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ON LARGE DEVIATIONS AND DISCONNECTION FOR RANDOM WALK AND RANDOM INTERLACEMENTS

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«Lorsqu'un regard s'élance: quel vol
par ces distances pures;
il faut la voix du rossignol
pour en prendre mesure.»

René Maria Rilke, *Les Quatrains Valaisans.*

To my parents.

Abstract

This thesis investigates various percolation problems for certain systems with strong correlations. More precisely, it is mainly concerned with two models: *random interlacements*, a model introduced by A.-S. Sznitman in [56], which exhibits a non-trivial phase transition as the level parameter of the model varies; and *the trace of simple random walk* on $\mathbb{Z}^d$, when $d \geq 3$.

In the first part, we derive a large deviation principle for the density profiles of occupation times of random interlacements at a fixed level in a large box of $\mathbb{Z}^d$, $d \geq 3$. As an application, we analyse the asymptotic behaviour of the probability that atypically high values of the density profile insulate a macroscopic body in a large box.

In the second part, we give a partial answer to a very interesting question: what is the (asymptotic) probability that a macroscopic body in $\mathbb{Z}^d$, $d \geq 3$, gets disconnected from infinity by the random interlacements (when the level is low enough) or by the trace of a single random walk? We derive asymptotic lower bounds on the probability for both problems. The proofs involve changes of measures which bring into play "tilted" random walks and "tilted" interlacements. These results are complemented by upper bounds proved in [62] in a similar set-up which are all conjectured to be tight, and are intimately connected with results concerning the "insulation" events in the first part.

Zusammenfassung

Diese Dissertation ist der Untersuchung und Klärung verschiedener Fragen zur Perkolation in bestimmten Systemen mit starken Korrelationen gewidmet. Genauer gesagt, geht es hauptsächlich um folgende zwei Modelle: *Zufällige Verflechtungen*, einem von A.-S. Sznitman in [56] eingeführten Modell, in dem sich bei Variierung des modellbestimmenden Niveauparameters ein nicht-trivialer Phasenübergang zeigt; und der *Spur der einfachen Irrfahrt* auf $\mathbb{Z}^d$ mit $d \geq 3$.

Im ersten Teil folgt die Herleitung eines Prinzips der grossen Abweichungen für das Dichteprofil der Besetzungszeiten zufälliger Verflechtungen auf fixiertem Niveau in einem grossen Würfel von $\mathbb{Z}^d$ mit $d \geq 3$. Als Anwendung ergibt sich eine Aussage zum asymptotischen Verhalten der Wahrscheinlichkeit, dass atypisch hohe Werte des Dichteprofils einen makroskopischen Körper in einem grossen Würfel abschirmen.

Im zweiten Teil wird eine partielle Antwort auf eine interessante Frage gegeben: Was ist die asymptotische Wahrscheinlichkeit, dass in $\mathbb{Z}^d$ mit $d \geq 3$ ein makroskopischer Körper vom Unendlichen abgetrennt wird, sei es durch zufällige Verflechtungen (auf genügend tiefem Niveau) oder durch die Spur einer einfachen Irrfahrt? Als Teilantwort werden untere Schranken für die Wahrscheinlichkeiten in beiden Fällen geliefert. Die Beweise beinhalten Masswechsel, die «geneigte» Irrfahrten und «geneigte» Verflechtungen ins Spiel bringen. Die hergeleiteten Resultate werden ergänzt durch die oberen Schranken, die in [62] in einem ähnlichen Rahmen bewiesen wurden. Diese Schranken, es wird vermutet, dass sie optimal sind, stehen in einem engen Zusammenhang mit Resultaten betreffend «Abschirmungsereignissen», wie sie im ersten Teil betrachtet werden.

Résumé

Cette thèse étudie divers problèmes de percolation pour certains systèmes à fortes corrélations. Plus précisément, elle porte principalement sur deux modèles : *les entrelacs aléatoires*, un modèle introduit par A.-S. Sznitman dans [56], qui présente une transition de phase non-triviale lorsque varie le paramètre de niveau du modèle; et *la trace de la marche aléatoire simple* sur $\mathbb{Z}^d$, lorsque $d \geq 3$.

Dans la première partie, nous démontrons un principe de grandes déviations pour les profils de densité des temps d'occupation des entrelacs aléatoires à un niveau fixe dans une grande boîte de $\mathbb{Z}^d$, $d \geq 3$. Comme application, nous analysons le comportement asymptotique de la probabilité que des valeurs anormalement élevées du profil de densité isolent un corps macroscopique dans une grande boîte.

Dans la deuxième partie, nous répondons partiellement à une question très intéressante : quelle est la probabilité (asymptotique) qu'un corps macroscopique dans $\mathbb{Z}^d$, $d \geq 3$, soit déconnecté de l'infini par les entrelacs aléatoires (quand le niveau est assez bas) ou par la trace d'une marche aléatoire simple ? Nous obtenons des bornes inférieures asymptotiques sur la probabilité pour les deux problèmes. Les démonstrations utilisent des changements de mesures, mettant en jeu des marches aléatoires « inclinées » et des entrelacs « inclinés ». Ces résultats sont complétés par des bornes supérieures démontrées dans [62] sous une configuration similaire qui sont tout conjecturées d'être optimales, et sont intimement liées avec des résultats concernant les événements d'« isolation » dans la première partie.

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X. L., Zürich, April 2016

Contents

Chapter 0

Introduction

0.1 Basics

This chapter aims at giving a brief introduction to the mathematical research I have conducted during my PhD studies. In its broadest sense, most of my research in this period has been motivated by the study of certain **critical phenomena** and **phase transitions** in models which lie at the intersection between probability theory and statistical physics.

More precisely, in my doctoral work I have been interested in **random walks** on graphs and **random interlacements**. These two models are instances of **percolation** models with long-range correlations which are closely related to the behaviour of simple random walks (**SRW**) on the Euclidean lattice or Brownian motion in Euclidean space. In the subsequent subsections I will briefly introduce these models respectively at the general mathematics level, describe the problem that I have worked on and sketch the idea of proof without touching too much of its details.

0.1.1 Percolation

In everyday context, *percolation* refers to the diffusion and filtering of fluids through porous materials: for example, to make coffee from a Moka pot, the water would need to “percolate” through grounded coffee beans. When it comes to mathematics this notion refers to the question whether a random set contains a connected component of infinite size. It has been a central subject in statistical physics for more than half a century. One primary reason is that many percolation models exhibit rich behaviour despite their relatively simple definition. Hence, statistical physicists and mathematicians hope to gain more knowledge on critical phenomena in the study of percolations.

The most basic example is Bernoulli (site) percolation on the $\mathbb{Z}^d$-lattice, in which each vertex is marked “open” (otherwise “closed”) independently with probability p, for some fixed $p \in [0,1]$. We denote by P_p the respective probability law on the configuration space $\{\text{open, closed}\}^{\mathbb{Z}^d}$.

For a given configuration, open vertices form “islands”, which we refer to as “clusters”. More precisely, two open sites belong to the same cluster if and only if there exists a nearest-neighbour path, purely consisting of open vertices, connecting them. A classical result states that for every $d \geq 2$, there exists a critical $p_c \in (0,1)$ (depending on d) such that for $p < p_c$, there is almost surely no infinite cluster, and for $p > p_c$,there is almost surely an infinite cluster. Moreover, in the case $p > p_c$, the infinite cluster is unique (however this requires quite some extra effort to prove). This model hence exhibits, in the language of physicists, a *non-trivial phase transition* at p_c.

However, to deepen the understanding of this phase transition, one can, for instance, further ask the question “what is the probability that the origin is connected in an open cluster to the boundary of a box of size N”. In fact, this probability decays as fast as an exponential function as N tends to infinity:

$$P_p[\{0 \longleftrightarrow \partial B_\infty(0,N)\}] \sim \exp(-cN^{c'}) \text{ for all } p < p_c, \tag{0.1}$$

where c, c' are positive constants depending on d only. In constrast, when $p > p_c$, it is known that

$$P_p[\{0 \longleftrightarrow \partial B_\infty(0,N)\}] \geq P_p[\{0 \longleftrightarrow \infty\}] \geq c(p). \tag{0.2}$$

Hence we say that this model exhibits a *sharp* phase transition.

However, beyond these basic properties, even in this classical model presented above, there are many crucial questions which are sill open, let alone for more general percolation models. The monograph [33] contains a very thorough discussion on the history and progress on percolation.

0.1.2 Random Walk

Random walk on graphs is one of the first probability models studied by mathematicians: as early as 1656 Blaise Pascal and Pierre Fermat already made discussions about the Gambler's Ruin problem (see [26]), which can be rephrased as a problem in random walks.

In modern mathematical language, the model of random walk is defined as follows: Let $G = (V, E)$ be an undirected graph, where V stands for the set of vertices and E for the set of edges. One should think of V as the collection of possible states of a system, thus the presence of an edge $e = (v, u)$ in E, where $u, v \in V$, indicates that it is permissible for the system to transit between states v and u. The system starts at a fixed vertex $x \in V$, and in the next step it moves to one of its neighbours uniformly chosen at random among all of them. This procedure is repeated indefinitely many times. We hence call this process a random walk and denote it by $(X_n)_{n\geq 0}$, where X_n stands for the state of the system at step n. We denote the law of the random walk started at x by P_x.

One very important example of this mode is the *simple random walk* on $\mathbb{Z}^d$, where (in the notation above) $V = \mathbb{Z}^d$ and E consists of all pairs of vertices of Euclidean distance 1. In this model, at each step, the random walk jumps to one of its neighbours with probability $(2d)^{-1}$. It is a classical result (see Theorem 4.1.1, p. 78 of [39]) that with probability one the random walk will return to where it starts if $d = 1$ or 2. In this case we call the random walk *recurrent*. However, when $d \geq 3$, there is a positive chance that the random walk will never return to the starting point. Moreover, with probability one, it will eventually "escape to infinity". In this case, we call the random walk *transient*.

A large class of random walk models falls into the frame of (reversible) *weighted random walk*, whose setup is slightly more complicated compared to the simple random walk. Simply put, one can construct such a model by giving weights to the edges and assigning the probabilities of jumping between the vertices accordingly. More precisely, we assign for each edge $(u, v) \in E$ a positive weight $\mu_{uv} = \mu_{vu}$ and denote by $\mu_u = \sum_{v \sim u} \mu_{uv}$ (i.e. the sum runs over all neighbouring vertices of u) the corresponding weight of this vertex. For all $u \in V$ we let the probability for the random walk to jump from u to v be μ_{uv}/μ_u, a quantity we refer to as the *transition probability*. A lot of information can be obtained from the setup with relatively little effort, such as the stationary/reversibility measure, which tell us how the equilibrium state of this random walk look like.

In the scope of my research, it is easier to consider the random walk in continuous time. This can be defined for any weighted random walk but for simplicity we restrict ourselves to the case of the simple random walk on $\mathbb{Z}^d$. For this purpose we imagine that at each lattice point u there is an alarm clock, which, after being set to work, rings at a random time with exponential distribution of parameter 1. Each time our random walk moves into one vertex we set the alarm clock to work, and when it rings, we let the random walk jump to one of its neighbours with probability $1/2d$ uniformly. When no ambiguity arises, we also denote by P_x the law of this weighted random walk started at x. From now on, whenever simple random walk appears, we actually mean simple random walk in continuous time. It is worth mentioning that one can easily define random walks with non homogeneous jumping time using this framework.

