarkable *The transformation of mathematics in the early Mediterranean world* (Netz, 2004).

work was so influential, that Luca Pacioli practically imported it as is into his work.[10]

The algebraic entities — the *cosa*, *censo* and *cubo* (the unknown, its square and its cube) — were sometimes (but by no means generally!) interpreted as geometric in introductory presentations and some applications. As 14^{th} century abbacist Maestro Dardi wrote, "The *cosa* is a linear length and is root of the *censo*, and one says *cosa*" (literally, thing) "because this name can be attributed to all things of the world in general. The *censo* would be a surface breadth and is the square of the cosa" ... "The *cubo* would be a corporeal magnitude, the body of which includes in itself the length of the *cosa* and the surface of the *censo*, and is called *cubo* according to the arithmetic of Boethius" ... "meaning such aggregation of numbers".[11] The algebraic terms here come from geometry and arithmetic, but have to do generally with "all things of the world". Indeed, the one interpretation of algebraic unknowns common to all abbacists is the economic one: they are used to represent monetary units and quantities of merchandise.

Since the *cosa* can be anything, practitioners were not always bound to respect the above allocation of algebraic terms to geometric dimensions. In fact, the notion of dimension was probably not terribly rigid in a culture that used such term as "quadro chubico" for volume (Paolo dell'Abbaco, 1964, 128), and sometimes measured volumes in terms of square *braccia* (*braccia* is a unit of length). By the time we reach Cardano we see one and two dimensional diagrams for algebraic cubic terms, and a square whose one side represents the second power of an unknown and the other its fourth power (Cardano, 1968, 21,52,238).

The correspondence between geometric and algebraic dimensions is not fixed in the work of Bombelli as well. He is apt to say such things as "the rectangle .i.l.g. will be a cube and the rectangle .i.l.f. will be $6\,\overset{1}{\smile}$",[12] denying any consistent relation between the algebraic and geometric hierarchies of powers and dimensions. He follows this conduct in his Book IV as well, where algebraic cubes can be drawn as squares with one side an algebraic square and the other an algebraic *cosa* (e.g. §28). While Book II does not attempt to geometrically instantiate quartic and biquadratic equations, Book

[10] See Gino Arrighi's introduction to della Francesca (1970).

[11] "La cosa è una lunghessa lineale ed è radicie del censo, e diciensi cosa perchè questo nome cosa si può atribuire a tutte le cose del mondo gieneralmente. Lo censo sie una anpiessa superficiale ed è quadrato della cosa, e diciesi censo da cerno cernis che sta per eleggiere, inperciò che el censo eleggie lo meçço proportionale in tra la cosa e'l cubo. Lo cubo sie una grossessa chorporale lo cuj chorpo inchiude in sè la lunghessa della cosa e lla superficie del censo, ed è ditta cubo sicondo l'arismetrica di Boetio da questo nome cubus cubi che tanto vuol dire quanto agreghatione di numerj" (Dardi, 2001, 37–38).

[12] "il paralellogramo .i.l.g. sarà un cubo ed il paralellogramo .i.l.f. sarà $6\,\overset{1}{\smile}$" (Bombelli, 1966, 229).

IV represents fourth powers as squares whose sides are algebraic squares (§§43,46).[13]

2.2. **Regimenting representation.** But a geometric representation of algebra, which has any regard for classical practice (as renaissance mathematics obviously had) must confront the issue of homogeneity. Cardano, for one, included a highbrow warning against the misunderstandings that failure to respect homogeneity may bring: "it is clear that they are mistaken", explains Cardano, "who say that if BH, for instance, is the value of" the unknown[14] "and GF is 3, the rectangle" formed by BH and GF "will be 3BH or triple BH. For it is impossible that a surface should be composed of lines" (Cardano, 1968, 34). But Cardano's way of actually dealing with homogeneity in his geometric proofs of algebraic rules was simply to turn a blind eye to its requirements. After making this very warning, Cardano ignores the difference between multiplying a line by a number and by another line, and very casually adds numbers, lines, areas and volumes as if they were all homogeneous (Cardano, 1968, e.g. 76,65,124).

Bombelli's practice, however, is much more carefully regimented. When Bombelli illustrates geometrically the solution of the problems '*Cose* equal number' and 'square(s) equal number' (§§22–24), he carefully equates products of lines with rectangles, and homogeneously applies rectangles to lines[15] or reduces them to squares.[16] Nevertheless, when constructing a line representing one of Euclid's special binomials (sums of number and square root), Bombelli has no problem writing "Let the line .a. be 16 square number, and the line .b. 12 non square number" ... "and let .c. be 144, the square of .b., and .d. 192, product of .a. and .b." (§56).[17] Does this mean that Bombelli, too, violated the requirements of homogeneity?

The answer is negative. Bombelli explicitly developed the means that would allow him to represent the product of lines as a line, while maintaining homogeneity. For instance, in §98 Bombelli poses the following question, a geometric version of a problem from his manuscript Book III: "Let the line .a.b. be given, which has to be divided into three parts in continued proportion in such a way that having found a line that would be equal to the product of the first and the second, the line .o. being the common

[13]The diagrams are missing from the manuscript treatment of these sections, but the text clearly shows that the diagrams inserted by Bortolotti are correct in this respect. Recent work by Marie Hélène Labarthe indicate that Pedro Nuñez applies similar practices.

[14]The English translation has x for Cardano's *res*, which I'd rather avoid.

[15]To apply a rectangle to a line is to construct another rectangle of the same area with one side equal to the given line.

[16]This accords with the notion of "strict homogeneity" attributed by Freguglia (1999) to Bombelli, but as we shall see below, this attribution has to be qualified and applied very cautiously.

[17]"Sia la linea .a. 16, numero quadrato, et la linea .b. 12, numero non quadrato, et minore de la .a., et sia la .c. 144, quadrato della .b., et la .d. 192, moltiplicatione dell'.a. in .b.".

measure, and the said line multiplied by the third, it would make a line equal to .g.l.".[18] The product of the first and second lines is a rectangle, and as such can be applied to a line, namely to the common measure .o.. The other side of this applied rectangle represents the product of the two original lines. The common measure need not be carried explicitly with the product line, but it is there implicitly as the other side of a rectangle, for otherwise homogeneity is violated. This practice makes the product line a hybrid between an independent line and a rectangle named by its non-common side. The final requirement posed by the problem may be simultaneously viewed as an equation confronting two lines (the product of the three parts and .g.l.), or as an equation between two boxes, where the first has one side equal to .o., and the second has two sides equal to .o. (and one equal to .g.l.). A line no longer simply stands for a line.

In other problems the common measure might not be mentioned under that name, but is still there. When in §111 Bombelli asks to find the line representing the square of a given line, he uses one of the given lines of the problem as a common measure. Indeed, when he verifies the solution by assigning lengths to lines, he assigns this line the length 1, and says that this is the value one should "always take".[19] But this "always" is immediately qualified. Geometrically, we may choose our unit as we wish, but arithmetically, our chosen common measure might already be given a non unit numerical length. And so Bombelli explains that if the line used as geometrical unit is not of length 1, then a common measure should be derived by rescaling[20] this given line according to its given length so as to produce a line of length 1. If we now perform the construction with this newly derived unit as the common measure, geometric practice will accord with the arithmetic score of line lengths.

We see here that Bombelli takes care to maintain homogeneity on the geometric level, and at the same time remains faithful to arithmetic considerations, to which homogeneous geometry in itself is blind. This practice distributes the length of a line between different co-extensive registers. Geometrically, the common measure should "always" be interpreted as 1. But arithmetically it may be assigned any value. The common measure ties

[18]"Sia data la linea .a.b. la quale si habbia a dividere in tre parti continoe, et proportionali in tal modo, che trovato una linea che sia pari alla potentia della prima nella seconda, essendo la linea .o. la commune misura, et la detta linea moltiplicata via la terza faccia una linea pari alla .g.l.". Note that the common measure could also be understood as unit measure. Indeed, Bombelli writes a few lines below that "the .o. is 1 always by rule" ("et la .o. 1, sempre per regola").

[19]"sempre pongasi la .a.b. essere 1".

[20]I use the term *rescaling* to refer to reducing or extending a line or an entire geometric structure by a given ratio.

together two mathematical orders, and is allowed to function on both levels, accepting and validating their difference. A single line — that of the common measure — may represent two values in two contexts at once.[21]

2.3. **The vicissitudes of regimentation.** Bombelli's geometric representation of algebra is provided with a more or less systematic setting. After reviewing some elementary Euclidean constructions (e.g. bisecting a line, reducing a rectilinear surface to a square, applying a rectangle to a line), Bombelli sets out to instantiate arithmetic and algebraic operations in geometric terms. Addition and subtraction are represented as concatenating and cutting off lines (§§15,16); the product of lines is represented as the rectangle that they contain (§17); dividing lines is done by introducing a common measure and using similar triangles (*Elements* VI.12) to draw a line whose ratio to the unit is as the ratio of the divided to the divisor (§18). As Bortolotti observed, this practice of division would later be reinvented by Descartes. Earlier in the text, in the context of applying a rectangle to a line, Bombelli uses a construction based on *Elements* III.35 (chords in a circle cut each other proportionally). This construction, too, is used further on to implement division with a common measure.