Another notion in random walks that plays an important role in my research especially in the study of random interlacements, which is to come up in the next subsection, is the capacity. Although it can be defined for any reversible transient Markov chain, again for simplicity we restrict ourselves to the discussion of simple random walk on $\mathbb{Z}^d$, $d \geq 3$, where the random walk is indeed transient. For K, a finite subset of $\mathbb{Z}^d$, we define the equilibrium measure for $x \in \partial_i K$ through the escape probability, where $\partial_i K$ stands for the inner boundary of K:

$$e_K(x) = P_x[X_t \text{ leaves } K \text{ at the first move and never returns}] \tag{0.3}$$

and consequently define the capacity of K as the total equilibrium measure of K

$$\text{cap}(K) = \sum_{x \in \partial_i K} e_K(x). \tag{0.4}$$

Let us end this subsection with this example illustrating the intimate relationship of the capacity with the study of random walks: when $d \geq 3$,

$$P_x[(X_t)_{t\geq 0} \text{ hits } K \text{ in finite time}] \sim \frac{c \cdot \text{cap}(K)}{|x|^{d-2}} \text{ as } |x| \to \infty. \tag{0.5}$$

0.1.3 Random interlacements

Random interlacements are an important model of percolation with long-range dependence (cf. the Bernoulli percolation where the status of each vertex is independent of those of other vertices). This model was recently introduced by Sznitman in [56] and has been intensively studied since then. Intuitively speaking, they consist of a random collection of trajectories on $\mathbb{Z}^d$, whose trace forms a random set, which visually resembles an interlacing fabric. A fundamental question is, under different densities, whether this "fabric" is "rainproof", or in the language of percolation, whether the complement of this set *percolates*.

Two major reasons make this model especially attractive to probabilists: the long-range dependence in this model which is more realistic for percolation and its connections to basic probabilistic models, such as Gaussian free field, fragmentation by random walk, etc. For example, in [6] and [65] it is employed to describes the local structure of a random walk on a large torus after having run up to a time proportional to the torus volume.

Now let us introduce this model in a slightly more formal manner. We consider the collection of random doubly-infinite nearest-neighbor trajectories on $\mathbb{Z}^d$, for $d \geq 3$, associated with a Poisson point process (with a suitable intensity measure). We denote by $\mathbb{P}$ the respective probability measure. A non-negative parameter u enters as a multiplicative factor in the intensity measure. The interlacements at level u, denoted $\mathcal{I}^u$, are then defined to be the union of traces of all paths which appear in the Poisson point process.

More often we wish to look at the local behaviours of random interlacements through a finite "window" $K \subset\subset \mathbb{Z}^d$. More precisely the law of $\mathcal{I} = \mathcal{I}^u \cap K$ can be described as follows: independently from each vertex x in $\partial_i K$ we start $\mathrm{Pois}(u \cdot e_K(x))$ independent random walks. Then $\mathcal{I}$ has the same law as record the union of the intersection with K of these traces (in fact by (0.4) and the property of Poisson distribution there are $\mathrm{Pois}(u \cdot \mathrm{cap}(K))$ trajectories in total).

The primary difficulty in the investigation of percolation problems on this model lies in the strong long-distance correlation (in contrast to the Bernoulli percolation introduced earlier this section where the status of each vertex is independent), which renders many classical methods used in the study of Bernoulli percolation invalid. However many powerful methods have been developed to overcome such difficulties, and fundamental results have been proved.

For any $u > 0$, $\mathcal{I}^u$ is known to be a translation invariant, ergodic and almost surely connected subset of $\mathbb{Z}^d$. However, its complement $\mathcal{V}^u = \mathbb{Z}^d \backslash \mathcal{I}^u$, known as the *vacant set*, undergoes a non-trivial percolative phase transition, in all dimensions $d \geq 3$, see [53, 56]. More precisely, there exists a parameter $u_* \in (0, \infty)$ such that $\mathcal{V}^u$ contains a unique (see [63]) infinite connected component almost surely for $u < u_*$ but contains almost surely only clusters of finite size for $u > u_*$.

In [53, 23] it was shown that an additional important critical value $u_{**} \in (0, \infty)$ exists. Informally, it is defined as the infimum over $u > 0$ for which the probability of the vacant cluster at the origin being large decays exponentially fast: when and only when $u > u_{**}$,

$$\mathbb{P}[\{0 \overset{\mathcal{V}^u}{\longleftrightarrow} \partial_i B_\infty(0, N)\}] \sim \exp(-cN^{c'}), \tag{0.6}$$

where $c, c' > 0$ depend on d and u only. It follows from the definition of u_{**} that $u_* \leq u_{**}$, however, it is an important open question whether u_{**} and u_* coincide, or equivalently, whether the phase transition is *sharp*.

For a detailed introduction to the model, see the monographs [13] and [23].

0.1.4 A very brief synopsis of my work

My works on the models mentioned above can be summarized as follows.

(i) **Large deviations for occupation time profiles of random interlacements.** A problem of particular importance in the study of random interlacements is large deviations of the density profile for the occupation time. This density profile turns out to be the right quantity linking random interlacements and Brownian interlacements. In Chapter 1, we prove a large deviation principle and analyzes the asymptotic behavior of the probability that atypically high values of the density profile insulate a macroscopic body in a large box.

(ii) **Disconnection by random interlacements and trace of SRW.** In Chapters 2 and 3, inspired by study of the "insulation" events in Chapter 1, we give lower bounds on the asymptotic probability that the trace of random interlacements (in [42]) or the trace of simple random walk on $\mathbb{Z}^d$ (in [43]) disconnects a macroscopic body. The proofs involve an interplay between the change-of-measure method and the analysis of "tilted" versions of random walks and random interlacements (which is defined through tilted random walks).

0.2 Large deviations for the density profile of occupation-measure of random interlacements

0.2.1 Introduction to the problem

Let us consider random interlacements on $\mathbb{Z}^d$, for $d \geq 3$.

Fix $u > 0$. The field of occupation times of random interlacements at level u is defined by the family of random variables $\{L_{x,u}\}_{x\in\mathbb{Z}^d}$, where $L_{x,u}$ is total duration spent at x by the trajectories of the random interlacements (see [57]). The occupation time field is an important object in the study of random interlacements. In particular, an isomorphism theorem relates it to the discrete Gaussian free field (see [59]).

Given a *closed box* $B \subseteq \mathbb{R}^d$, we are interested in the density profile of the occupation times at level u in the rescaled discrete box $(NB) \cap \mathbb{Z}^d := \{Nx \ : \ x \in B\} \cap \mathbb{Z}^d$, given for $N \geq 1$ by

$$\rho_{N,u} = \frac{1}{N^d} \sum_{x\in(NB)\cap\mathbb{Z}^d} L_{x,u}\, \delta_{\frac{x}{N}}. \tag{0.7}$$

We view $\rho_{N,u}$ as a random element of $M_+(B)$, the set of positive measures on B. As a consequence of the ergodic theorem, almost surely $\rho_{N,u}$ converges weakly to um_B, where m_B stands for the restriction of the Lebesgue measure to B.

0.2.2 Results

In Chapter 1, we establish a large deviation principle (**LDP**) for $\rho_{N,u}$, which shows that for every fixed $u > 0$,

(0.8) $\rho_{N,u}$ satisfies an LDP on $M_+(B)$ at speed N^{d-2} with an explicit, convex, good rate function $\frac{1}{d}I_u$.

We also derive an LDP for the density profile of the occupation-time measure of Brownian interlacements in all dimensions $d \geq 3$

as both an intermediate step in and an important byproduct of the above LDP. This is done by exploiting the scaling property of the occupation times of Brownian interlacements (we refer the readers to [60] for an introduction of Brownian interlacements). Instead of $\rho_{N,u}$ we consider the random measure $\nu_{L,\alpha}$ on B, such that for any bounded measurable f on B, the integral $\langle \nu_{L,\alpha}, f\rangle$ equals

$$\big\langle \nu_{L,\alpha}, f \big\rangle = \frac{1}{L^d} \int_{LB} f\Big(\frac{y}{L}\Big)\, \mathcal{L}_\alpha(dy), \text{ for } L \geq 1, \alpha \geq 0. \tag{0.9}$$

We show in Corollary 1.3.3 that for $\alpha > 0$,

(0.10) as $L \to \infty$, the laws of $\nu_{L,\alpha}$ on $M_+(B)$ satisfy a large deviation principle at speed L^{d-2} with the convex good rate function I_α (see (1.3)).

As an application of the above LDP (0.8), we analyse the asymptotic behaviour of the probability that high values of the (smoothed-out) density profile (i.e. "level sets") insulate a large macroscopic body. Given $\delta > 0$ a sub-box $B_0 \subset B$ whose distance from B^c is at least δ, a compact subset $K \subset B_0$ and a positive number a, we study the probability that a regularised version of the density profiles above level a "disconnects K from ∂B_0". We denote by $\mathcal{D}_{a,\delta}$ the collection of such profiles and show that for $a > u$,

(0.11)

$$\text{i)} \quad \limsup_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}[\rho_{N,u} \in \mathcal{D}_{a,\delta}] \leq -\tfrac{1}{d}\big(\sqrt{a} - \sqrt{u}\big)^2 \mathrm{cap}_{\mathbb{R}^d}(K),$$

$$\text{ii)} \quad \liminf_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}[\rho_{N,u} \in \mathcal{D}_{a,\delta}] \geq -\tfrac{1}{d}\big(\sqrt{a} - \sqrt{u}\big)^2 \mathrm{cap}_{\mathbb{R}^d}(K^\delta),$$

where $\mathrm{cap}_{\mathbb{R}^d}(\cdot)$ stands for the **Brownian capacity** (a quantity from the potential theory of Brownian motions) and K^δ for the closed δ-neighbourhood of K. Interestingly, this LDP has some similar flavour to the results of the same type with respect to the Gaussian free field (see [11]), although their proofs are quite different.

0.3 Disconnection problem by random interlacements and simple random walk

0.3.1 Introduction

In this project the question of asymptotic bounds on the probability that a macroscopic body gets disconnected by a random subset of $\mathbb{Z}^d$, $d \geq 3$ is studied. More precisely, we are interested in random subsets which exhibit long-range dependence, such as the trace of random interlacements in the percolative regime (of the vacant set) $0 < u \leq u_{**}$, or the trace of simple random walk.

Consider first the case of random interlacements. Let $K \subset \mathbb{R}^d$ be a compact subset and define its discrete blow-up by $K_N = \{x \in \mathbb{Z}^d;\ \exists z \in K \text{ s.t. } \|x - zN\|_\infty \leq 1\}$. Then, we wish to study the probability of the event

$$A_N = \{K_N \overset{\mathcal{V}_u}{\nleftrightarrow} \infty\} \equiv \left\{ \begin{array}{l} K_N \text{ is not connected to infinity by a} \\ \text{nearest-neighbor path in } \mathcal{V}^u \end{array} \right\}.$$

It is worth noting that this problem is not only interesting *per se*, but also shares some intimate connection to the type of insulation events studied in Chapter 1 (see (0.11) and above) and enables us to understand the power of the large deviation principle of occupation-time profile (see (0.8) obtained therein.

Similarly, in the case of simple random walk started at the origin, let $\mathcal{V}$ denote the set of points in $\mathbb{Z}^d$ which are not in its trace. Then, we define $\overline{A}_N = \{K_N \overset{\mathcal{V}}{\nleftrightarrow} \infty\}$, and wish to understand its probability.

0.3.2 Disconnection by random interlacements

In Chapter 2 we derive an asymptotic lower bound for the probability of A_N. More precisely, for $u \in (0, u_{**}]$ we have

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}} \log\big(\mathbb{P}[A_N]\big) \geq -\frac{1}{d}\big(\sqrt{u_{**}} - \sqrt{u}\big)^2 \mathrm{cap}_{\mathbb{R}^d}(K). \tag{0.12}$$

The main strategy in proving (0.12) is to combine a change of measure argument together with an entropic bound. For large N, we construct a new probability measure $\widetilde{\mathbb{P}}_N$, corresponding to "**tilted random interlacements**", which look like a collection of two-sided paths of space-inhomogeneous Markov chain (instead of simple random walk in the original version of random interlacements).