Next Bombelli considers root extraction (§19). This is performed either by Euclid's semicircle construction of the geometrical mean between a given line and a common measure (*Elements* VI.13, without explicit reference) or the variation of the same construction, where the unit is taken on the given line, rather than appended to it (a variation later used by Descartes in his *Regulae*). Bombelli then relates this operation to reducing a rectangle or a sum of squares to a square. Cubic roots (§20) are then extracted through a trial-and-error method based on constructing similar right angle triangles that Bombelli, following Barbaro's commentary on Vitruvius, attributes to Plato (or his disciples) (Bombelli, 1966, 47,228), and that is known from Pappus' *Mathematicae Collectiones*.[22] Bombelli promises two methods for the extraction of cubic roots, but provides only one. A second method appears in Book I of the print edition, and is based on superposing right-angled rulers (Bombelli, 1966, 48). Bombelli relates the extraction of a

[21]It is interesting to note that Bombelli's virtuosity concerning homogeneity confused even as deep and insightful an editor as Ettore Bortolotti (his rampant whiggishness notwithstanding). In the notes to §110 Bortolotti says that Bombelli's rectangle .e.b.f. should be a box, because it equals what Bortolotti reconstructs as X^3. But in fact Bortolotti's X^3 is for Bombelli not a box whose sides are equal to an unknown X, but a square whose one side is the *cosa* (Bortolotti's X), and the other side the square of the *cosa* reduced to a line, with .b.f. serving as unit measure (in Bortolotti's terms: $X^2/\overline{bf}$). In Bortolotti's terms, then, the correct algebraic model for the term equated with .e.b.f. would be $X^3/\overline{bf}$, and homogeneity is indeed maintained if .e.b.f. is a rectangle. This shows how subtle Bombelli's practice of regimenting homogeneity was, and how evolved was Bombelli's skill of tacitly retaining the two-dimensionality represented by a single line.

[22]See Giusti (1992, 305–306) for more details.

cubic root to finding two mean proportionals between a given line a common measure as well as to reducing a box to a cube.

These building blocks allow Bombelli to translate arithmetic operations into geometric ones. However, we should note, as did Enrico Giusti (1992, 311–312), that the use of these building blocks is not terribly consistent. We must acknowledge that the strategy of reducing rectangles to lines and employing explicit common measures is in fact not central to Bombelli's work.[23] Bombelli usually maintains homogeneity in the classical way, which does not require such manoeuvres. We should also note that Bombelli's practice here is in fact in line with the choices of Descartes, who would later develop the same technology to deal with homogeneity, but, again like Bombelli, would rarely use it to work outside the classical practice of homogeneity (Bos, 2001, 299). Van Ceulen, the other pre-Cartesian author who used a unit measure to reduce a product of lines to a line, had his contemporary editor insert a note insisting that the product of two lines is an area (Bos, 2001, 156). We see that bypassing homogeneity by common measures was hard to digest, even when it was explicitly introduced in a rigorous manner. In fact, the haphazard approach represented by Cardano enjoyed more popularity, as those who didn't mind homogeneity followed its path, while those who did mind homogeneity were uncomfortable even with the Cartesian approach.[24]

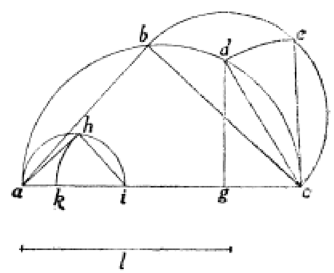
But Bombelli's deviations from his expository building blocks for instantiating algebra in geometric terms are not restricted to issues of homogeneity management. For example, Bombelli frequently uses the Pythagorean theorem, which is not mentioned in the expository part. In the context of geometrically instantiating algebraic solutions to problems imported from his manuscript Book III Bombelli's approach is sometimes more creative than a strict combination of his expository building blocks would suggest. For instance in §94 (figure 1), when algebra instructs to divide the square on .b.c. by twice .a.c., rather than actually double .a.c. and divide using intersecting chords or proportional triangles as in the introduction, Bombelli draws the line .c.e. whose square is half the square on .b.c., and then takes advantage of the fact that .c.e. is a mean proportional between .a.c. and the result of the division, drawing this result (.g.c.) using one of the diagrams for producing a mean proportional.

The point of this example is not to nit-pick on Bombelli's commitment to the procedures presented in his exposition. I include these details to provide an example of how Bombelli follows the respective idiosyncratic practicalities

[23]While I am puzzled by Giusti's claims that modelling division via proportional triangles "is somewhat hidden" and that division by intersecting chords "is abandoned" (Giusti, 1992, 312), it is true that these diagrams rarely depend on an explicit assignment of the role of a unit to any particular segment.

[24]Newton, for instance, wrote: "Multiplication, division and such sorts of computation are newly received into geometry, and that unwarily and contrary to first design". Quoted in Roche (1998, 79).

Figure 1. Bombelli (1929, §94): geometric idiosyncracies with respect to algebra



of algebra and geometry, and ends up with different procedures even when geometry is there supposedly to reproduce an algebraic practice.[25]

These reservations notwithstanding, Bombelli's concern with regimenting homogeneity and representing algebra by geometry demonstrates two majors points of my overall argument. First, a rigorous representation of algebra by geometry does not necessarily entail establishing univocal links between the two. Indeed, we saw above how algebraic powers could correspond to various geometric dimensions, how arithmetic operations received various geometric interpretations (e.g. root extraction interpreted as reducing a rectangle to a square and as finding a mean proportional with a common measure), and how the algebraic and geometric strata of expression maintained their own irreducible particularities (as in the description of §94 above, where a geometric construction, meant to represent an algebraic procedure, depended on the opportunities provided by the geometric diagram, rather than on the introductory gemetrico-algebraic 'dictionary'). Second, the interaction of geometry and algebra forces each stratum to express more than it had before the encounter. We do not have a simple double expression, where each geometric sign expresses a geometric entity and an algebraic one. What we

[25]At the same time, in some cases of reconstructing algebraic solutions geometrically the diagrams follow algebraic orders very strictly. In §99, for example, the algebraic rule requires doubling one of the given quantities, and then dividing it by 2 (when Bombelli constructed the quadratic equation that solved the algebraic version of this problem, twice the given quantity appeared as a coefficient in the equation, and the solution rule then required this coefficient to be halved — hence the subsequent multiplication and division by 2). The geometric reconstruction follows the algebraic rule so faithfully, that it insists on reproducing the subsequent and mutually-cancelling doubling and halving. A further interesting attempt to express arithmetic information geometrically occurs when extracting the root of binomials. In §63, when subtracting two roots whose ratio is rational (in Bombelli's language, the numbers under the root are to each other as "a square number to a square number" — "come da numero quadrato a numero quadrato"), the roots are modelled as a a square removed from a square. But in the next section, where the ratio between the subtracted roots is not rational, they are modelled as a rectangle removed from a square.

have is a relative sliding of multiple algebraic and geometric interpretations under the same geometric sign. Indeed, as we saw in the previous subsection, a single geometric line could represent a line or a rectangle, a geometric unit measure or an arbitrary arithmetic length. [26]

We will return to this latter point in the fourth section of this paper. But first let's expand the former point, and explore the functional (rather than semantic) relations that Bombelli constructs between geometry and algebra.

3. The functional relations between geometry to algebra

The first geometric representation of an algebraic operation in Bombelli's Book II is preceded by the following statement: "And while this science is arithmetic (as it was called by Diophantus the Greek author and the Indians), nevertheless it does not follow that one can't *provare* it all by geometric figures (as does Euclid in the second, sixth, tenth)" books of the *Elements*.[27] According to the convincing arguments of Sabetai Unguru and David Rowe (1981–1982) Euclid saw things quite differently. But for the Renaissance revisionists the *provare*-bility of arithmetic by geometry was taken for granted. What must be carefully analysed here, however, is the precise use that Bombelli makes of the verb *provare*.

Provare could potentially mean 'prove', 'demonstrate', 'test', 'verify' and a myriad of shades in between. In this section we will follow the different ways that Bombelli relates geometry and algebra. In some cases geometry

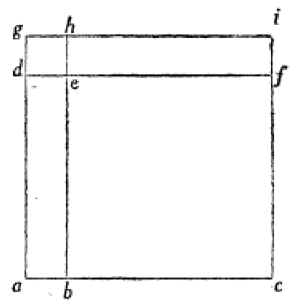
[26]It is interesting to recall here Lacan's reaction in his *Instance of the letter* to de Saussure's representation of the signifier and the signified strata as two parallel horizontal wave figures correlated by vertical lines. Rather than horizontal strata of 'upper water' (signifier) and 'lower water' (signified) anchored to each other by the vertical dashes, Lacan suggests reading the diagram as one horizontal stream flowing under another, while vertical drops of rain flow between them (Lacan, 2006).

[27]"E benchè questa scientia sia Arimetica (come la chiamano Diofante Autore Greco e li Indiani) però non resta che il tutto non si possi provare per figure Geometriche (come fa Euclide nel secondo, sesto, decimo)" (Bombelli, 1966, 184–185). Bombelli rarely quotes explicitly from Euclid — almost all explicit quotations are found in the first 18 sections of Book IV, where Bombelli introduces his basic geometric constructions. From the reference to Euclid's VI.12 in Bombelli's §18 we can infer that Bombelli used either a Greek version or an edition of Zamberti's Latin translation from the Greek (Campanus' Latin translation from an Arabic source has the Greek VI.10 as his VI.12). Nevertheless, Bombelli's diagram in §18 is not a reproduction of the Greek diagram, but an ad-hoc diagram adapted to his specific needs. In fact, Bombelli's list of algebraic sources (quoted above) suggests a thorough bibliographic research, and if this research extended to geometry too, it is likely that Bombelli consulted several versions of the *Elements*, and was not committed to any one particular edition. Medieval translators and commentators had already conflated arithmetic and algebra (e.g. Barlaam's commentary on Book II), but Bombelli's reduction of Euclid's binomials to sums of roots or of a number and a root is closest to what we observe in Tartaglia's Italian translation, which is based on an integration of Campanus' and Zamberti's translations, but which takes a further step toward an arithmetisation of Euclidean geometry (Malet, 2006). Bombelli's step, in turn, towards such arithmetisation is bolder still, as he considers Euclid's entire Book X as covered by the arithmetic of his own Book I (Bombelli, 1966, 9).

provides an independent *justification* of algebraic manoeuvres. In others geometry serves as *instantiation* or reconstruction of algebraic operations in lines rather than characters. In yet other cases a geometric diagram serves as a common pivotal *translation* of distinct algebraic expressions, which ties them together as equivalent. Finally, we must not neglect the cases where geometric diagrams serve as *accompaniment* for algebraic problems, either to manifest two approaches to a single problem without an attempt to relate them, or to supply a visual accompaniment without any specific functional purpose. I will review these themes in detail so as to demonstrate the multiplicity inherent in Bombelli's relating of algebra and geometry. I hope that this multiplicity may serve as a guiding framework for understanding geometry-algebra relations in wider contexts as well.