Intuitively speaking, the measure $\widetilde{\mathbb{P}}_N$ forces a "local level" of interlacements corresponding to $u_{**} + \epsilon$ to generate a strongly non-percolative "fence" surrounding K_N. This yields

$$\lim_{N\to\infty} \widetilde{\mathbb{P}}_N(A_N) = 1. \tag{0.13}$$

The rigorous proof of the last fact is obtained by a local comparison at a mesoscopic scale between the occupied set of tilted interlacements and standard interlacements at a level exceeding u_{**}. This is achieved through an analysis on the capacity and equilibrium measure associated with the tilted interlacements.

In the meantime, we keep the change of measure at a "low entropic cost"; that is, we show that the entropy of $\widetilde{\mathbb{P}}_N$ with respect to $\mathbb{P}_u$, denoted by $H(\widetilde{\mathbb{P}}_N|\mathbb{P}_u)$ satisfies a relation of the type

$$H(\widetilde{\mathbb{P}}_N|\mathbb{P}_u) \lessapprox \frac{u_{**}}{d}\mathrm{cap}_{\mathbb{R}^d}(K)N^{d-2}. \tag{0.14}$$

The combination of both properties of $\widetilde{\mathbb{P}}_N$ then gives the result by applying a classical inequality relating probability and entropy.

The recent article [62] studies the question of upper bounds for the probability of an event similar to A_N and shows that there exists some $\overline{u} > 0$ such that for any $M > 1$ and every $0 < u < \overline{u}$

$$\begin{aligned}&\limsup_{N\to\infty} \frac{1}{N^{d-2}} \log\Big(\mathbb{P}\big[B_N \overset{\mathcal{V}^u}{\nleftrightarrow} (B_{MN})^c\big]\Big)\\ &\leq -\frac{1}{d}\big(\sqrt{\overline{u}} - \sqrt{u}\big)^2 \mathrm{cap}_{\mathbb{R}^d}([-1,1]^d),\end{aligned} \tag{0.15}$$

where $B_N = [-n,n]^d \cap \mathbb{Z}^d$.

It was shown in [62] that $0 < \overline{u} \leq u_*(\leq u_{**} < \infty)$. It is plausible, but unproven at the moment, that actually $\overline{u} = u_* = u_{**}$. If this is indeed the case, the asymptotic lower bound (0.12) obtained in [42] matches the asymptotic upper bound (0.15) obtained in [62], i.e.,

$$\lim_{N\to\infty} \frac{1}{N^{d-2}} \log(\mathbb{P}[A_N]) = -\frac{1}{d}\big(\sqrt{u_*} - \sqrt{u}\big)^2 \mathrm{cap}_{\mathbb{R}^d}(K), \tag{0.16}$$

which implies that the result is tight. Consequently, the comparison between (0.11), (0.12) and (0.15) suggests that the large deviations of the occupation-time profile do capture the main mechanism through which $\mathcal{I}^u$ disconnects a macroscopic body from infinity.

Interestingly, the tilted interlacements, the protagonist of this work, might indeed offer a microscopic model (in the discrete set-up), for the type of "Swiss cheese" picture proposed in [9], which is a study of the moderate deviations of the volume of the Wiener sausage (although the relevant modulating functions in [3] and in this work correspond to distinct variational problems and are different).

0.3.3 Disconnection by simple random walk

In Chapter 3, an asymptotic lower bound for the probability of $\overline{A}_N$ is derived. More precisely, it is shown that

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}} \log\big(P_0[\overline{A}_N]\big) \geq -\frac{u_{**}}{d} \mathrm{cap}_{\mathbb{R}^d}(K), \tag{0.17}$$

where P_0 is the probability measure of a simple random walk started from the origin.

Remarkably, this problem can be formally regarded as a limiting case of disconnection by random interlacements, when $u \to 0$. Moreover, the proof of (0.17) brings into play random interlacements as well as a suitable strategy to implement disconnection. As in the proof of (0.12) the proof uses the change of measure technique combined with an entropy bound. In this case, the new measure $\widetilde{P}_N$ is defined via a fine-tuning of Radon-Nikodym derivative, which yields a "**tilted walk**". In essence, this walk evolves in the first stage as a finite-range Markov chain with a certain generator up to a deterministic time, and then continues as a simple random walk in the second stage. A specific choice for the deterministic time enables to achieve entropy-efficiently the effect that the tilted walk ;eaves a trace close to K_N which behaves locally like the trace of random interlacements with a level slightly larger than u_{**}. This creates a "fence", separating K_N from infinity.

The main obstruction in the proof is that unlike the model studied in Chapter 2, we only have a single trajectory at our disposal. In addition, the state space and the generator of the tilted walk in the first stage are both highly space non-homogeneous. This makes the extraction of the necessary independence much more delicate in comparison with the case of random interlacements, requiring a fair amount of accurate understanding on the behavior of the tilted walk.

Similar to the case of random interlacements, see the discussion around (0.15), in [62] an asymptotic upper bound involving $\overline{u}$ for events of this type was derived. In addition, if $\overline{u} = u_* = u_{**}$, then the lower and upper bounds for simple random walk are tight. Furthermore, in this case, the constant u_* stemming from the study of interlacement percolation becomes in fact a key characteristic quantity for the description of the disconnection properties of the simple random walk *itself*.

In the case of (above) (0.17), one can also wonder whether one actually has the following asymptotics (possibly with some regularity assumption on K)

$$\lim_{N\to\infty} \frac{1}{N^{d-2}} \log(P_0[\overline{A}_N]) = -\frac{u_*}{d} \mathrm{cap}_{\mathbb{R}^d}(K). \tag{0.18}$$

0.4 Outlook

0.4.1 Brownian interlacements

The model of Brownian interlacements is the continuous counterpart of random interlacements. Recently introduced by Sznitman in [60], it is defined as the Poissonian cloud of doubly-infinite continuous Brownian trajectories in the d-dimensional Euclidean space, $d \geq 3$. Brownian interlacements bear similar properties, for instance long-range dependence, to random interlacements. This model plays a crucial role in both the study of the asymptotic behaviors of various aspects of random interlacements, and the interconnection of random interlacements, loop soups and Gaussian free fields. For $r > 0$, we observe the union of points whose distance from the trajectories are at most r and study its geometry as well as that of its complement. It is expected that these sets would exhibit similar percolative properties and plan to study them. In particular, one can hope that the study of this model will shed some light on other models of continuous percolation.

0.4.2 Random interlacements in Two dimensions

Although there is no direct analogue of random interlacements in $d = 2$, one can consider models with similar features. In [18] a possible way of constructing random in two dimensions was proposed. It would be interesting to investigate the problem of percolation phase transition in this case. One may also naturally wonder whether such models and results concerning random interlacements translate into some new (and possibly insightful) concepts related to two-dimensional random geometry.

0.4.3 Branching interlacements

It is a very natural question whether it is possible to define a random spatial process which combines branching random walks (or branching Brownian motion) with random interlacements (respectively Brownian interlacements). For such models one can ask many question such as percolation properties of the model, existence of shape theorems and large deviation results for disconnection. In addition, such a model can create a connection between fragmentation process and branching random walk on torus, similar to [6, 7, 64]. Currently in this direction there is an ongoing work, see [4].

Chapter 1

Large deviations for occupation time profiles of random interlacements

We derive a large deviation principle for the density profile of occupation times of random interlacements at a fixed level in a large box of $\mathbb{Z}^d$, $d \geq 3$. As an application, we analyze the asymptotic behavior of the probability that atypically high values of the density profile insulate a macroscopic body in a large box. As a step in this program, we obtain a similar large deviation principle for the occupation-time measure of Brownian interlacements at a fixed level in a large box of $\mathbb{R}^d$, and we derive a new identity for the Laplace transform of the occupation-time measure, which is based on the analysis of certain Schrödinger semi-groups.

1.0 Introduction

Random interlacements have been instrumental in the analysis of various questions concerning the disconnection or the fragmentation created by random walk trajectories, see [14], [55], [64]. The existence of a non-trivial phase transition for the percolative properties of the vacant set of random interlacements, when one increases the level u of the interlacement, plays an important role in their analysis. As it turns out, the level u of random interlacements can also be measured by means of the random field of occupation times, which, in the case of $\mathbb{Z}^d$, $d \geq 3$, is stationary, ergodic, and has average value u. In this work, we study the large deviations of the density profile of this random field in a large box of $\mathbb{Z}^d$. As an application of the general large deviation principle we obtain, we analyze the asymptotic behavior of the probability that atypically high values of the density profile insulate a macroscopic body in a large box. One may naturally wonder whether such type of large deviations of the occupation-time profile actually captures the main mechanism for an atypical disconnection of a macroscopic body from infinity by the random interlacements, when the vacant set is in a percolative regime. In the course of our program, we derive a similar large deviation principle for the occupation-time measure of Brownian interlacements at a fixed level, in a large box of $\mathbb{R}^d$, $d \geq 3$. The scaling invariance of Brownian interlacements permits to recast this problem in terms of general Cramér theory, and our results rely on a new identity for the Laplace transform of the occupation-time measure, which is based on the analysis of Schrödinger semi-groups.

We now discuss our results in more detail. We consider continuous time random interlacements on $\mathbb{Z}^d$, $d \geq 3$. In essence, this is a Poisson point process on a certain state space consisting of doubly infinite $\mathbb{Z}^d$-valued trajectories marked by their duration at each step, modulo time-shift. A non-negative parameter u comes as a multiplicative factor of the intensity measure of this Poisson point process, which is defined on a certain canonical space, see [57], which we denote here by $(\overline{\Omega}, \overline{\mathcal{A}}, \overline{\mathbb{P}})$. The field of occupation times of random interlacements at level u is denoted by $L_{x,u}(\omega)$, for $x \in \mathbb{Z}^d$, $u \geq 0$, $\omega \in \overline{\Omega}$. It records the total duration spent at x by the trajectories modulo time-shift with label at most u in the cloud ω, see [57].

Given a *closed box* $B \subseteq \mathbb{R}^d$ (by this we mean that B is the product of d non-degenerate compact intervals in $\mathbb{R}$), a central object of interest in this work is the density profile of the occupation times at level u in the large discrete box $(NB) \cap \mathbb{Z}^d$:

$$\rho_{N,u} = \frac{1}{N^d} \sum_{x \in (NB) \cap \mathbb{Z}^d} L_{x,u}\, \delta_{\frac{x}{N}}, \ \text{ for } \ N \ge 1, u \ge 0. \tag{1.1}$$

We view $\rho_{N,u}$ as a random element of $M_+(B)$, the set of positive measures on B, which we endow with the weak topology generated by $C(B)$, the set of continuous functions on B, and with its corresponding Borel σ-algebra. As a consequence of the ergodic theorem, see (1.98), $\overline{\mathbb{P}}$-a.s., $\rho_{N,u}$ converges weakly to $u\, m_B$, where m_B stands for the restriction of the Lebesgue measure to Borel subsets of B. In Theorem 1.5.8, we establish a large deviation principle for $\rho_{N,u}$, which shows that for $u > 0$,

(1.2)
as $N \to \infty$, the laws of $\rho_{N,u}$ on $M_+(B)$ satisfy a large deviation principle at speed N^{d-2} with the convex, good rate function $\frac{1}{d}\, I_u$,

where for $v > 0$ and $\mu \in M_+(B)$, we have defined

(1.3)

$$I_v(\mu) = \begin{cases} +\infty, & \text{if } \mu \text{ is not absolutely continuous w.r.t. } m_B, \\ \inf\Big\{\frac{1}{2}\displaystyle\int_{\mathbb{R}^d} |\nabla\varphi|^2 dy;\ \varphi \in H^1(\mathbb{R}^d), \\ \qquad \varphi = \sqrt{\dfrac{d\mu}{dm_B}} - \sqrt{v}, \ \text{ a.e. on } B\Big\}, \\ \qquad \text{if } \mu \text{ is absolutely continuous w.r.t. } m_B \\ \qquad \text{(and the infimum of the empty set equals } +\infty). \end{cases}$$

In other words, I_v is a non-negative, convex, lower semi-continuous function, with compact level sets $\{I_v \le a\}$, for $a \ge 0$, and for any open subset O and closed subset C of $M_+(B)$, we have

(1.4)

i) $\displaystyle \limsup_N \frac{1}{N^{d-2}} \log \overline{\mathbb{P}}[\rho_{N,u} \in C] \le -\inf_C \frac{1}{d}\, I_u$, and

ii) $\displaystyle \liminf_N \frac{1}{N^{d-2}} \log \overline{\mathbb{P}}[\rho_{N,u} \in O] \ge -\inf_O \frac{1}{d}\, I_u\,.$

In the course of proving Theorem 1.5.8, we derive a large deviation principle for the density profile of the occupation-time measure of Brownian interlacements at level $\alpha > 0$ in a large box LB of $\mathbb{R}^d$, $d \geq 3$, which is of independent interest. Letting $\mathcal{L}_\alpha$ stand for the random Radon measure of occupation times in $\mathbb{R}^d$ of Brownian interlacements at level α, see (1.43) (or Section 2 of [60]) we consider in place of $\rho_{N,u}$ the random measure $\nu_{L,\alpha}$ on B, such that for any bounded measurable f on B, the integral $\langle \nu_{L,\alpha}, f\rangle$ equals

$$\langle \nu_{L,\alpha}, f \rangle = \frac{1}{L^d} \int_{LB} f\Big(\frac{y}{L}\Big)\, \mathcal{L}_\alpha(dy), \text{ for } L \geq 1, \alpha \geq 0. \tag{1.5}$$

We show in Corollary 1.3.3 that for $\alpha > 0$,

(1.6) as $L \to \infty$, the laws of $\nu_{L,\alpha}$ on $M_+(B)$ satisfy a large deviation principle, at speed L^{d-2} with the convex good rate function I_α (see (1.3).