3.1. **Geometric justification of algebra.** One aspect of the relation between geometry and algebra in Bombelli's text goes back to the Arabic sources. This is the justification of algebraic rules for solving quadratic equations by geometric diagrams. In §49, for example, Bombelli justifies a rule for squaring the sum of roots by drawing a square, whose sides are the sum of those roots (see figure 2). The square is divided into the square of the first root (.g.e.), the square of the second (.c.e.) and the two rectangles, each representing the product of the roots (.i.e. and .a.e.). The mood of the explanation is so geometric that the congruence between the two rectangles is established not by arithmetic commutativity, but by Theorem II.4 of the *Elements*. Nevertheless, this geometric mood is complemented by assigning numerical values to the lines, without which the diagram could not specifically represent a sum of roots.

FIGURE 2. Bombelli (1929, §49): Geometric justification of an arithmetic rule



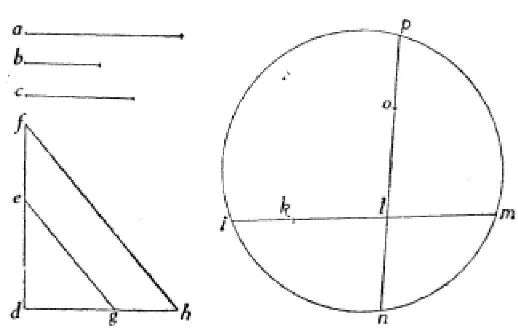
Similarly, many of the rules of Books I and II concerning operating on binomials and solving equations receive (at least an attempted) geometric justification in Book IV. These diagrams are not new, and go back to Cardano, the abbacists, Latin sources and Arabic sources. Quite a few of these diagrams were included in the print edition of Books I and II. Geometry appears to be a required footing for algebra to stand on.

Bombelli extends this geometric footing to specific algebraic problems imported from the manuscript Book III as well. §75, for example, translates the division of 20 *scudi* in two parts, such that $\frac{2}{3}$ of the first part equals $\frac{3}{4}$ of the second (Bombelli, 155?, 17r), into the following geometric problem. Given a line, divide it into two parts, such that if we take a portion of the first part according to the ratio between the given lines .a. and .b., and a portion of the second according to the ratio between the given lines .c. and .d., then the resulting portions turn out equal. The geometric solution that Bombelli provides does not imitate the procedure of the algebraic solution from the manuscript Book III, and its justification depends not on arithmetic, but on elementary proportion theory (*Elements* VI.12, which is not referenced explicitly here, but is referred to earlier in Book IV). The geometric analysis is followed by supplementing the diagram with numerical data identical to those in the problem from the manuscript Book III. The end result is of course the same as well, but the intermediary numbers involved in the Book IV solution are different from those in Book III, as the geometric solution is not a replication of the algebraic moves. Geometry thus provides an independent verification for algebra.

3.2. **Geometric instantiations of algebra.** If we go on to §78, however, we get another side of the story. The problem here is to divide a line .a. in two parts, such that the rectangle made of one part and the given line .b. equals the rectangle made of the other part and the given line .c..

First we are presented with a properly geometric solution. On the left hand part of figure 3 the line .d.f. equals .a., .d.g. equals .c. and .g.h. equals .b.. The lines .g.e. and .f.h. are parallel. Euclidean proportion theory guarantees that .e. divides .d.f. in the required way. If we were to plug in the numerical values from the original algebraic question into the diagram, and derive the same result that we had obtained algebraically, we would obtain, as above, a geometric verification of the algebraic result.

FIGURE 3. Bombelli (1929, §78): geometric justification and geometric instantiation



But Bombelli does not do that. Instead, Bombelli writes: "and since the rule of algebra shows another solution for it, I didn't want to hold back from presenting it".[28] Bombelli's next solution for the same problem depends on the right hand part of Figure 3. Here .i.l. equals .a., .l.m. equals .c., and .l.p. equals the sum of .b. (equal to .l.o.) and .c. (equal to .o.p.). Euclidean circle theory (*Elements* III.35), interpreted by Bombelli as a means of implementing division, yields that .l.n. equals .a. times .c. divided by the sum of .b. and .c.. Setting .k.l. equal to .l.n., claims Bombelli, obtains the desired partition of the given line.

What is the justification for this claim? Here Bombelli makes no geometric attempt to show that the rectangle contained by .i.k. and .l.o. equals that contained by .k.l. and .o.p.. What Bombelli actually did here is take the algebraic solution procedure from the corresponding problem in the manuscript Book III (Bombelli, 155?, 119r), and implement it geometrically. Reformulating Bombelli's solution anachronistically and replacing his numbers by the letters a, b and c (but remaining faithful to his step by step reasoning), the algebraic solution says that the parts are x and $a - x$, and since we require that $bx = c(a - x)$, our equation reduces to $(b + c)x = ac$, which yields $x = \frac{ac}{b+c}$. The diagram simply applies Bombelli's geometric mechanism for dividing ac by $b + c$ to produce the result .l.n..

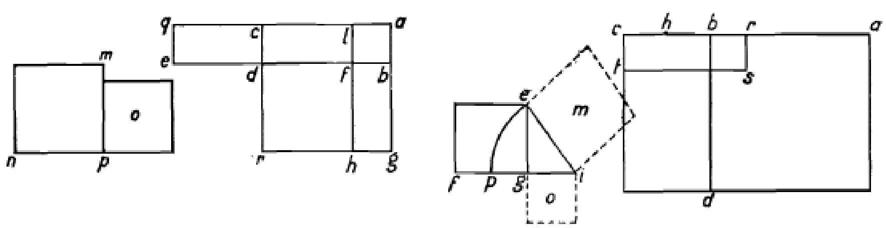
Instead of justifying the construction geometrically, Bombelli verifies that the constructed line represents the correct result by providing numerical values for the lines. In the example above .a. is given the value 11, .b. is 6 and .c. is 4. Therefore .l.p. is 10, $.l.n.$ is $\frac{11 \cdot 4}{10}$ (note that this is an arithmetic claim, and not a claim immediately expressible in Euclidean terms!), and the resulting parts are $4\frac{2}{5}$ and $6\frac{3}{5}$. This result is explicitly verified to satisfy the terms of the problem, namely, that multiplying the former part by .b. and the latter by .c. yields equal results. But, I emphasise, there is no attempt here to argue geometrically why the rectangle contained by .k.l. and .b. equals the rectangle contained by .k.i. and .c.. If there is any justification for this procedure, it is to be found in the manuscript Book III algebraic solution (compounded by the endorsement of intersecting chords as modelling division).

In fact, for the most part, Bombelli's geometric solutions of problems imported from Book III reproduce his algebraic constructions, rather than justify them by an independent geometric procedure. In some cases the algebraic rule is quoted explicitly, while in others only a reference to the relevant problem of the manuscript Book III is included. If there are explanations added, these are usually clarifications as to how the geometric manipulations mimic the algebraic procedure (e.g. end of §§94,95), and not geometric arguments for the soundness of the solution.

[28] "Et perchè la regola de l'algebra ne mostra un'altra dimostratione non ho voluto restare di metterla".

This double gesture — *justification* and *instantiation* — is not restricted to problems imported from Book III, but occurs in the context of rules for solving polynomial equations as well. In Bombelli (1966, 195) and in the manuscript §25, the solution of the quadratic is justified by the standard diagram borrowed from Arabic sources (figure 4 left). But next to this well known diagram appears a new diagram, which, as far as I can tell, is new to the abbacist and the Latin contexts. This diagram geometrically constructs the solution of the quadratic equation, rather than justify it. Indeed, if we write the equation anachronistically as $x^2 + bx = c$, then the area .e.f. represents the number c, and the side of the area .o. represents $\frac{b}{2}$. By the Pythagorean theorem the line .i.e. is the root of the sum of .e.f. and .o. ($\sqrt{\left(\frac{b}{2}\right)^2 + c}$). Taking .i.p. equal to .i.e., and subtracting the side of .o. (which, recall, is $\frac{b}{2}$), we obtain .p.g., which we can anachronistically write as the well known solution formula $-\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + c}$. I emphasise again: the left hand diagram explains geometrically why the standard rule actually solves the equation; the right hand diagram instantiates the rule by a construction that follows its arithmetic steps, but makes no attempt to justify the claim that it is in fact the sought line.

Figure 4. Bombelli (1966, 195–196): From geometric justification to geometric instantiation



3.3. **Common geometric translation of distinct algebraic entities.** But to reduce Bombelli's geometric representations to the distinct roles of justification and instantiation of algebra would be inattentive to the details. In the sections dealing with the solution of cubics and quartics (§§32–48) the geometric representations typically do not quite do either.

In the case of cubics, for example (a treatment adapted from Cardano), the algebraic cube is represented by a geometric cube, and the various other algebraic elements (numbers, *Cose*, squares) are represented by slabs added to the cube or removed from it. When areas or volumes are said to equal or cancel each other — unlike in the case of summing roots quoted above, and in contrast with Cardano's preferred practice — Bombelli's argument does not depend on Euclidean theorems, but on an arithmetic calculation of the sizes of the areas or volumes involved. Then, the transformations required to solve the cubic (change of variable or completion to cube) are presented as different algebraic models for capturing the same geometric diagram. What we have here is not a geometric construction *instantiating* the bottom line of the algebraic solution, nor a *justification* of algebra by a strictly geometric argument, but a geometric diagram that binds different algebraic expressions by serving as their common *translation* — a geometric pivot that enables superposing different algebraic expressions on each other. For example, in §31, the diagram serves to show that the solution of the equation '1 $\overset{3}{\smile}$ equals 6 $\overset{1}{\smile}$ +9' equals the sum of the solution of '$\frac{1\overset{6}{\smile}+8}{1\overset{3}{\smile}}$ equals 9' and of 2 divided by that solution; in §35 the diagram serves to show that the solution of the equation '1 $\overset{3}{\smile}$ +6 $\overset{2}{\smile}$ equals 81' is related to the solution of the equation '1 $\overset{3}{\smile}$ equals 12 $\overset{1}{\smile}$ +65' by a shift of 2 (the text drags a silly scribal or calculation error and has 64 for 65).[29]

An even more explicit expression of the role of geometric diagrams as pivots binding different algebraic expressions is provided in the sections dealing with transforming quadratic equations. For example, in Bombelli (1966, 204) the equation '$\overset{2}{\smile}$ +6 $\overset{1}{\smile}$ equals 16' is related to the equation '$\overset{2}{\smile}$ equals 6 $\overset{1}{\smile}$ +16' not by algebraically substituting $\frac{6}{1\overset{1}{\smile}}$ for $\overset{1}{\smile}$, but by showing that both equations translate the same diagram of a rectangle of area 16 whose one side is longer than the other by 6 — the difference lying in the side modelled as $\overset{1}{\smile}$. But this translation of algebraic language into geometry is useful precisely because it is *not* one-to-one. Different algebraic texts (the original and reduced equations) are translated by the same geometric figure — only the parts of the figure are modelled differently by each algebraic interpretation. The single reference or translation for the two different algebraic texts serves as a pivot that identifies the distinct algebraic equations beyond their formal differences, and allows for *seeing* them as solving the same problem.