As an application of the large deviation principle (1.2), we analyze the asymptotic behavior of the probability that high values of the (smoothed-out) density profile insulate a large macroscopic body. We consider a regularization f_N of $\rho_{N,u}$ obtained by the convolution of $\rho_{N,u}$ with a continuous probability density φ_δ supported in the closed Euclidean ball of radius δ centered at the origin of $\mathbb{R}^d$. Given a compact subset K of a sub-box B_0 of B (at distance at least δ from B^c), and a positive number a, we are interested in the event $\{\rho_{N,u} \in \mathcal{D}_{a,\delta}\}$ (see (1.157) for the precise definition), where the level set $\{f_N \geq a\}$ "disconnects K from ∂B_0". We show in Theorems 1.6.2 and 1.6.4 that when $a > u$,

(1.7)

$$\text{i) } \limsup_N \frac{1}{N^{d-2}} \log \overline{\mathbb{P}}[\rho_{N,u} \in \mathcal{D}_{a,\delta}] \leq -\frac{1}{d}\big(\sqrt{a} - \sqrt{u}\big)^2 \mathrm{cap}(K),$$

$$\text{ii) } \liminf_N \frac{1}{N^{d-2}} \log \overline{\mathbb{P}}[\rho_{N,u} \in \mathcal{D}_{a,\delta}] \geq -\frac{1}{d}\big(\sqrt{a} - \sqrt{u}\big)^2 \mathrm{cap}(K^\delta),$$

where $\mathrm{cap}(\cdot)$ stands for the Brownian capacity (see below (1.41)) and K^δ for the closed δ-neighborhood of K. As $\delta \to 0$, one knows that $\mathrm{cap}(K^\delta) \downarrow \mathrm{cap}(K)$, see Remark 1.6.5 1), so that upper and lower bounds in (1.7) become identical. Actually, one can let δ slowly tend to 0 so that the corresponding upper and lower bounds match with the right-hand side of (1.7) i), see Remark 1.157 2).

The asymptotics (1.7) has an interesting consequence. There is an intuitive strategy to ensure that the level set $\{f_N \geq a\}$ disconnects K from ∂B_0 (or in other words that $\{\rho_{N,u} \in \mathcal{D}_{a,\delta}\}$ occurs). Roughly speaking, it consists in inducing a suitable increase of the rate of the Poisson distribution of the number of bilateral trajectories with label at most u, which enter $(NK^\delta) \cap \mathbb{Z}^d$. The lower bound (1.7) ii) shows that this intuitive strategy is sub-optimal, see Remark 1.6.5 4). In essence, this strategy leads to a version of (1.7) ii) where $(\sqrt{a} - \sqrt{u})^2$ is replaced by the strictly bigger quantity $a \log \frac{a}{u} - a + u$.

Further, it is known that the vacant set $\mathcal{V}^u$ of random interlacements at level u undergoes a phase transition between a percolative phase when $u < u_*$, and a non-percolative phase when $u > u_*$, for a certain critical level $u_* \in (0, \infty)$ (see [56], [52], and also [24], [46] for recent developments). When $u < u_*$, the vacant set of random interlacements is in a percolative regime. In the context of Bernoulli percolation, disconnecting a large macroscopic body in the percolative phase would involve an exponential cost proportional to N^{d-1} (and surface tension), in the spirit of the study of the presence of a large finite cluster at the origin, see p. 216 of [33], and Theorem 2.5, p. 16 of [12]. In the present context, one may wonder whether large deviations of the density profile, as in (1.7), with an exponential cost proportional to N^{d-2}, capture the main mechanism ensuring that a macroscopic body gets disconnected from infinity by the interlacement at level u, when $u < u_*$. We refer to Remark 1.6.5 5) and to [43] for more on this topic.

It is also of interest to point out that the large deviation principle (1.2) has some similar flavor to results of [11] concerning the Gaussian free field (although the approaches in the two articles are quite different). This feature is in line with the isomorphism theorem of [59], which relates the field of occupation times of random interlacements to the square of Gaussian free fields.

Let us give some comments concerning proofs. The large deviation principle (1.6) (concerning the profile of the Brownian occupation-time measure) is used as a step in the proof of (1.2). Due to the scaling property of $\mathcal{L}_\alpha$, see (1.45), $\nu_{L,\alpha}$ has the same distribution as $\widetilde{\nu}_{L,\alpha}$, the restriction to B of $\frac{1}{L^{d-2}}\mathcal{L}_{\alpha L^{d-2}}$. Since $\mathcal{L}_\alpha$ has stationary and independent increments, (1.6) can be proved by means of general Cramér theory and sub-additivity, see for instance [20], p. 252. In this process, one important ingredient is a new identity for the Laplace functional of $\mathcal{L}_\alpha$, which is based on methods of Schrödinger semi-groups, see [15], [16], [17]. Indeed, we show in Theorem 1.2.2 that for $\alpha \geq 0$, and any bounded measurable function V on $\mathbb{R}^d$, with compact support,

$$\mathbb{E}[e^{\langle \mathcal{L}_\alpha, V\rangle}] = e^{\alpha\Gamma(V)}, \tag{1.8}$$

where

$$\Gamma(V) = \int_{\mathbb{R}^d} V\,dy + \sup_{\varphi\in L^2(\mathbb{R}^d)} \Big\{2\int_{\mathbb{R}^d} V\varphi\,dy + \int_{\mathbb{R}^d} V\varphi^2 dy - \mathcal{E}(\varphi,\varphi)\Big\} \in (-\infty,+\infty], \tag{1.9}$$

and for $\varphi \in L^2(\mathbb{R}^d)$,

$$\mathcal{E}(\varphi,\varphi) = \begin{cases} \frac{1}{2}\displaystyle\int_{\mathbb{R}^d} |\nabla\varphi|^2 dy, & \text{if } \varphi \in H^1(\mathbb{R}^d), \\ +\infty, & \text{otherwise,} \end{cases} \tag{1.10}$$

stands for the Dirichlet form attached to the Brownian semi-group (acting on $L^2(\mathbb{R}^d)$), see also below (1.51). Both members of (1.8) may well be infinite. Remarkably, and unlike Proposition 2.6 of [60] (see also Remark 1.2.1 below), (1.8) is an identity between extended numbers in $(0,+\infty]$, which does not require any smallness assumption on V. We also refer to Remark 1.2.3 for the discrete space counterpart of this identity.

The proof of the main large deviation principle (1.2) on the density profile $\rho_{N,u}$ appears in Theorem 1.5.8. It relies on sub-additivity and naturally splits into a lower bound and an upper bound. On the one hand, the lower bound (proved in Theorem 1.5.4) first relies on a lower bound stemming from sub-additivity. It then exploits a combination of the fact that the Brownian occupation-time measure is a scaling limit of occupation times of random interlacements on $\mathbb{Z}^d$ (as proved in [60] and recalled in (1.99) below), and the key large deviation result on $\frac{1}{\alpha}\,\mathcal{L}_\alpha$ restricted to a box (proved in Section 3). On the other hand, the upper bound (proved in Theorem 1.5.5) combines an upper bound stemming from sub-additivity, involving a discrete version of the functional in (1.3), and an estimate in the spirit of Gamma-convergence (see Chapter 7 of [19]), which compares the large N behavior of a sequence of variational problems on the scaled lattices $\frac{1}{N}\,\mathbb{Z}^d$ to a suitable continuous-space variational problem.

The asymptotic bounds (1.7) on the probability of insulation of a large macroscopic body by high values of the (smoothed-out) occupation-time profile (see Theorems 1.6.2 and 1.6.4) are direct applications of the main large deviation principle (1.2) and of the structure of the rate function, see Lemma 1.81.

Let us now describe the organization of this article. Section 1 introduces further notation, collects material concerning Schrödinger semi-groups, and recalls some properties of Brownian interlacements. The main objective of Section 2 is to establish the identity (1.52) in Theorem 1.2.2. Some of the key consequences of this identity appear in the Corollaries 1.2.4 and 1.2.5. The discrete space situation is discussed in Remark 1.2.3. Section 3 derives in Theorem 1.3.2 a large deviation principle for the Brownian interlacement case, which plays a central role. Its application to the proof of (1.6) appears in Corollary 1.3.3. The main properties of the rate function I_v are collected in Lemma 1.3.1. The short Section 4 describes the (scaled) discrete space set-up and some of the results following by the methods of Section 2 in this context, see Theorem 1.4.1 and Corollary 1.4.2. Section 5 is devoted to the proof of (1.2), see Theorem 1.5.8. The lower bound appears in Theorem 1.5.4 and the upper bound in Theorem 1.5.5. The main sub-additive estimates are contained in Corollary 1.5.3, and the relevant form of the scaling limit of occupation times in Lemma 1.5.1. The last Section 6 contains the proof of (1.7), see Theorems 1.6.2 and 1.6.4. Extensions are discussed at the end, in Remark 1.6.5.

Finally, let us explain the convention we use concerning constants.

We denote by $c, c', \overline{c}$ positive constants changing from place to place, which simply depend on d. Numbered constants $c_0, c_1, \dots$ refer to the value corresponding to their first appearance in the text. Dependence of constants on additional parameters appears in the notation.

1.1 Some useful facts about Schrödinger semi-groups and Brownian interlacements

In this section, we first introduce some further notation. We collect some rather classical properties of Schrödinger semi-groups and gauge functions, which will be useful in the next section, see Proposition 1.1.2 and Lemma 1.1.3. Moreover, we recall some properties of Brownian interlacements on $\mathbb{R}^d$. Further facts concerning continuous time random interlacements on $\mathbb{Z}^d$ will appear in Section 4. Throughout, we tacitly assume that $d \geq 3$.

We write $|\cdot|$ and $|\cdot|_\infty$ for the Euclidean and the supremum norms on $\mathbb{R}^d$. We denote by $B(y,r)$ the closed Euclidean ball with center $y \in \mathbb{R}^d$ and radius $r \geq 0$. We write $B_\infty(y,r)$ in the case of the supremum norm. Given $A, B \subseteq \mathbb{R}^d$, we denote by $d(A,B) = \inf\{|y-y'|;\, y \in A,\, y' \in B\}$ the mutual Euclidean distance of A an B. When $A = \{y\}$, we simply write $d(y,B)$ in place of $d(\{y\},B)$. We define $d_\infty(A,B)$ and $d_\infty(y,B)$ analogously, with $|\cdot|_\infty$ in place of $|\cdot|$. The shorthand notation $K \subset\subset \mathbb{R}^d$, resp. $K \subset\subset \mathbb{Z}^d$, means that K is compact subset of $\mathbb{R}^d$, resp. a finite subset of $\mathbb{Z}^d$. We denote by $f_+ = \max\{f, 0\}$, $f_- = \max\{-f, 0\}$, the positive and negative part of a function f. We routinely write $\langle \nu, f \rangle$ to denote the integral with respect to a measure ν of a measurable, non-negative, or ν-integrable, function f. When f, h are measurable functions on $\mathbb{R}^d$ such that $|fh|$ is Lebesgue-integrable, we write $\langle f, h \rangle = \int_{\mathbb{R}^d} f(y)h(y)dy$. We denote by $\|f\|_\infty$ the supremum norm of the function f, and by $\|f\|_p$ its L^p-norm ($1 \leq p < \infty$). We specify the relevant L^p-space in the notation when there might be some ambiguity, and write for instance $\|f\|_{L^p(\mathbb{R}^d, dy)}$ or $\|f\|_{L^p(B,\nu)}$.

We denote by W_+ the subspace of $C(\mathbb{R}_+, \mathbb{R}^d)$ of continuous $\mathbb{R}^d$-

valued trajectories tending to infinity at infinite times. We write $(X_t)_{t\geq 0}$ for the canonical process, and denote by $(\theta_t)_{t\geq 0}$ the canonical shift. We endow W_+ with the σ-algebra $\mathcal{W}_+$ generated by the canonical process. Given an open set U of R^d, $w \in W_+$, we write $T_U(w) = \inf\{s \geq 0; X_s(w) \notin U\}$, for the exit time of U. When F is a closed subset of $\mathbb{R}^d$, we write $H_F(w) = \inf\{s \geq 0;\ X_s(w) \in F\}$, and $\widetilde{H}_F(w) = \inf\{s > 0; X_s(w) \in F\}$, for the respective entrance, and hitting times of F. We assume $d \geq 3$, so that Brownian motion on $\mathbb{R}^d$ is transient, and we view P_y, the Wiener measure starting from $y \in \mathbb{R}^d$, as defined on $(W_+, \mathcal{W}_+)$. We denote by E_y the corresponding expectation. When ρ is a finite measure on $\mathbb{R}^d$, we write P_ρ for the Wiener measure with "initial distribution" ρ and E_ρ for the corresponding expectation. We write $p(t,y,y') = (2\pi t)^{-\frac{d}{2}} \exp\{-\frac{|y-y'|^2}{2t}\}$, with $t > 0$, $y, y' \in \mathbb{R}^d$, for the Brownian transition density.

We now recall some properties of the Schrödinger semi-groups we consider here. We denote by $L^\infty_c(\mathbb{R}^d)$ the space of bounded measurable functions V on $\mathbb{R}^d$, which vanish outside a compact subset of $\mathbb{R}^d$. Given $V \in L^\infty_c(\mathbb{R}^d)$, we introduce the Schrödinger semi-group attached to V, namely the strongly continuous self-adjoint semi-group on $L^2(\mathbb{R}^d, dy)$, see Proposition 3.3, p. 16 of [54],

(1.11)
$$\begin{aligned} R^V_t f(y) &= E_y\big[f(X_t)\, e^{\int_0^t V(X_s)ds}\big],\ \text{for } f \in L^2(\mathbb{R}^d), t \geq 0, y \in \mathbb{R}^d, \\ &= \int_{\mathbb{R}^d} r_V(t,y,y')\, f(y')dy',\ \text{when } t > 0, \end{aligned}$$

where

$$r_V(t,y,y') = p(t,y,y')\, E^t_{y,y'}\Big[\exp\Big\{\int_0^t V(X_s)ds\Big\}\Big], \quad \text{for } t > 0, y, y' \in \mathbb{R}^d, \tag{1.12}$$

is a symmetric function of y, y', see Proposition 3.1, p. 13-14 of [54], which is jointly continuous, see Proposition 3.5, p. 18 of [54], and $E^t_{y,y'}$ stands for the expectation corresponding to $P^t_{y,y'}$, the Brownian bridge measure in time t from y to y', see p. 137-140 of [54]. As an immediate consequence of (1.12),

$$(1.13)\qquad \begin{aligned} e^{-\|V\|_\infty t} p(t,y,y') &\le r_V(t,y,y') \\ &\le e^{\|V\|_\infty t}\, p(t,y,y'),\ \text{for } t>0, y,y' \in \mathbb{R}^d. \end{aligned}$$

We now turn to the discussion of the Green operators corresponding to the Schrödinger semi-groups. We thus consider $V \in L^\infty_c(\mathbb{R}^d)$ as above and define

$$(1.14)\qquad \begin{aligned} G_V f(y) &= E_y\Big[\int_0^\infty f(X_s)\, e^{\int_0^s V(X_u)du} ds\Big] \\ &= \int_{\mathbb{R}^d}\int_0^\infty r_V(s,y,y')\, f(y') ds\, dy', \end{aligned}$$

for f a measurable non-negative function on $\mathbb{R}^d$, and $y \in \mathbb{R}^d$.

When $V=0$ (so $r_V(t,y,y') = p(t,y,y')$) we simply write G, and recover the usual Green operator attached to Brownian motion

(1.15)
$$Gf(y) = E_y\Big[\int_0^\infty f(X_s)ds\Big] = \frac{\Gamma(\frac{d}{2}-1)}{2\pi^{\frac{d}{2}}} \int_{\mathbb{R}^d} \frac{1}{|y-y'|^{d-2}}\, f(y')dy.$$

We introduce in the (classical) lemma below a condition corresponding to the so-called sub-criticality (of $\frac{1}{2}\Delta + V$) see p. 145, as well as pp. 129, 136, 150 of [45]. Our assumptions are slightly different and we briefly sketch the proof for the reader's convenience.

Lemma 1.1.1. *(recall $V \in L^\infty_c(\mathbb{R}^d)$) Assume that*

(1.16) $G_V W(y_0) < \infty$, *for some* $y_0 \in \mathbb{R}^d$ *and some* $[0,1]$*-valued, measurable function* W *not a.e. equal to* 0.

Then, for any bounded open set U,

(1.17) $G_V\, 1_U$ *is a bounded function.*

Moreover, in the notation of (1.11), the Schrödinger semi-group

$$(1.18)\qquad \begin{array}{l}(R_t^V)_{t\ge 0} \textit{ is a strongly continuous semi-group}\\ \textit{of self-adjoint contractions on } L^2(\mathbb{R}^d).\end{array}$$

Proof. We first sketch the proof of (1.17). We note that

$$(1.19)\qquad r_V(1,y_0,y) \ge c(V,y_0,y_1)\, r_V\Big(\frac{1}{2},y_1,y\Big), \text{ for } y_0,y_1,y\in\mathbb{R}^d,$$

with $c(V,\cdot,\cdot)$ locally bounded away from zero. Indeed, this follows from (1.13) and the inequality $p(1,0,z)/p(\frac{1}{2},z_1,z) = 2^{-\frac{d}{2}}\exp\{\frac{1}{2}\,|z-2z_1|^2 - |z_1|^2\} \ge 2^{-\frac{d}{2}}\exp\{-|z_1|^2\}$, for $z,z_1\in\mathbb{R}^d$, combined with translation invariance (set $z=y-y_0$, $z_1=y_1-y_0$). Thus, for $y_1\in\mathbb{R}^d$, we find that the above inequality together with the semi-group property yields that

$$(1.20)\qquad \begin{array}{l}\infty > G_V W(y_0) \overset{(1.14)}{\ge} \displaystyle\int_0^\infty \int r_V(s+1,y_0,y)\, W(y)dy\, ds \ge\\ c(V,y_0,y_1)\displaystyle\int_{\frac{1}{2}}^\infty \int r_V(t,y_1,y)\, W(y)dy\, dt,\end{array}$$

which combined with (1.13) (for the values $t\in(0,\frac{1}{2}]$) implies that $G_V W$ is locally bounded. In addition, by the semi-group property, we see that for $y\in\mathbb{R}^d$,

$$(1.21)\qquad G_V W(y) \ge \int_0^\infty \int r_V(s+1,y,y')\, W(y')dy'ds \ge G_V 1_U(y)\, \inf_{z\in U} R_1^V W.$$

By our assumption on W, the last term is positive, and hence, $G_V 1_U$ is locally bounded. Choosing $K \subset\subset \mathbb{R}^d$ containing U and the support of V, the strong Markov property yields,

$$(1.22)\qquad G_V 1_U(y) = E_y\big[(G_V 1_U)(X_{H_K}), H_K<\infty\big], \text{ for any } y\in\mathbb{R}^d,$$

so $G_V 1_U$ is bounded and (1.17) follows.

We now turn to the proof of (1.18). For $\varphi\in L_c^\infty(\mathbb{R}^d)$ we denote by $dE_{\varphi,\varphi}(\lambda)$ the spectral measure of φ (and E is a spectral resolution of the identity of the generator of $(R_t^V)_{t\geq 0}$, see for instance Theorems 13.30 and 13.37, pp. 348, 360 of [49]).We find by (1.17) that

$$\infty > \int_0^\infty dt \langle R_t^V \varphi, \varphi \rangle = \int_0^\infty dt \int_{\mathbb{R}} e^{-\lambda t} dE_{\varphi,\varphi}(\lambda), \tag{1.23}$$

Hence, $dE_{\varphi,\varphi}$ gives no mass to $(-\infty, 0]$, and therefore for $\varphi \in L_c^\infty(\mathbb{R}^d)$,

(1.24)
$$\langle R_t^V \varphi, R_t^V \varphi \rangle = \int_0^\infty e^{-2\lambda t} dE_{\varphi,\varphi}(\lambda) \leq \int_0^\infty dE_{\varphi,\varphi}(\lambda) = \|\varphi\|^2_{L^2(\mathbb{R}^d)}.$$

Since $L_c^\infty(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, the claim (1.18) follows. □

We now recall some properties of the gauge function

$$\gamma_V(y) = E_y\big[e^{\int_0^\infty V(X_s)ds}\big], \text{ for } y \in \mathbb{R}^d \tag{1.25}$$

(note that $V \in L_c^\infty(\mathbb{R}^d)$ and the integral is finite due to transience, or more precisely, to the fact that P_y is a probability on W_+, see above (1.11)). As the next proposition shows, the gauge function is closely related to the Schrödinger semi-group (attached to V) via its Green operator. We refer to [15], [16], and [17], for much more on the subject.

Proposition 1.1.2. *(recall $V \in L_c^\infty(\mathbb{R}^d)$) The condition (1.16) is equivalent to*

(1.26) γ_V *is not identically infinite (Gauge Condition).*

If (1.16), or equivalently (1.26), holds, then

(1.27)
γ_V *is a bounded continuous function on* $\mathbb{R}^d$ *tending to* 1 *at infinity,*

and

$$\gamma_V = 1 + GV\gamma_V = 1 + G_V V. \tag{1.28}$$

Proof. To show that (1.16) implies (1.26), we use the identity

$$e^{\int_0^t V(X_s)ds} = 1 + \int_0^t V(X_s)\, e^{\int_0^s V(X_u)du} ds, \text{ for } t \geq 0. \tag{1.29}$$

By Fatou's lemma, (1.29) implies that $\gamma_V(y) \leq 1 + G_V|V|(y)$, which is a bounded function of y, by (1.17). The fact that γ_V coincides with the last expression of (1.28) follows by dominated convergence.

To prove that (1.26) implies (1.16), either V_+ ($= \max(V,0)$) vanishes a.e., so that $r_V \leq p$ and (1.16) holds, or else, by Theorem 2.8, p. 4651 of [15], $\|G_V V_+\|_\infty < \infty$ and V_+ is not a.e. equal to 0. This implies (1.16) (choosing $W = V_+ \wedge 1$). Thus (1.16) and (1.26) are equivalent.