[29]Freguglia (1999) also studies Bombelli's geometric representations of solutions of algebraic equations. He states that for second and third degree equations Bombelli provides geometric step-by-step justifications of the algebraic processes of solution as well as geometric constructions of the solutions that do not justify the algebraic rule. For fourth degree equations, Freguglia claims, Bombelli only provides a partial geometric justification of the algebraic solution process. But as far as I can see, this analysis is imprecise. For second degree equations Bombelli provides a geometric *justification* of the solution and then a distinct geometric *instantiation* (see figure 4); for third degree equations he provides a geometric *translation* of the original equation to a reduced one and a distinct geometric *accompaniment* that constructs a solution independently of the algebraic procedure (see below); and for fourth degree equations Bombelli provides only a geometric *translation* of the original equation to a reduced one. Freguglia links Bombelli's different representation strategies to the problem of homogeneity and of geometric representation of high powers of the unknown. But my analysis of Bombelli's treatment of homogeneity undermines this explanation. In fact, Bombelli had the means to provide all kinds of geometric representations to all relevant equations. His choices had more to do with complexity than with inherent limitation of representation techniques.

But there's another way in which geometry can tie together different algebraic objects: it can serve as a common representation that bridges their conceptual divisions. In §§38–42, for example, a single diagram accompanies five different reductions of equations of the form 'cube, squares and *Cose* equal number' to simpler cubic equations. Each reduced form is of a different kind, in parallel to the treatment in Book II. The fact that a single diagram manages to represent all cases weakens the organising principle structuring Book II, which considers different forms of cubic equations as distinct cases that require separate treatment.

A similar impact is had by §44 and §45, which share the same diagram, and where the latter, claims Bombelli, "is no different from the former except in that the square number .u. is added to the square .e.f., and in the other subtracted".[30] That a common diagram can represent different problems that differ by a sign renders these problems less obviously distinct. The point here is not simply that a single 'general' diagram can represent several 'specific' algebraic cases; the point here is that a single diagram can tie together algebraic elements that are considered essentially different (in the former example different kinds of cubic equations, in the latter addition and subtraction). Such common geometric representations may have helped Bombelli (on top of the exhaustion expressed explicitly in the conclusion of his Book II) to eventually let go of treating separately all different reductions of quartic equations and of distinguishing all subtractive and additive changes of variable required for these reductions.

Another instance of geometry bridging algebraic differences occurs in the context of the principle setting irrational and rational number apart as differing in 'nature' (Bombelli, 1966, 13). Indeed, when Bombelli explains how to geometrically extract a square root, he notes that it "doesn't suffer the difficulties that it suffers in numbers; because one will always find the root of any given line" (§19).[31] Given that for Bombelli any quantity can be represented by a line, the division between numbers that have discrete roots and those that do not becomes much less substantial.[32]

3.4. **Geometric accompaniment of algebra.** So far we saw that geometry sometime served as a *justification* for algebra, sometimes as *instantiation* of an algebraic solution procedure, and sometimes as a *translation* or pivot tying together different algebraic terms. But these functions, which are not always easy to set apart, do not exhaust Bombelli's ways of relating algebra and geometry.

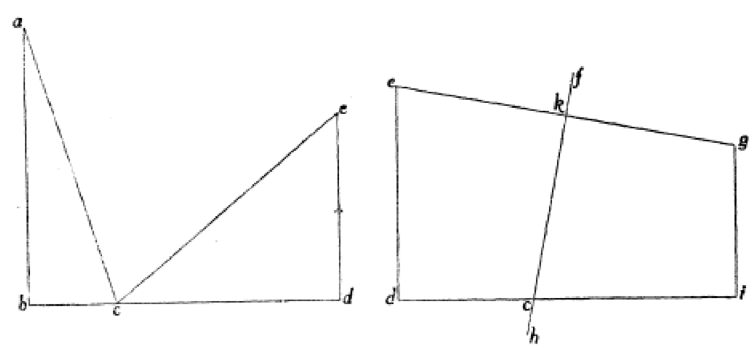
[30]"non è differente da la prima se non in questo, che il quadrato numero .u. si aggiungeva con il quadrato .e.f., et in questo si cava.".

[31]"il quale creatore non patisce le dificultà, che pate nel numero; perchè sempre si troverà il creatore d'ogni preposta linea, essendo noto la comune misura".

[32]See Wagner (2010) for an overview of Bombelli's division of numbers according to their natures and of the undermining of this very division in his algebraic practice.

Problems §123–124, for example, both require to find the point .c. equidistant from .e. and .a. (left hand side of figure 5). The solution of §123 models the line .b.c. as an unknown, and uses the Pythagorean theorem to derive a quadratic equation for it. The solution in §124 uses the right hand diagram, and constructs the solution geometrically, having nothing to do with the algebra (the trick is to construct .k.c., an orthogonal bisector of .e.g., and use the Pythagorean theorem). This independent geometric construction is not accompanied by numerical values for verifying that it, too, would lead to the same solution as the algebraic argument. It is simply there as a counterpoint, accompanying the algebra, without any attempt to project it on the algebra or tie them together (another example for a geometric accompaniment independent from the algebraic solution is the construction of a solutions to cubic equations by means of lines, discussed in section 4.2 below).

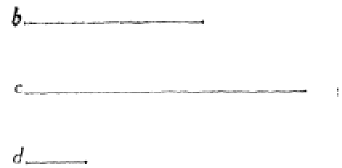
FIGURE 5. Bombelli (1929, §§123–124): Independently accompanying algebraic and geometric solutions of a geometric problem



In other cases the relation between geometry and algebra is weaker still. I'm referring, for example, to the demonstration of rules for adding and subtracting roots. While in figure 2 we saw one of these rules represented by a partitioned square, the diagrams for the other rules (figure 6) are nothing but disconnected lettered lines. Book III also has one of those odd diagrams, where three lines of equal lengths marked .a., .b. and .c. accompany a Diophantine problem (problem 140).

These examples perhaps appear more odd than they should. The Euclidean tradition insisted on drawing diagrams for all problems and theorems, even arithmetic problems and theorems, and even those where the diagrams were nothing but a bunch of independent lettered lines. Ian Mueller suggested that in such cases "the diagram plays no real role" ... "except possibly as a mnemonic device for fixing the meaning of the letters" (Mueller, 1981, 67). Reviel Netz asserts that arithmetic diagrams "reflect a cultural assumption, that mathematics *ought* to be accompanied by diagrams" (Netz,

FIGURE 6. Bombelli (1929, §51): Non-functional geometric diagram accompanying for algebraic rule



1999, 42). To appreciate the role of diagrams in Bombelli's text, we must not neglect this tradition of diagrams, which are there to illustrate, not to argue. [33]

One more accident allows us to better appreciate the role of diagrams in Bombelli's work. In §72 the triangle .d.e.g. is given the numerical side lengths 1 for .d.e., 8 for .d.g. and 4 for .e.g.. These data contradict the triangle inequality, as $1 + 4 < 8$, and therefore cannot describe a genuine Euclidean triangle. But since the geometric instantiation was not practiced as a faithful illustration of the data (many other diagrams are disproportional compared to the numerical data), Bombelli could go on with the example unhindered.

3.5. **So what is the relation between algebra and geometry?** If the multiplicity of relations between geometry and algebra documented above appears confusing, it is probably because we take too much critical distance. We must remember: algebra had already had its diagrams when it came to Italy. The Latin tradition, as well as some abbacus treatises, retained the geometric diagrams that accompanied the Arabic solution of quadratic equations and led up to a geometric understanding of the solution of the cubic equation. On the other hand, many abbacists set algebra and geometry apart, and developed the former independently. When this maturing algebra came across geometry again, it brought about an arithmetic understanding of some of Euclid's geometric books and algebraic solutions of geometric problems.

It is therefore not surprising that Bombelli, who inherited all these different traditions, came to piece geometry and algebra together in various different way, and, according to his own quotation above, felt that algebra and geometry belonged together. This togetherness of algebra and geometry did not depend on a unique, one-directional relation of *justification*, *instantiation*, *translation* or *accompaniment*. For people like Bombelli, it seems, algebra was geometry's younger — but not for all that entirely dependent — sister, and these siblings were believed to play best when they were allowed to play together. Bombelli was intent on exploring and diversifying

[33] For an elaboration of this point see my Deleuzian analysis of classical Greek diagrams (Wagner, 2009).

the playgrounds available for the common games of geometry and algebra. For an author as mathematically proficient as Bombelli, this didn't come at the expense of rigour.[34]

4. The geometry of what's not quite there

Bombelli's *algebra linearia* becomes much more challenging when we consider not only its various relations with algebra's established entities, but also its original manner of representing more challenging algebraic objects whose existence or status was not yet properly settled. Here we can see the genuine synergy inherent in the core of *algebra linearia*: its treatment of negative entities, their roots, algebraic unknowns and their powers. A proper understanding of Bombelli's approach and achievements depends on this hitherto neglected aspect of his work. A careful study of this aspect will show us how Bombelli's bringing together of geometry and algebra generated new mathematical practices, hybrid geometrico-algebraic entities, grounds for endorsing questionable mathematical entities, and multiple ways of reading a given sign.

The first subsection will deal with the geometric representation and endorsement of negative magnitudes. The second subsection will deal with the endorsement of expressions involving roots of negative numbers and with the underlying implicit hypothesis of co-expressivity between algebra and geometry. The last two subsections will study the hybrid gemetrico-algebraic practices built around representations of unknown magnitudes, and the multiple vision of geometric signs that these practices depended on.

4.1. **The geometry of missing things.**

I argue in Wagner (2010) that the Renaissance *meno* cannot be properly reduced to subtraction. But in the context of geometric representations of algebra the subtraction interpretation seems more defensible, although not entirely exhaustive. Cardano, for one, uses the term 'add negatively' for turning from addition to something between a subtraction and an addition of negative geometric elements.[35] Bombelli's practice is similarly ambiguous. When he says that some areas

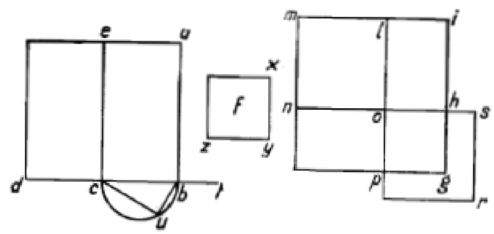
[34]But that's a very local interpretation. Seen from a wider epistemological-historic perspective, the conjunction of geometry and algebra may have had to do with more wide ranging trends. For one, the regulated projection of one domain of signs on another is, drawing on Kristeva, a semiotic technique for producing an effect of truth (*vraisemblance*) since the birth of the novel in the 15^{th} century (see *La productivité dite texte* in Kristeva (1969); for how this theory works in a mathematical context see my Wagner (2009)). Moreover, the emphasis on geometrically observing algebra, rather than just symbolically writing it, can be seen as related to the role of vision in the epistemology of early modern science. I don't pursue either direction here, because Bombelli does not provide enough textual evidence to substantiate either interpretation. But Bombelli did not write in scientific isolation, and the general trends of the period should not be ignored.

[35]"addito per m̃.", where m̃ is short for *minus* (Cardano, 1968, e.g. 54,97). Note that the latter expression was revised in the later editions of the book so as to use unambiguous subtraction talk.

are *meno*, they can be, in that context (e.g. §§26,27,46,47), interpreted as subtracted geometric magnitudes rather than as added negative geometric magnitudes.

One of the most interesting points of ambiguity occurs around the diagram reproduced in figure 7. Bombelli explains that "from the square .g.h.o. one removes the square .r.o.s." (where the latter is bigger than the former). "And because we're missing the gnomon .h.r.p.g. and we have the surface .f., it is necessary that the said gnomon be equal to that surface".[36] It is clear that Bombelli allows for the subtraction of a larger area from a smaller one, and that the result is a "missing" area. But this missing area is positive, in as much as it can be compared to another positive area. Here geometric representation turns what's missing into something present. By way of a similar geometric treatment, the absence or presence of the gnomon becomes a relative position, rather than an ontological characteristic of a geometric element. Again we witness the role of diagrams as pivots, translating different arithmetic entities into a common referent.[37]

FIGURE 7. Bombelli (1966, 203): Subtracting a larger area from a smaller one



4.2. **The geometry of sophistic things.** The first appearance of roots of negative numbers in Book II occurs in the context of quadratics. A solution involving a root of a negative number is presented on a par with another "sophistic method",[38] that of simply replacing the impossible subtraction in the solution rule by an addition.[39] A few pages later, in the context of the

[36]"del quadro .g.h.o. si levi il quadro .r.o.s." ... "E perchè ci manca il gnomone .h.r.p.g. et habbiamo la superficie .f., di necessità bisogna che il detto gnomone sia pari ad essa superficie" (Bombelli, 1966, 202–203).

[37]This representation should be contrasted with Cardano's diagram illustrating a situation where 40 were to be subtracted from 25, if one applied the usual solution rule to the quadratic equation 'square plus 40 equals 10 things'. There, all that the diagram shows is a square of area 25 built on half a line of length 10 (Cardano, 1968, Ch. 37).

[38]"modo sofistico" (Bombelli, 1966, 201).

[39]This approach might be explained by the work of Marco Aurel, who taught and published in Valencia. In his *Despertador de ingenios, Libro Primero de Arithmetica Algebratica* (1552), he offers as solution to the irreducible case of $x^2 + c = bx$ the negative counterpart of Bombelli's "sophistic" suggestion, namely $-\frac{b}{2} - \sqrt{\frac{b^2}{4} + c}$ (this is, of course,

bi-quadratic without real solutions, solutions involving roots of negatives are not mentioned at all, not even under the mark sophistry. It is simply claimed that "such case could not be solved because it concerns the impossible".[40]

But when we turn to cubic equations, Bomobelli's approach is very different. The context here is equations of the kind 'Cube equals *Tanti* and number'. Applying Dal Ferro's and Tartaglia's solution rule for some instances of such equations — Cardano's so called irreducible case — yields an expression involving roots of negative numbers. Bombelli's most renowned achievement is his endorsement and analysis of such solutions. In the manuscript, Bombelli justifies his approach on pragmatic grounds: such solutions arise from the same rule that worked in the reducible cases, and Bombelli can often enough transform them into correct real solutions (Bombelli, 155?, 72v). Bombelli's prime example (in anachronistic notation) is the equation $x^3 = 15x + 4$, whose solution, according to the received solution rule and Bombelli's method for extracting cubic roots of binomials, is $\left(2+\sqrt{-121}\right)^{1/3} + \left(2-\sqrt{-121}\right)^{1/3} = \left(2+\sqrt{-1}\right) + \left(2-\sqrt{-1}\right) = 4$. But the derivation of correct real solutions cannot alone account for Bombelli's endorsement of solutions involving roots of negative numbers, because a few pages further on such a solution is endorsed, even though Bombelli can't rewrite it as a real solution (Bombelli, 155?, 76v).[41]

By the time the print edition was ready, Bombelli could provide another reason for endorsing solutions for cubic equations that involve roots of negative numbers. He explains: "and although to many this thing will seem eccentric — for I too was of this opinion some time ago, having the impression that it should be more sophistic than true — nevertheless I sought until I found the demonstration, which will be written down below, that indeed this can be shown in lines, that moreover in these operations it works

anachronistic notation). If one postulates that the product of two isolated negative numbers is negative (a postulate that Bombelli took up in his manuscript; see my Wagner (2010)), one indeed obtains a correct solution (this observation is derived from a talk by Fàtima Romero Vallhonesta delivered at the PASR conference in Ghent on August 27, 2009). Bombelli's reference to this way of solving quadratic equations, as well as his manuscript reference to the negative result of the product of isolated negative numbers, might suggest that the "certain spaniard" ("certo spagnuolo") mentioned among Bombelli's sources in his introduction was not the Portuguese Pedro Nuñez, as asserted by Bortolotti, but in fact Marco Aurel.

[40] "Ma se non si potrà cavare il numero del quadrato della metà delle potenze, tal capitolo non si potrà agguagliare per trattarsi dell'impossibile" (Bombelli, 1966, 207).

[41] In contemporary terms the solution is $\left(3+\sqrt{-720}\right)^{1/3} + \left(3-\sqrt{-720}\right)^{1/3} - 3$. Bombelli's techniques do not allow him to simplify the cubic root in this expression.

without any difficulty, and often enough one finds the value of the *Tanto* as number".[42]

But what exactly is this demonstration "in lines"? Bombelli's argument is as follows. First Bombelli shows that if one has a sum of cubic roots of the anachronistic form $\left(a+\sqrt{-b}\right)^{1/3}+\left(a-\sqrt{-b}\right)^{1/3}$, then a cubic equation can be constructed, which, given the received solution rule, would produce that sum as its solution (Bombelli, 1966, 226). Then Bombelli shows geometrically that such cubic equations must have solutions (Bombelli, 1966, 227–228). Since he can show that the questionable algebraic expression solves an equation, and since, moreover, he can draw a solution for that same equation geometrically, Bombelli concludes that the geometrically found "length of the *Tanto* will also be the length of" the sum of "the two cubic roots above".[43] Bombelli concludes that the algebraic solution of the equation must coincide with the geometric construction, and that the latter therefore validates the former.

This argument, however, is not without its difficulties. The first difficulty is one that is explicitly addressed by Bombelli. Indeed, Bombelli's geometric solution is a planar construction, which, given a segment and an area representing the coefficients of the equation, a unit measure, and two right angled rulers, yields a segment representing the solution. Bombelli rejects possible objections to the use of right angled rulers by noting that a planar solution to a solid problem must use advanced tools[44] and by relying on the authority of no less than Plato and Archytas.

A second difficulty, which Bombelli fails to address, concerns the conditions of viability of the geometric construction. The solid diagram for solving cubic equations (Bombelli, 1966, 226), derived from the one introduced by Cardano, fails to solve the irreducible case, as Bombelli and Cardano explicitly note. The novelty in Bombelli's planar diagram is precisely that it circumvents this difficulty. However, Bombelli makes no effort to show that his diagram is indeed constructible for all possible coefficients of the relevant kind of cubic equation. This is all the more unsettling, as the next kind of cubic equation that Bombelli treats does not always have a positive solution, but while Bombelli is well aware of this fact, and provides a precise arithmetic solvability condition (Bombelli, 1966, 231), he makes no attempt to point out the geometric obstruction restricting his construction. Bombelli does not raise the question of whether his former geometric construction is

[42] "Et benchè a molti parerà questa cosa stravagante, perchè di questa opinione fui ancho già un tempo, parendomi più tosto fosse sofistica che vera, nondimeno tanto cercai che trovai la dimostratione, la quale sarà qui sotto notata, sì che questa ancora si può mostrare in linea, che pur nelle operationi serve senza difficultade alcuna, et assai volte si trova la valuta del Tanto per numero" (Bombelli, 1966, 225).