To prove (1.27), we already know from the discussion below (1.29) that γ_V is a bounded function. It is continuous by the Corollary, p. 150 of [17], or (ii) in Theorem 4.7, p. 115 of the same reference. The fact that γ_V tends to 1 at infinity follows from the first equality of (1.28), which we prove now. To derive the first equality of (1.28) (and this will compete the proof of Proposition 1.1.2), we use the identity

$$e^{\int_0^t V(X_s)ds} = 1 + \int_0^t V(X_s)\, e^{\int_s^t V(X_u)du} ds, \text{ for } t \geq 0. \tag{1.30}$$

Integrating with respect to P_y and using the Markov property yields

$$E_y\big[e^{\int_0^t V(X_s)ds}\big] = 1 + E_y\Big[\int_0^t V(X_s)\, E_{X_s}\big[e^{\int_0^{t-s} V(X_u)du}\big] ds\Big], \tag{1.31}$$

for all $y \in \mathbb{R}^d$. By (1.17) and (1.29) we see that the inner expectation is uniformly bounded, and converges to $\gamma_V(X_s)$ as $t \to \infty$. The first equality of (1.28) now follows by dominated convergence. The second equality results from the discussion below (1.29), and the proof of Proposition 1.1.2 is now complete. □

The following approximation lemma will be helpful in Sections 2 and 3. We say that r and r' in $(1,\infty)$ are conjugate exponents when

$$\frac{1}{r} + \frac{1}{r'} = 1. \tag{1.32}$$

Lemma 1.1.3. *Consider $r > \frac{d}{2}$, and $V, V_n, n \geq 1$, in $L_c^\infty(\mathbb{R}^d)$, which all vanish outside $K \subset\subset \mathbb{R}^d$. Assume that (1.16) holds for V and that*

$$\lim_n \|V - V_n\|_{L^r(\mathbb{R}^d)} = 0. \tag{1.33}$$

Then, for large n, (1.16) holds for V_n, and

$$\gamma_{V_n} \text{ converges uniformly to } \gamma_V \text{ on } \mathbb{R}^d. \tag{1.34}$$

Proof. By Theorem 2.17, p. 4660 of [15] (Super Gauge Theorem), for some $p > 1$, pV satisfies (1.26) and hence (1.16). Thus, denoting by q the conjugate exponent of p,

$$\begin{aligned} |\gamma_{V_n}(y) - \gamma_V(y)| \leq\ & E_y\big[\big|e^{\int_0^\infty (V_n - V)(X_s)ds} - 1\big| e^{\int \infty_0 V(X_s)ds}\big] \\ \overset{\text{Hölder}}{\leq}\ & E_y\big[\big|e^{\int_0^\infty (V_n - V)(X_s)ds} - 1\big|^q\big]^{\frac{1}{q}}\, \big\|\gamma_{pV}\big\|_\infty^{\frac{1}{p}}, \end{aligned} \tag{1.35}$$

for all $y \in \mathbb{R}^d$ (and $\|\gamma_{pV}\|_\infty$ is finite by (1.27)).

Thus, choosing $2m \geq q$, with $m \geq 1$ integer, and setting $\Delta_n = V_n - V$, we find that the $(2m)$-th power of the first term in the last line of (1.35) is smaller than
(1.36)

$$E_y\big[\big(e^{\int_0^\infty \Delta_n(X_s)ds} - 1\big)^{2m}\big] = \sum_{\ell=0}^{2m} \binom{2m}{\ell}(-1)^{2m-\ell} E_y\big[e^{\int_0^\infty \ell\Delta_n(X_s)ds}\big].$$

The claim (1.34) will thus follow once we show that for each fixed $\ell \geq 0$,

$$\sup_{y\in\mathbb{R}^d} \big|E_y\big[e^{\ell\int_0^\infty \Delta_n(X_s)ds}\big] - 1\big| \underset{n}{\longrightarrow} 0\,. \tag{1.37}$$

Expanding the exponential and using the Markov-property, we can bound the absolute value in (1.37) by $\sum_{k\geq1} \ell^k \|G\,|\Delta_n|\,\|_\infty^k$.

By assumption, $r > \frac{d}{2}$, so, the conjugate exponent r' belongs to $(1, \frac{d}{d-2})$. Then, for $z \in \mathbb{R}^d$,

$$
\begin{aligned}
G|\Delta_n|(z) &\overset{(1.15)}{=} c \int \frac{|\Delta_n(y)|}{|y-z|^{d-2}}\, dy \\
&\overset{\text{Hölder}}{\leq} c\Big(\int_K \frac{1}{|y-z|^{(d-2)r'}}\, dy\Big)^{\frac{1}{r'}} \|\Delta_n\|_r \\
&\leq \ c(r,K)\, \|\Delta_n\|_r
\end{aligned}
\tag{1.38}
$$

(for the last bound, where the dependence in z has disappeared, one considers the smallest $R \geq 1$ such that $B(0,R) \supseteq K$, and looks separately at $z \notin B(0,2R)$, or $z \in B(0,2R)$ and hence $K \subseteq B(z,3R)$). So, we see that for $\ell \leq 2m$,

$$
\begin{aligned}
&\sup_{y\in\mathbb{R}^d} \big|E_y\big[e^{\ell\int_0^\infty \Delta_n(X_s)ds}\big] - 1\big| \\
&\leq \sum_{k\geq 1} (2m)^k \|G\,|\Delta_n|\,\|_\infty^k \\
&\leq \frac{c(r,K)\,2m\|\Delta_n\|_r}{(1-c(r,K)\,2m\|\Delta_n\|_r)_+} \ \underset{n}{\overset{(1.15)}{\longrightarrow}}\ 0.
\end{aligned}
\tag{1.39}
$$

This proves (1.37) and (1.34) follows. □

We now recall some properties of Brownian interlacements, and refer to Section 2 of [60] for more detail. Brownian interlacements correspond to a certain Poisson point process on a state space, which is the product $W^* \times \mathbb{R}_+$, where W^* stands for the space of continuous trajectories from $\mathbb{R}$ into $\mathbb{R}^d$, tending to infinity at plus and minus infinite times, modulo time-shift. The intensity measure of this Poisson point process is the product of a certain σ-finite measure ν on W^* (see Theorem 2.2 of [60]), with the Lebesgue measure $d\alpha$ on $\mathbb{R}_+$. Informally, this point process corresponds to a cloud of doubly-infinite trajectories modulo time-shift having each a non-negative label (the $\mathbb{R}_+$-component of $W^* \times \mathbb{R}_+$). The Poisson point process is defined on a certain canonical space $(\Omega, \mathcal{A}, \mathbb{P})$, see (2.23) of [60]. We collect below some properties of Brownian interlacements, which we will use here.

Given $K \subset\subset \mathbb{R}^d$, $\alpha \geq 0$, and $\omega \in \Omega$, one considers the point measure on W_+, denoted by $\mu_{K,\alpha}(\omega)$, which collects for all bilateral trajectories with label at most α, which enter K at some point, their forward trajectories after their first entrance time in K. Then, see (2.25) of [60],

$$\begin{array}{c}\mu_{K,\alpha} \text{ is a Poisson point process on} \\ W_+ \text{ with intensity measure } \alpha P_{e_K},\end{array} \tag{1.40}$$

where

$$e_K(dy) \text{ stands for the equilibrium measure of } K. \tag{1.41}$$

The equilibrium measure of K is a finite measure concentrated on ∂K, and its total mass is called the capacity of K, see pp. 58, 61 of [54]. Moreover,

$$\langle e_K, Gf \rangle = \langle 1, f \rangle \ \text{ if } f \in L_c^\infty(\mathbb{R}^d) \text{ vanishes outside } K. \tag{1.42}$$

One also introduces the occupation-time measure $\mathcal{L}_\alpha(\omega)$ of Brownian interlacements at level $\alpha \geq 0$ in the cloud ω. It is the Radon measure on $\mathbb{R}^d$, which to each $A \in \mathcal{B}(\mathbb{R}^d)$ gives a mass equal to the total time spent in A by all trajectories modulo time-shift with label at most α in the cloud ω. In particular, when $V \in L_c^\infty(\mathbb{R}^d)$ vanishes outside $K \subset\subset \mathbb{R}^d$, one has for $\alpha \geq 0$, and $\omega \in \Omega$,

$$\begin{array}{c}\big\langle \mathcal{L}_\alpha(\omega), V \big\rangle = \big\langle \mu_{K,\alpha}(\omega), f_V \big\rangle, \text{ where} \\ f_V(w) = \displaystyle\int_0^\infty V\big(X_s(w)\big) ds, \text{ for } w \in W_+.\end{array} \tag{1.43}$$

The intensity measure of $\mathcal{L}_\alpha$ equals $\alpha\, dy$, cf. (2.38) of [60], that is

$$\mathbb{E}\big[\big\langle \mathcal{L}_\alpha, V \big\rangle\big] = \alpha \int_{\mathbb{R}^d} V\, dy, \text{ for all } V \in L_c^\infty(\mathbb{R}^d). \tag{1.44}$$

Moreover, $\mathcal{L}_\alpha$ has an important scaling property, see (2.43) of [60],

$$\mathcal{L}_{\lambda^{2-d}\alpha} \overset{\text{law}}{=} \lambda^2 h_\lambda \circ \mathcal{L}_\alpha, \text{ for } \lambda > 0, \tag{1.45}$$

where $h_\lambda \circ \mathcal{L}_\alpha$ stands for the image of $\mathcal{L}_\alpha$ under the homothety of ratio λ on $\mathbb{R}^d$.

One also has an expression for the Laplace transform of $\mathcal{L}_\alpha$ in the "neighborhood of the origin" (see Proposition 2.6 of [60]). Namely for $V \in L_c^\infty(\mathbb{R}^d)$ such that $\|G\,|V|\,\|_\infty < 1$, one has

$$\mathbb{E}[e^{\langle \mathcal{L}_\alpha, V\rangle}] = \exp\{\alpha\langle V, (I - GV)^{-1}1\rangle\} \tag{1.46}$$

(the assumption on V ensures that $I - GV$ operating on $L^\infty(\mathbb{R}^d)$ has a bounded inverse). In the next section we will derive identities that bypass the smallness assumption on V in (1.46), and remain true even when the left-hand side of (1.46) explodes.

We close this section with a lemma about Poisson point processes, which will be helpful in the next section. We consider a measurable space $(E, \mathcal{E})$.

Lemma 1.1.4. *Let μ be a Poisson point process on E with finite intensity measure η (i.e. $\eta(E) < \infty$), and let Φ: $E \to \mathbb{R}$ be a measurable function. Then, one has*

$$\mathbb{E}[e^{\langle \mu, \Phi\rangle}] = \exp\Big\{\int_E (e^\Phi - 1)d\eta\Big\} \tag{1.47}$$

(this is an identity between numbers in $(0, +\infty]$).

Proof. Set $\Phi_n = \sum_{\mathbb{Z}\ni k<n2^n} \frac{k}{2^n}\, 1\{\frac{k}{2^n} \le \Phi < \frac{k+1}{2^n}\}$, for $n \ge 1$, so that Φ_n is measurable, $(-\infty, n]$-valued, and Φ_n increases to Φ, as $n \to \infty$. Then, for each n, classically, see for instance [48], p. 130-134,

$$\mathbb{E}[e^{\langle \mu, \Phi_n\rangle}] = \exp\Big\{\int_E (e^{\Phi_n} - 1)d\eta\Big\}$$

($< \infty$, since $\Phi_n \le n$ and $\eta(E) < \infty$), and (1.47) follows by monotone convergence. □

1.2 Laplace functional of occupation-time measures of Brownian interlacements

In this section, with the help of Schrödinger semi-groups techniques (in particular Lemma 1.1.1 and Proposition 1.1.2 of the previous section), we derive an identity for the Laplace functional of $\mathcal{L}_\alpha$, see Theorem 1.2.2. This identity plays an important role for the identification of the rate function governing the large deviation principle for the occupation-time profile of Brownian interlacements in a large box, which we derive in the next section. We state two consequences of the basic identity (1.52), see Corollaries 1.2.4 and 1.2.5. In Remark 1.2.3, we discuss the corresponding identity one obtains in the case of continuous-time random interlacements on a transient weighted graph.