[43] "e trovata che si haverà la longhezza del Tanto sarà ancora la longhezza delle due R.c. legate proposte" (Bombelli, 1966, 226).

[44] See Bos (2001, Ch.3–4) for the context of such an argument.

or is not restricted by obstructions, and leaves his claim of general solvability without critical examination.

There's a further difficulty that Bombelli fails to address, which concerns the correspondence between the arithmetic and geometric representations of the solution. Bombelli is well aware of negative solutions of cubic equations. He uses such solutions to derive positive solutions of other equations, and sometimes even considers them independently.[45] But Bombelli never explicitly raises the possibility that his questionable sum of cubic roots might capture a negative solution, rather than the positive solution that he constructed geometrically.

Bombelli is obviously eager to make his solution for the cubic acceptable. To that end he is willing to use questionable construction methods, avoid dealing with the viability conditions of his diagrams, and suppress negative solutions. But most important here is that on top of all that Bombelli takes another crucial step on this already shaky ledge. Bombelli uses his argument above to endorse an algebraic entity, which his geometric construction does not actually draw. Nowhere does the diagram pick up roots of negative numbers, either directly or inside the cubic root of a binomial. In the language we introduced above, the geometric construction *accompanies* the algebraic solution, rather than *justify*, *instantiate* or even *translate* it. The diagram does indeed construct a line satisfying the terms of the equation, but its construction has nothing to do with the algebraic rule of solution and its roots of negative numbers. The speculation that the algebraic solution is identical to the line constructed in the diagram is snuck in through the back door without an explicit account.

The relation between geometry and algebra here can be qualified in a finer manner, if we observe Bombelli's remark that a certain cubic equation allowed him to trisect an angle, and that this fact led him to keep attempting (in vain) to transform that equation into one that he could solve without roots of negative numbers. His conclusion was that "it is impossible to find such general rule" for solving cubics without roots of negatives.[46] The point here is the tension that Bombelli expresses between being able to draw a solution and not being able to write it down in traditional arithmetic terms. This tension is something that Bombelli finds so hard to sustain, that he concludes by allowing an expression involving roots of negative numbers as

[45]See Wagner (2010) for details.

[46]"Sì che (quanto al mio giuditio) tengo impossibile ritrovarsi tal regola generale" (Bombelli, 1966, 245). The discussion probably refers to §135 of Book V, where the trisection of an angle for the construction of a regular nine-gon is reduced to a cubic equation. This is not the same cubic equation as cited in Book II, and there's no attempt in Book V to solve this cubic equation with or without roots of negative numbers, but we must recall that Book V is the least complete among the books of *L'algebra*. Note also that this algebraic-geometric reflection, unlike the one concerning the general plane geometric solution of 'Cube equals *Tanti* and number', was already present in Bombelli's manuscript (Bombelli, 155?, 88r), and may have therefore factored into his original endorsement of solutions involving roots of negative numbers.

an algebraic representation of the geometric solution. Recall, in contrast, that in the case of quadratic equations, where solutions involving roots of negative numbers could not be drawn geometrically or reduced to verifiable real solutions, Bombelli rejected them as sophistry. Bombelli's underlying conviction thus reveals itself: if you can draw it, you should be able to express it algebraically, even if the expression looks like gibberish. To put it into a catch phrase: 'What you see is what you say'.

If we are to understand Bombelli's articulation of the algebra-geometry nexus, we should acknowledge the intimacy granted here to algebra and geometry. It's not just about justification, instantiation, translation or accompaniment. It's not just about two strata of expression interacting with each other in order to bring the best out of both. It's about a deeply underlying assumption of co-expressivity: what's expressible in the one domain, should be expressible in the other. Geometric visibility does not only guarantee reality, it should also guarantee algebraic expressibility. Without such underlying assumption, Bombelli's argument above would not have forced him to integrate roots of negative numbers into his mathematics.

The logic that short-circuits visibility and expressibility is, of course, an issue that deserves an independent tracking across the history of science. But here we'll restrict ourselves only to validating that it works in both directions. The other direction of the principle that 'what you see is what you say' is the principle that what you say should be visible as well. According to this principle algebraic unknowns should be expressible geometrically. And this is where we're turning next.

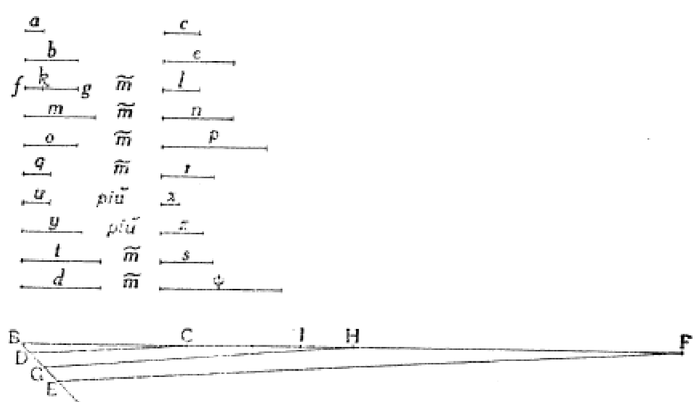
4.3. **The geometry of the unknown: the rule of three.** Geometric modelling of unknown magnitudes is not new. The very diagrams that since Al-Khwarizmi accompanied the solution of the quadratic equation did just that. The unknown line was represented by an arbitrary line, and so were lines representing known magnitudes, without necessarily respecting the proportions between the eventual value of the unknown and the given known values. But Bombelli is interested in doing better than that.

Let's go back to figure 4. The left hand side is the typical diagram traceable to Arabic sources, where .a.b. is the unknown line and .b.g. is a known line. The right hand side of the diagram, as explained above, is not another justification of the algebraic solution, but a geometric instantiation of the solution rule. The unknown square .a.b.d. (standing for the square of the unknown) and rectangle .d.b.c. (standing for the known coefficient .b.c. times the unknown) are to be equated to the known .f.g.e. (a number). The construction derives .p.g. as the value of the unknown. Since this is obviously disproportional with respect to .a.b., the previous representation of the unknown, Bombelli redraws the right hand side of the right hand diagram as the rectangle .r.s.t.

This strategy is not typical of the tradition, nor is it typical of Bombelli. It stands out more as an expression of a problem than as a way of dealing with it. When including known and unknown lines in the same diagram, and then using the known lines to derive a representation of the unknown, the unknown can no longer be drawn arbitrarily. Bombelli is concerned with this problem and attempts to derive a more satisfactory form of representation.

Bombelli's alternative representation is as ingenious as it is simple, and the fact that it was not systematically replicated (as far as I know) was, I feel, a genuine loss to techniques of mathematical representation. One of the most interesting examples occurs in figure 8. This geometric problem is presented by Bombelli as an analogue of Problem 41 from the manuscript Book III, which reads: "Two people have money, and find a purse in which there was as much money as the first person had, and this first person says to the other: if I had the purse, and $\frac{1}{3}$ of yours, and 2 more, I would have twice as much as your remainder. The second says: if I had the purse, by giving you 4 of mine, I would have two times and a half as much as you. One asks how much money was in the purse and how much each of them had".[47] In the geometric version of Book IV §86, the money of each person is replaced by an unknown line, $\frac{1}{3}$ is replaced by the ratio between the lines .a. and .b., and 2 and 4 are replaced by the lines .c. and .e. respectively.

FIGURE 8. Bombelli (1929, §86): adding known and unknown lines



The solution begins by selecting a line .f.g. "as one wishes, which would play the role that the *cosa* plays for numbers",[48] and which stands for the second person's original money (note that Bombelli mentions the algebraic version only after the geometric statement and solution are concluded; I

[47]"Due hanno denari, et trovano una borsa ne la quale era tanti denari quanti haveva il primo, et dice esso primo à l'altro: se io havessi la borsa con $\frac{1}{3}$ de tuoi, et 2 puì," [sic] "io haveria duo tanti del tuo rimanente. Dice il secondo: se io havessi la borsa, con il dare a te 4 de miei, io haverei due volte et mezzo quanto te. Si domanda quanti denari era ne la borsa et haveva ciascuno da se" (Bombelli, 155?, 122r).

[48]"Pigliasi una linea a beneplacito la quale farà l'effetto che fa la cosa nel numero".

conflate the two problems to bring out the analogy and make things easier to follow). But Bombelli warns that "the unknown lines can never be added or subtracted from the known".[49] And so in figure 8, when .f.g. is rescaled according to the given ratio .a. : .b. to form .f.k., and then the known .c. is subtracted from the remainder .k.g. (this represents taking $\frac{1}{3}$ of the second person's money and 2 more), .f.k. (the result of the rescaling) is drawn on .f.g., but the subtracted .l. (which equals the known .c.) is drawn separately. If one continues the geometric modelling of the protagonists' manipulations and wades through their geometric representations[50], one eventually obtains that the unknown .d. minus .y. is to equal the known .z. plus .ψ.

Only at this point does Bombelli bring known and unknown lines into contact via the bottom part of the diagram. On the 'unknown axis' (this is *not* Bombelli's term) we draw .B.D. equal to the difference between the unknown .d. and .y., and .B.E. equal to our original unknown .f.g.. On the 'known axis' we draw .B.C. equal to the sum of .z. and .ψ.. Now, as .B.C. is supposed to equal .B.D., drawing the parallel lines .D.C. and .E.F. yields .B.F., which, according to Euclidean proportion theory, is the sought value of the unknown .f.g..

Bombelli does not justify his procedure here, because he has already done so earlier, in §73, where this rescaling technique was explicitly related to the rule "*of three*".[51] In the context of this example, the rule of three applies as follows. After the geometric construction is concluded, Bombelli assigns to the known lines the original numerical values from Book III, and to the unknown line .B.E.=.f.g. he assigns the tentative numerical value 3. Then, following the steps of the geometric construction, .B.D. ends up being $\frac{3}{4}$ and .B.C. ends up as $9\frac{1}{2}$. Now, since $\frac{3}{4}$ (.B.D.) is actually supposed to stand for $9\frac{1}{2}$ (.B.C.), then, according to the rule of three, .B.E. is actually supposed to stand for $\frac{3 \cdot 9\frac{1}{2}}{\frac{3}{4}} = 38$, which is the length of .B.F..

[49] "Et notasi, che mai le linee incerte, non si possano aggiungere nè cavare con le certe".

[50] This goes as follows. The difference between .k.g. and .l. corresponds to the second person's remainder in the first person's narrative. To retrieve what the first person originally had according to his narrative, double the latter to produce the difference between the unknown .m. an the known .n., subtract from the latter the unknown .f.k. and the known .c. to get the difference between the unknown .o. and known .p., and then divide in half to get the difference between the unknown .q. and known .r.. Now that we have a representation of the first person's original money, we follow the second person's narrative. We add to the latter the known .e. to get the sum of the unknown .u. and the known .x., which corresponds to the first person's money at the end of the second person's narrative. Rescaling by the ratio of 5 : 2 we get the sum of the unknown .y. and the known .z., which corresponds to the second person's money at the end of his own narrative. According to that narrative, this sum must also correspond to the result of adding .f.g. to .q. minus .r. (represented as the unknown .t. minus the known .s.) and then removing the known .e.. This final magnitude is represented by the unknown .d. minus the known .ψ.. The bottom line is that the unknown .y. and known .z. equal the unknown .d. minus the known .ψ..

[51] "la regola della proportione chiamata *del tre*".

Just as the arbitrary numerical choice 3 was a place holder for the correct solution 38, so is the line .B.E.=.f.g. a place holder for to correct solution .B.F.. The number 3 didn't simply stand for itself, and neither did the line .f.g.. This form of representation enables a line to stand for its possible rescaling with respect to other given lines. As long as we keep the known magnitudes and unknown magnitudes along different 'axes' or 'dimensions', there's no risk of error. The point is that just as the rule of three (as I argue in Wagner (2010)) evolved into an abbacist practice of seeing numbers as other numbers, here we learn to see lines as other lines.

The double vision required to understand the diagram extends to the side-by-side setting of known and unknown lines in the upper part of the diagram. The two scales are confronted, but not confused. And this separation occurs in the numerical instantiation that follows the geometric construction as well. Given the numerical values he assigns, Bombelli obtains that .t. minus .s. is "$4\frac{1}{2}$ *meno* 3", but this is never interpreted as $1\frac{1}{2}$, because $4\frac{1}{2}$ is our place holder for an unknown magnitude, while 3 is a known magnitude. Indeed, the equivalent term in the algebraic solution of Book III is $1\frac{1}{2}\overset{1}{\smile} - 3$ (recall that .f.g., which stands for the unknown, was assigned the value 3 in the arithmetical verification of the geometric version of this problem, and so $1\frac{1}{2}\overset{1}{\smile} = 4\frac{1}{2}$). Moreover, in §85, solved by a similar technique, a certain difference is reconstructed as "2 *meno* 3", but this difference is not marked as negative, as the 2 is a place holder for an unknown magnitude. Indeed, in the diagram, the lines representing this difference have a positive difference, as is the case if we substitute the final result for these lines.

The bottom line of this geometric representation technique is that one is trained in seeing lines as tentative place holders and as representing more than their visible lengths, not only with respect to a rescaling of the entire diagram (as is already the case in Euclidean diagrams), but also relative to other lines in the same diagram. What you get is more than what you see. Lines belong to various systems of relations with respect to other lines, and these relations are invisible for an eye not trained in this hybrid practice of algebra in lines.

The confrontation of known and unknown lines described above starts as reproducing cossist algebra, but then ends up as a geometric representation of the rule of three. Similarly, §§119–120 juxtapose in the context of the same problem cossist algebra, geometric rescaling and a quadratic version of the rule of three (where rescaling a given magnitude by a certain factor rescales another magnitude by that factor squared; here this applies to the relation between the side of a triangle and its area). While in terms of arithmetic/algebraic practice the rule of three and linear cossist algebra are distinguishable, Bombelli's geometric representation makes it difficult to set them apart. This technique of representation serves as a common reference that helps bind the rule of three with algebra, and may have contributed to the eventual absorption of the older practice into the newer one.

4.4. **The geometry of the unknown: *cosa*.** But another step is required if one is to render cossist algebra geometrically visible. According to one interpretation, geometric representations of powers of the *cosa* were restricted to the line-square-cube scale. Beyond that, algebra was not geometrcially instantiated.[52] However, as we saw above, not only is Bombelli willing to represent higher algebraic powers by lower geometric dimensions, he also has the technique to rigorously reduce higher geometric dimensions to lines. This is combined explicitly in §21, where an elegant spiral-like diagram starts from a unit segment and reconstructs each power of the *cosa* as the geometric mean between the previous power and the next.

The geometric representation of higher powers in Books IV and V of *L'algebra* is treated in several ways. The first treatment occurs in the context of geometric translations of algebraic solutions of polynomial equations, following the Arabic sources and Cardano's diagrams discussed above (see Subsection 3.3). In this treatment higher powers of the *cosa* correspond to higher geometric dimensions. The second treatment concerns 'arithmetic geometry', where geometric problems are presented in terms of numerical line lengths. There the unknown line is modelled as *cosa*, and the geometric situation is expressed by an algebraic problem, which is then solved using algebraic rules. This was an abbacist practice whose state of the art exposition belongs to Piero della Francesca (Bombelli's innovation here is restricted to the often included geometric instantiation of the algebraic solution, see Subsection 3.2).

But Bombelli's third geometric treatment of *cosa* powers, however rare, is the most interesting. The four problems §§102,104,122 and 131 go beyond the two forms of geometric treatment mentioned in the previous paragraph. These geometric problems are analysed without recourse to specific numerical values, they model higher powers of *cosa* as lines, and they manage to provide not only a geometric instantiation of the algebraic solution, but also a geometric representation of the equations themselves and of their algebraic reduction to a canonical form. This modelling technique requires a delicate and specialised way of seeing.

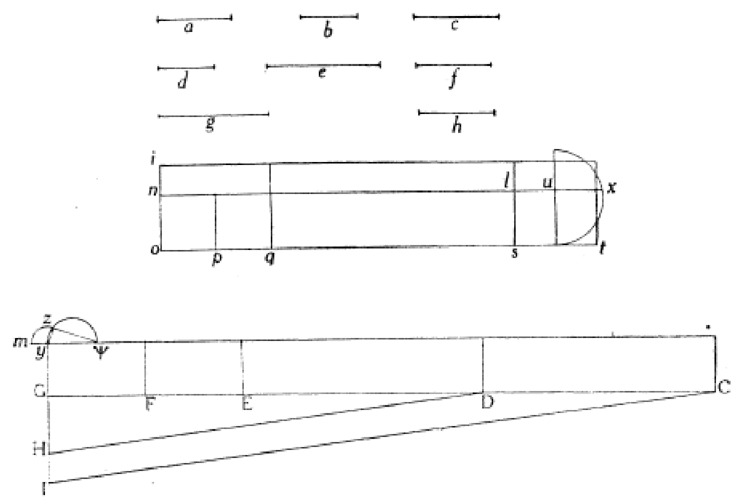
Let's have a close look at §102.[53] The problem asks for three lines, which are represented in figure 9 as follows. The unknown .d. is the first line sought by the problem. .e. is twice .d., and together with .f., which equals the given .a., constitutes the second line sought by the problem. The third line sought

[52]See, for example, della Francesca (1970, 91), where Piero makes this comparison explicit for the first three powers of the *cosa*, reconstructs the fourth power as "two squares" ("doi quadrati"), but then neglects a further geometrisation of the higher powers that he names and defines.

[53]§104 starts with the same approach, but then skips directly to a geometric instantiation of the algebraic solution. §§122 and 131 have blank spaces for diagrams in the manuscript, and the missing diagrams are difficult to reconstruct from the text.

by the problem is such that Rect(third line, .b.)[54] (where .b. is given) equals the rectangle formed by the previous two lines, namely Rect(.d., .e. plus .f.). Finally, the ratio between .d. and the sum of all three sought lines should equal the ratio between the given .c. and .G.C..

FIGURE 9. Bombelli (1929, §102): multiplying known and unknown lines



To make things easier to handle, Bombelli represents the unknown .d. as equal in length to the known .b. This choice is possible because .d. and .b. belong to different scales — unknown and known lines. That they appear to be equal doesn't commit us to assume that they actually are. As long as they're kept apart, any choice is as good as any other. Now that .b. equals .d. "in length",[55] the third line required above (such that Rect(third line, .b.) equals Rect(.e. plus .f., .d.)) is equal "in length" to .e. plus .f., and is represented by .g. plus .h.. But Rect(.g. plus .h., .b.) must equal Rect(.e. plus .f., .d.) not only in area but also in scale. Since .e. and .d. are of the scale of the unknown (corresponding to the algebraic *cosa*), and .f. and .b. are known (corresponding to numbers), .g. must be scaled like the second power of the unknown (algebraic *censo*), and .h. must be of the scale of the unknown (*cosa*). This way both rectangles are sums of *censi* and *cose*. In the corresponding algebraic problem 92 of the manuscript Book III (Bombelli, 155?, 134v), the line .d. is modelled as $\overset{1}{\smile}$,

[54]By 'Rect(line1, line2)' I refer to the rectangle built from the two lines. This is not Bombelli's notation, and I use it here to make things easier to follow. The same goes for the 'Sum(line1, line2, ...)' and 'line1:line2' notations.

[55]"in lunghezza"

.e. plus .f. as $2\overset{1}{\smile}$ +4 and .g. plus .h. as $2\overset{2}{\smile}$ +4 $\overset{1}{\smile}$ (.b. has no explicit algebraic equivalent, and serves as the geometric unit measure).