We begin with the observation that the Laplace functional of $\mathcal{L}_\alpha$ naturally involves the gauge function. We consider $V \in L_c^\infty(\mathbb{R}^d)$ and recall that γ_V stands for the gauge function, see (1.25). When V vanishes on K^c, with $K \subset\subset \mathbb{R}^d$, as in (1.43), we can express $\langle \mathcal{L}_\alpha, V\rangle$ in terms of the Poisson point process $\mu_{K,\alpha}$ introduced above (1.40), and use Lemma 1.1.4 to compute the exponential moment of $\langle \mathcal{L}_\alpha, V\rangle$. We find

$$\begin{aligned} &\mathbb{E}[e^{\langle \mathcal{L}_\alpha, V\rangle}] \\ (1.48)\qquad &\overset{(1.43)}{=} \mathbb{E}[e^{\langle \mu_{K,\alpha}, f_V\rangle}] \overset{(1.40),(1.47)}{=} e^{\alpha\langle e_K, E_x[e^{\int_0^\infty V(X_s)ds}]-1\rangle} \\ &\overset{(1.25)}{=} e^{\alpha\langle e_K, \gamma_V - 1\rangle}, \text{ for } \alpha \geq 0 \end{aligned}$$

(note that this is an identity between numbers in $(0,\infty]$, and there is no smallness assumption on V).

Remark 1.2.1. It is well-known that the gauge function γ_V (and hence the left-hand side of (1.48)) may very well be infinite, see for instance [45], p. 227, and pp. 166, 167. This feature complicates the study of the Laplace transform of $\mathcal{L}_\alpha$. When (1.16) (or equivalently (1.26)) holds, we know from (1.27), (1.28) that γ_V is bounded continuous and $\gamma_V - 1 = GV\gamma_V$. By (1.42) we thus find that

$$\mathbb{E}[e^{\langle \mathcal{L}_\alpha, V\rangle}] = e^{\alpha\langle V, \gamma_V\rangle}, \text{ for } \alpha \geq 0, \text{ when (1.16) holds.} \tag{1.49}$$

In particular, when $\| G|V| \|_\infty < 1$, then (1.26) holds (see for instance (2.6) in [57]), and $I - GV$ operating on $L^\infty(\mathbb{R}^d)$ has a bounded inverse. So, by the first equality in (1.28), $\gamma_V = (I - GV)^{-1}1$, and we find $\langle V, \gamma_V\rangle = \langle V, (I - GV)^{-1}1\rangle$. In this fashion we recover (1.46) out of (1.49) when $\| G|V| \|_\infty < 1$. □

We introduce a notation for the logarithm of the Laplace functional of $\mathcal{L}_1$. For $V \in L_c^\infty(\mathbb{R}^d)$, we set

$$\Lambda(V) = \frac{1}{\alpha} \log \mathbb{E}[e^{\langle \mathcal{L}_\alpha, V\rangle}] \in (-\infty, +\infty] \tag{1.50}$$

(and $\alpha > 0$ is arbitrary by (1.48)). We also recall from (1.9) the notation

$$\Gamma(V) = \int_{\mathbb{R}^d} V\, dy + \sup_{\varphi \in L^2(\mathbb{R}^d)} \{2\langle V, \varphi\rangle + \langle V\varphi, \varphi\rangle - \mathcal{E}(\varphi, \varphi)\}, \tag{1.51}$$

where $\mathcal{E}(\varphi, \varphi)$ denotes as in (1.10) the Dirichlet form attached to the Brownian semi-group acting on $L^2(\mathbb{R}^d)$ (i.e. $(R_t^{V=0})_{t\geq 0}$ in the notation of (1.27)), see for instance [54], p. 26. Further, we will also use the notation $\mathcal{E}(\varphi, \varphi)$, when φ belongs to the extended Dirichlet space $\mathcal{F}_e$ (consisting of functions that are a.e. limits of sequences in $H^1(\mathbb{R}^d)$ that are Cauchy for $\mathcal{E}(\cdot, \cdot)$), see Chapter 1 §5 of [29].

We are now ready for the key identity of this section. The arguments we use are general and easily adapted to the context of continuous-time random interlacements on transient weighted graphs, as explained in Remark 1.2.3 below.

Theorem 1.2.2.

$$\Lambda(V) = \Gamma(V), \text{ for all } V \in L_c^\infty(\mathbb{R}^d). \tag{1.52}$$

Proof. We first assume $\Lambda(V) < \infty$ and show the identity (1.52). By (1.48), we know that (1.26), and hence (1.16) hold. By (1.49) we have

$$(1.53) \qquad \Lambda(V) = \langle V, \gamma_V \rangle \overset{(1.28)}{=} \langle V, 1 \rangle + \langle V, G_V V \rangle.$$

Similarly, we have

$$(1.54) \qquad \Gamma(V) = \langle V, 1 \rangle + \Psi(V),$$

where

$$\begin{aligned} \Psi(V) &= \sup_{\varphi \in L^2(\mathbb{R}^d)} \{2\langle V, \varphi \rangle + \langle V\varphi, \varphi \rangle - \mathcal{E}(\varphi, \varphi)\} \\ &= \sup_{\varphi \in L^2(\mathbb{R}^d)} \sup_{\varepsilon > 0} \{2\langle V, \varphi \rangle + \langle (V - \varepsilon)\varphi, \varphi \rangle - \mathcal{E}(\varphi, \varphi)\} \\ &= \sup_{\varepsilon > 0} \sup_{\varphi \in L^2(\mathbb{R}^d)} \{2\langle V, \varphi \rangle + \langle (V - \varepsilon)\varphi, \varphi \rangle - \mathcal{E}(\varphi, \varphi)\}. \end{aligned}$$

We know by (1.18) that the Schrödinger semi-group $(R_t^V)_{t \geq 0}$ is a semi-group of self-adjoint contractions on $L^2(\mathbb{R}^d)$. Its quadratic form is $\mathcal{E}(\varphi, \varphi) - \langle V\varphi, \varphi \rangle$, see for instance [15], p. 4654. Then, by Lemma 4.4, p. 22 of [54], and also below (4.10), p. 23 of the same reference, we have

$$(1.55) \qquad \sup_{\varphi \in L^2(\mathbb{R}^d)} \{2\langle V, \varphi \rangle + \langle (V - \varepsilon)\varphi, \varphi \rangle - \mathcal{E}(\varphi, \varphi)\} = \langle V, G_{V-\varepsilon} V \rangle,$$

where $G_{V-\varepsilon}$ is defined as in (1.14) with V replaced by $V - \varepsilon$. Note that the left-hand side of (1.55) is a decreasing function of ε, so the same holds for the right-hand side. As a result, we find that

$$(1.56) \qquad \Psi(V) = \sup_{\varepsilon > 0} \langle V, G_{V-\varepsilon} V \rangle = \lim_{\varepsilon \to 0} \langle V, G_{V-\varepsilon} V \rangle = \langle V, G_V V \rangle,$$

using dominated convergence and (1.17) in the last step. We have thus shown that

$$\Gamma(V) \overset{(1.54)}{=} \langle V, 1 \rangle + \Psi(V) \overset{(1.56)}{=} \langle V, 1 \rangle + \langle V, G_V V \rangle \overset{(1.53)}{=} \Lambda(V),$$

that is, (1.52) holds when $\Lambda(V) < \infty$.

We now assume that $\Gamma(V) < \infty$, and show (1.52). Since $\Gamma(V) < \infty$, we have

$$\sup_{\varphi \in L^2(\mathbb{R}^d)} \langle V\varphi, \varphi\rangle - \mathcal{E}(\varphi, \varphi) \le 0 \tag{1.57}$$

(otherwise the supremum in (1.51) would be infinite). Since $\mathcal{E}(\varphi, \varphi) - \langle V\varphi, \varphi\rangle$ is the quadratic form associated to the strongly continuous self-adjoint Schrödinger semi-group $(R_t^V)_{t\ge 0}$ on $L^2(\mathbb{R}^d)$, it follows that $R_t^V, t \ge 0$, are contractions.

We first discuss the case where $V_- = 0$ a.e.. If $V = 0$ a.e., then (1.52) is immediate. Otherwise, V_+ is not a.e. equal to 0. We then apply the same considerations as below (1.54) to find that for $\varepsilon > 0$, $\sup_{\varphi\in L^2(\mathbb{R}^d)}\{2\langle V, \varphi\rangle + \langle (V-\varepsilon)\varphi, \varphi\rangle - \mathcal{E}(\varphi,\varphi)\} = \langle V, G_{V-\varepsilon}V\rangle$, and that this quantity increases to $\sup_{\varphi\in L^2(\mathbb{R}^d)}\{2\langle V, \varphi\rangle + \langle V\varphi, \varphi\rangle - \mathcal{E}(\varphi,\varphi)\}$, as ε tends to 0. Coming back to the definition of $\Gamma(V)$ in (1.51), we find that

$$\infty > \Gamma(V) = \langle V, 1\rangle + \lim_{\varepsilon\to 0}\langle V, G_{V-\varepsilon}V\rangle = \langle V, 1\rangle + \langle V, G_V V\rangle, \tag{1.58}$$

where we used monotone convergence in the last step. Hence, (1.16) holds and $\Lambda(V) < \infty$, by (1.49) and (1.27). The identity (1.52) follows from the first part of the proof.

If instead V_- is not a.e. equal to 0, we define for $\lambda \in [0, 1]$,

$$V_\lambda = (1-\lambda)V - \lambda V_- = V - \lambda V_+ \ \big(\in L_c^\infty(\mathbb{R}^d)\big), \tag{1.59}$$

so that V_λ increases to V as λ decreases to 0. Note that

(1.60)
$$2\langle V, \varphi\rangle + \langle V\varphi, \varphi\rangle - \mathcal{E}(\varphi,\varphi) \ge 2\langle V_\lambda, \varphi\rangle + \langle V_\lambda\varphi, \varphi\rangle - \mathcal{E}(\varphi,\varphi) + A_\lambda,$$
where
$$A_\lambda = \inf_\varphi \lambda\{2\langle V_+, \varphi\rangle + \langle V_+\varphi, \varphi\rangle\} \ge \lambda \inf_{u\ge 0}\{-2\langle V_+, 1\rangle^{\frac12} u + u^2\} > -\infty$$

(in the second line, we used the Cauchy-Schwarz Inequality in $L^2(V_+dy)$ to write that $\langle V_+, \varphi\rangle \ge -\langle V_+, 1\rangle^{\frac12}\langle V_+\varphi, \varphi\rangle^{\frac12}$, and took a lower bound over $u = \langle V_+\varphi, \varphi\rangle^{\frac12}$).