We can already see the multiple vision, a geometrico-algebraic hybrid, required to understand the relations between lines in this diagram. Now let's see how things are put together. Recall that the problem requires that the ratio .d.:Sum(.d., .e. plus .f., .g. plus .h.) equal the known ratio .c.:.G.C.. In problem 92 of Book III The ratio .c.:.G.C. is modelled as 20 : 300, and the ratio .d.:Sum(.d., .e. plus .f., .g. plus .h.) as $\overset{1}{\smile}$: 2 $\overset{2}{\smile}$ +7 $\overset{1}{\smile}$ +4. This translates to an equality between Rect(.d., .C.G.) and Rect(.c., Sum(.d., .e. plus .f., .g. plus .h.)). The former rectangle is the bottom rectangle of the diagram, where .G.y. equals the unknown .d.. The latter rectangle is the middle one in the diagram, where .o.i. equals the given .c., .o.q. equals the *censi* .g., .q.s. equals the *cose* .d., .e. and .h., and .s.t. equals the known .f.. In the algebraic version of Book III the equation of the rectangles is mirrored by the cross multiplication yielding the equation $40 \overset{2}{\smile} +140 \overset{1}{\smile} +80 = 300 \overset{1}{\smile}$.

Now the task is to compare the two rectangles, balancing at the same time their areas and scales. Bombelli's trick here is elementary as it is ingeniously elegant. Both rectangles are rescaled according to the known ratio between .b. and .c.. But the middle rectangle is rescaled along the known vertical dimension, reducing its height from .o.i. to .o.n., whereas the lower rectangle is rescaled along the known horizontal dimension, reducing its length from .G.C. to .G.D.. Now both rectangles have the same height in terms of length (but not in terms of scale). The middle rectangle is divided into *censi*, *cose* and known areas, whereas the bottom is strictly *cose*. Note that this normalisation move does not have an algebraic equivalent in the solution of the algebraic version in Book III. As with the choice of .d. equal in length to .b., it is a geometric artefact of the geometrico-algebraic hybrid.

Since the middle rectangle includes the *cose* rectangle .q.s.l., which is of the same scale as the lower rectangle, this can be subtracted from both rectangles, leaving the rectangles .o.q.n. and .s.t.x. in the middle and .E.G.y. below. This manoeuvre reflects the subtraction of *cose* terms from both sides of the algebraic equation to obtain an equation of the form *censi* and numbers equal *cose*. The resulting equation in problem 92 of Book III is $40 \overset{2}{\smile} +80 = 160 \overset{1}{\smile}$.

The last normalisation step is to reduce the *censi* to a single square. Algebraically this is the division of the quadratic equation by its leading coefficient 40. Geometrically this is the rescaling of all rectangles so that .o.q.n. becomes a square, that is, rescaling by the ratio .b.:.o.q., which, due to the representation of .b. as equal to .d. and the equality between .o.q. and twice .d., is exactly $\frac{1}{2}$. Note that the extra geometric rescaling and the choice of .b. for the role of geometric unit measure led to a difference between the geometric and algebraic rescaling: in the algebraic model we rescale once by a factor of $\frac{1}{40}$, whereas in the geometric model we first rescaled along one dimension by the ratio of .b.:.c. ($\frac{1}{20}$) and then along the other dimension by the ratio of .b.:.d. ($\frac{1}{2}$). The resulting equation is $1 \overset{2}{\smile} +2 = 4 \overset{1}{\smile}$

We are finally in a situation where the single *censo* .n.p.o. plus the known .s.l.u. equal the *cose* .F.G.y.. This is a normalised equation, and can be treated geometrically as in the left hand side of figure 4 above. But Bombelli skips directly to the geometric instantiation of the solution rule as in the right hand side of figure 4. He draws the known .ψ.y., half the coefficient of the *cose* (2 in the algebraic equation), finds u.x., the root of the known rectangle .s.l.u. ($\sqrt{2}$ in the algebraic equation), and then copies it as .ψ.z. to form the right angle triangle .y.ψ.z.. Now .y.z. is the root of the difference between the squares of .ψ.z. and .ψ.y. ($\sqrt{2^2 - 2}$), which is copied as .y.m.. Finally this root is added to the known .ψ.y.. The end result is .ψ.m. ($2 + \sqrt{2^2 - 2}$), the sought value of the original unknown .d..

It is, however, crucial to note that despite the analogies with the algebraic model, the geometric treatment includes no assignment of arithmetic values to line lengths. The solution is algebraico-geometric, but strictly non-arithmetic.

The point of pursuing this last example in such detail is of course to bring out the multiple vision of lines according to their lengths and scales that's required to pursue Bombelli's diagrams correctly on the arithmetic, algebraic and geometric levels. I made a point of highlighting the shifts that algebraic practice undergoes when Bombelli translates it into a geometrico-algebraic hybrid. It is this multiple, co-expressive, but not entirely congruent vision of several mathematical strata, which renders Bombelli such an insightful mathematician.

5. Conclusion

Together with a recognition of the hybridisation of algebraic and geometric points of view in the above diagrams, it is important to appreciate the multiplicity and deferral inherent in each of the two perspectives of this double vision. In subsection 4.3 we saw that a segment/number may not stand for its length/value, but as a place holder for another; in subsection 4.4 we saw that a line may carry a scale or dimension that turns its apparent relation to other lines problematic, but not for all that meaningless (recall that scale-independent equality of lengths and areas did play an important role in the last solution above); in subsection 2.2 we saw that a line can be seen a an implicit rectangle whose other side is a unit measure, and that the geometric unit role can be played by any line, but this is no longer the case where arithmetic values are imposed on lines. All this is compounded by the *sine qua non* of algebra: that the unknown stand by definition for a deferred value.

Interesting as they are, Bombelli's inventive geometric representations are a minor and isolated strategy. I do not mean to say that they have never been repeated by later mathematicians; I mean that they have been used locally, sketchily, heuristically and without a rigorous articulation. The canonical co-representation of algebra and geometry is the one emanating from Descartes: instead of Bombelli's single unknown line that can take the place of various different values and be interpreted as belonging to various different scales, Descartes' successors draw all possible lines together, so to speak side by side. To put this last claim more clearly, note that in Bombelli's representation each side of the equation $40 \overset{2}{\smile} +140 \overset{1}{\smile} +80 = 300 \overset{1}{\smile}$ is represented by two rectangles, whose scales are yet to be determined, and therefore stand for a range of possible values. In the post-Cartesian representation, however, each of the vertical lines that cover the space between the curve $y = 40x^2 + 140x + 80$ and the x-axis captures one possible value of the term $40x^2 + 140x + 80$, each of the vertical lines that cover the space between the curve $y = 300x$ and the x-axis captures one possible value of the term $300x$, and the intersection of the curves captures the one value of x for which the former is equal to the latter. Instead of representing arbitrarily the desired length and deferring the determination of its scale, we post-Cartesians put the infinity of possible representations side by side, and extract the one that's required. With the post-Cartesian representation, multiple vision is disambiguated. Instead of seeing one thing as many, we review all possibilities at a glance.

The transition from Bombelli's representation technique to the post-Cartesian one, which dominated early modern mathematics, can perhaps be situated in a more general context: the production of modern techniques for exhaustive representation and observation rather than a representation through a token instance.[56] But the view of many underdetermined things under one sign is not for all that abandoned. When one extends one's scope, one can recognise that the pair of post-Cartesian curves above may contract the representation of many different empirical phenomena; that it may serve as a token for a general technique rather than a specific example; that it may be read as hiding several compressed dimensions (the x can always be reinterpreted, for example, as a projection of two independent variables onto their sum). The story of mathematical representation is a story played with hybrids, contractions and disambiguations. The range of hybridisations and contractions is never confined or charted in advance, and the disambiguation is never entirely exhaustive.

The transition from Bombelli's *algebra linearia* to post-Cartesian analytic geometry is also a process of normalising the relations between geometry and algebra. Whereas Bombelli simultaneously built on several traditions of representations (Greek, Arabic, Latin and abbacist), and observed relations of justification, instantiation, translation and accompaniment (which were

[56] I'm referring here to recent work by Lorraine Daston presented in her talk at the History of Science Society meetings in Pittsburgh in 2008, but the connection I'm making is yet to be historically validated.

mixed far more casually than this crude division suggests), the following centuries tended toward a more foundational approach. Rigorous practices were developed that tried to impose a single, well regimented set of relations between algebra and geometry. Not that foundational attempts ever suppressed the practice of plural relations between algebras and geometries, but they often suppressed an explicit account of these multiple relations, as well as the idiosyncratic residues that the isomorphisms constructed to reduce one to the other kept leaving behind.

Why is this important for historians of mathematics? When 19^{th} and most 20^{th} century historians considered Greek mathematics, all they could see was geometric algebra — geometry was conceived of as a technique for representing (perhaps even concealing) algebraic knowledge. This view was so thoroughly entrenched that historians, who reacted against it, had to dislodge not only the claim that classical Greek geometry was a coded algebra, but also the weaker claim that Greek geometry contained traces of algebraic thinking. Some of the arguments used to establish such claims show that if one allows some traces of algebra into our interpretation of Greek geometry, one will end up algebraising one's interpretation to an extent that effectively recreates geometric algebra.[57] But historians should be careful of extending such arguments beyond their intended scope. Outside the classical Greek framework (and occasionally at its boundaries) geometry can be algebraic to various different extents without necessarily recreating geometric algebra. One does not simply have two possibilities: pure geometry and geometric algebra. As Bombelli shows, there's very much between these options and beyond.

Bombelli's case is a fine demonstration of the fact that one can rigorously handle various different ways of relating algebra and geometry without giving up on the specificities of either. Geometry and algebra are conceived as co-expressive; what the one can show the other should be able to say and vice-versa. But this co-expressivity does not turn into a hierarchy, a reduction or a complete isomorphism. Each algebraic sign and each geometric line stands for more than a single well regimented value chosen from among a confined space of interchangeable choices. The techniques of translation are many, and leave idiosyncracies behind. They require specialised multiple vision and lead to hybrid and expanded algebraic and geometric practices. But, most importantly, these multiplicities are not a problem — they are, precisely, (what enables finding) solutions.

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[57]I'm referring, of course, to the ground breaking work of Sabetai Unguru and his colleagues, e.g. Unguru & Rowe (1981–1982), which went as far as asserting the *death* of geometric algebra in Hebrew, Greek and Latin at the end of their paper.

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