Since $\Gamma(V) < \infty$ by assumption, we see that

$$\Gamma(V_\lambda) < \infty, \text{ for } \lambda \in [0,1]. \tag{1.61}$$

Moreover, for $0 < \lambda \leq 1$, we have

(1.62)
$$2\langle V, \varphi\rangle + \langle V\varphi, \varphi\rangle - \mathcal{E}(\varphi,\varphi) \geq -2\langle V_-, \varphi\rangle + \langle V_\lambda\varphi, \varphi\rangle - \mathcal{E}(\varphi,\varphi) + B_\lambda,$$
$$\text{where } B_\lambda = \inf_\varphi 2\langle V_+, \varphi\rangle + \lambda\langle V_+\varphi, \varphi\rangle > -\infty,$$

by a similar argument as below (1.60). This shows that

$$\infty > \sup_{\varphi \in L^2(\mathbb{R}^d)} \{-2\langle V_-, \varphi\rangle + \langle V_\lambda\varphi, \varphi\rangle - \mathcal{E}(\varphi,\varphi)\}. \tag{1.63}$$

In addition, by (1.61) and the argument below (1.57), the Schrödinger semi-group $(R_t^{V_\lambda})_{t\geq 0}$ is a strongly continuous semi-group of self-adjoint contractions on $L^2(\mathbb{R}^d)$. From the argument below (1.54), we see that the above supremum in (1.63) equals $\lim_{\varepsilon\to 0}\langle V_-, G_{V_\lambda - \varepsilon} V_-\rangle$ $= \langle V_-, G_{V_\lambda} V_-\rangle$ (using monotone convergence for the last equality), and this quantity is finite. Hence, V_λ satisfies (1.16) and (see below (1.58)) $\Lambda(V_\lambda) < \infty$. So, by the first part of the proof,

$$\Lambda(V_\lambda) = \Gamma(V_\lambda)(< \infty), \text{ for } 0 < \lambda \leq 1. \tag{1.64}$$

By monotone convergence in (1.50), we see that

$$\lim_{\lambda \downarrow 0} \Lambda(V_\lambda) = \Lambda(V). \tag{1.65}$$

On the other hand, by (1.51) and (1.59), $\Gamma(V_\lambda)$ is a supremum of affine functions of $\lambda \in [0,1]$, which is finite when $\lambda = 1$ (since $V_{\lambda=1} \leq 0$) and when $\lambda = 0$ (by assumption). Hence, it is a convex, lower semi-continuous, finite function on $[0,1]$, which is therefore continuous, so that

$$\lim_{\lambda \to 0} \Gamma(V_\lambda) = \Gamma(V). \tag{1.66}$$

This implies that (1.52) holds and completes the proof of Theorem 1.2.2. □

Remark 1.2.3. The proof of Theorem 1.2.2 is easily adapted to the case of continuous time random interlacements on a transient weighted graph E. In this set-up, see for instance Section 1 of [59], one has a countable, locally finite, connected graph, with vertex set E, endowed with non-negative symmetric weights $c_{x,y} = c_{y,x}$, for $x, y \in E$, which are positive exactly when x, y are distinct and $\{x, y\}$ is an edge of the graph. The induced continuous time random walk is assumed to be transient. It has exponential holding times of parameter 1, and its discrete skeleton has transition probability

$$p_{x,y} = \frac{c_{x,y}}{\lambda_x}, \text{ where } \lambda_x = \sum_{z \in E} c_{x,z}, \text{ for } x, y \in E. \tag{1.67}$$

The continuous time random interlacements on the weighted graph can now be defined as a Poisson point process on a space of doubly-infinite E-valued trajectories, tending to infinity at plus and minus infinite times, marked by their duration at each step, modulo time-shift, see Section 1 of [59]. The field of occupation-times at level $u \geq 0$ in $x \in E$ corresponds to

(1.68)
$$L_{x,u}(\omega) = \frac{1}{\lambda_x} \times \text{ the total duration spent at } x \text{ by trajectories modulo time-shift, with label at most } u \text{ in the cloud } \omega$$

(durations of successive steps of a trajectory are described by independent exponential variables of parameter 1, but occupation times at x get rescaled by λ_x^{-1}).

In this set-up, one introduces for $V\colon E \to \mathbb{R}$, finitely supported

(1.69)
$$\Lambda(V) = \frac{1}{u}\, \overline{\mathbb{E}}\Big[\exp\Big\{\sum_{x \in E} L_{x,u} V(x)\Big\}\Big] \text{ (this does not depend on } u > 0 \text{ by the corresponding calculation to (1.48)),}$$

and

$$(1.70)\qquad \Gamma(V) = \langle V, 1\rangle + \sup_{\varphi \in C_c(E)} \big\{2\langle V, \varphi\rangle + \langle V\varphi, \varphi\rangle - \mathcal{E}(\varphi, \varphi)\big\},$$

where $\langle f, g\rangle$ stands for $\sum_{x\in E} f(x)\, g(x)$ whenever the sum converges absolutely, 1 denotes the constant function equal to 1 on E, $C_c(E)$ stands for the set of finitely supported functions on E, and $\mathcal{E}$ for the Dirichlet form

$$(1.71)\qquad \mathcal{E}(f, f) = \frac{1}{2} \sum_{x,y\in E} c_{x,y}\big(f(y) - f(x)\big)^2 \,(\in [0, \infty])$$

for $f\colon E \to \mathbb{R}$. Let us mention that one can also replace $C_c(E)$ in (1.70) by the extended space of the Dirichlet form $\mathcal{E}$ (see Chapter 1 §5 of [29]). The proof of Theorem 1.2.2 adapted to the present context yields

Theorem 1.2.2'.

$$(1.72)\qquad \Lambda(V) = \Gamma(V), \text{ for all finitely supported } V\colon E \to \mathbb{R}.$$

We now derive some corollaries of Theorem 1.2.2, which will play an important role in the next section. As above (1.1), B is a closed box (i.e. a compact subset of $\mathbb{R}^d$ that is the product of d possibly different non-degenerate compact intervals in $\mathbb{R}$), and m_B, as above (1.2), the restriction of Lebesgue measure to B. We write $L^p(B)$ as a shorthand for $L^p(B, dm_B)$, when $1 \le p < \infty$. Given $\widetilde{\varphi} \in L^2(B)$, we introduce

$$(1.73)\qquad \widetilde{\mathcal{E}}_B(\widetilde{\varphi}, \widetilde{\varphi}) = \inf_{\varphi\in L^2(\mathbb{R}^d)} \{\mathcal{E}(\varphi, \varphi);\ \varphi = \widetilde{\varphi} \text{ a.e. on } B\},$$

the so-called *trace Dirichlet form* on B, see [29], pp. 265, 266. We will often drop the subscript B, when this causes no ambiguity. One knows that $\widetilde{\mathcal{E}}_B(\widetilde{\varphi}, \widetilde{\varphi})$ is finite precisely when $\widetilde{\varphi}$ is a.e. equal to the restriction to B of a quasi-continuous function $\overline{\varphi}$ in the extended Dirichlet space $\mathcal{F}_e$, and in this case $\varphi(y) = E_y[\overline{\varphi}(X_{H_B}),\ H_B < \infty]$ belongs to $\mathcal{F}_e$ and $\widetilde{\mathcal{E}}_B(\widetilde{\varphi}, \widetilde{\varphi}) = \mathcal{E}(\varphi, \varphi)$.

Let us also note that $L^1_+(B) \stackrel{\text{def}}{=} \{f \in L^1(B);\ f \ge 0,\ m_B\text{-a.e.}\}$ is a closed convex subset of $L^1(B)$ (endowed with the norm topology).

Corollary 1.2.4. *For $V \in L^\infty_c(\mathbb{R}^d)$ vanishing outside B, one has*

$$\text{(1.74)} \quad \Lambda(V) = \sup_{h \in L^1_+(B)} \Big\{ \int_B V h\, dm_B - \widetilde{\mathcal{E}}(\sqrt{h}-1, \sqrt{h}-1)\Big\}, \text{ and}$$

$$\text{(1.75)} \quad h \in L^1_+(B) \longrightarrow \widetilde{\mathcal{E}}(\sqrt{h}-1, \sqrt{h}-1) \in [0, +\infty] \text{ is a convex lower semi-continuous function.}$$

Proof. We consider $V \in L^\infty_c(\mathbb{R}^d)$ vanishing outside B. Then, by (1.52) we have

(1.76)
$$\begin{aligned}
\Lambda(V) &= \sup_{\varphi \in L^2(\mathbb{R}^d)} \Big\{ \int V\, dy + 2\int V\varphi\, dy + \int V\varphi^2 dy - \mathcal{E}(\varphi,\varphi)\Big\} \\
&= \sup_{\varphi \in L^2(\mathbb{R}^d)} \Big\{ \int V(1+\varphi)^2 dy - \mathcal{E}(\varphi,\varphi)\Big\} \\
&\stackrel{(1.73)}{=} \sup_{\widetilde{\varphi} \in L^2(B)} \Big\{ \int_B V(1+\widetilde{\varphi})^2 dm_B - \widetilde{\mathcal{E}}(\widetilde{\varphi},\widetilde{\varphi})\Big\}
\end{aligned}$$

(since $V = 0$ outside B).

Now, the function $\rho(u) = |1+u| - 1$ satisfies $\rho(0) = 0$ and $|\rho(u) - \rho(v)| \le |u - v|$, for $u, v \in \mathbb{R}$, and by [29], pp. 4, 5, we have for $\widetilde{\varphi} \in L^2(B)$, $\widetilde{\psi} = \rho(\widetilde{\varphi})$, $\widetilde{\mathcal{E}}\,(\widetilde{\psi},\widetilde{\psi}) \le \widetilde{\mathcal{E}}(\widetilde{\varphi},\widetilde{\varphi})$, and $(1+\widetilde{\psi})^2 = (1+\widetilde{\varphi})^2$. As a result, we find that

(1.77)
$$\Lambda(V) = \sup\Big\{ \int V(1+\widetilde{\psi})^2 dm_B - \widetilde{\mathcal{E}}(\widetilde{\psi},\widetilde{\psi});\ \widetilde{\psi} \in L^2(B),\ 1+\widetilde{\psi} \ge 0 \ \text{ a.e.}\Big\}$$

(the above argument shows that $\Lambda(V)$ is smaller or equal to the right-hand side of (1.77), but by the last line of (1.76), $\Lambda(V)$ is also bigger or equal to the right-hand side of (1.77)).

Setting $h = (1+\widetilde{\psi})^2$, we obtain a bijection between $\{\widetilde{\psi} \in L^2(B);\ 1+\widetilde{\psi} \ge 0, \text{ a.e.}\}$, and $L^1_+(B)$, and the claim (1.74) follows. As for

(1.75), it is proved by a similar argument as in ii), below (4.2.64), p. 135 of [22] (see also Theorem 6.2.1 and (1.3.18), (1.4.8) in [29]). □

The next corollary brings us a step closer to the identification of what will be the rate function of the large deviation principle, which we derive in the next section. We tacitly identify the set $L^\infty(B)$ of bounded measurable function on B with the set $\{V \in L^\infty_c(\mathbb{R}^d);\ V = 0 \text{ on } B^c\}$, and recall that (see below (1.1)) $C(B)$ stands for the space of continuous functions on B (identified with the set of functions vanishing outside B, with continuous restriction to B).

Corollary 1.2.5. *For $h \in L^1_+(B)$, one has*

$$\begin{aligned} \widetilde{\mathcal{E}}_B(\sqrt{h}-1,\sqrt{h}-1) &= \sup_{V\in L^\infty(B)} \Big\{\int V h\, dm_B - \Lambda(V)\Big\} \\ &= \sup_{V\in C(B)} \Big\{\int V h\, dm_B - \Lambda(V)\Big\}. \end{aligned} \tag{1.78}$$

Proof. Since $L^\infty(B, dm_B)$ is the dual of $L^1(B, dm_B)$, the first equality follows from (1.74), (1.75) and the duality formula in Theorem 2.2.15, p. 55 of [22], or Lemma 4.5.8, p. 152 of [20], together with the fact we can replace $L^\infty(B, dm_B)$ by $L^\infty(B)$, since the quantity under the supremum in the right-hand side of the first equality coincides for V and V', if $V = V'$, m_B-a.e. (recall (1.44)).

The quantity on the last line of (1.78) is obviously smaller or equal to the right-hand side of the first equality. Fix h in $L^1_+(B)$. Our claim will thus follow, once we show that for any $V \in L^\infty(B)$ with $\Lambda(V) < \infty$ and $\varepsilon > 0$, one can find $W \in C(B)$ such that

$$\int W h\, dm_B - \Lambda(W) \ge \int V h\, dm_B - \Lambda(V) - \varepsilon. \tag{1.79}$$

We thus pick $r > \frac{d}{2}$, as in Lemma 1.1.3, and choose $V_n \in C(B)$

(extended to be equal to 0 on B^c) such that

$$(1.80) \quad \begin{cases} \text{i)} & \|V_n - V\|_{L^r(\mathbb{R}^d)} \to 0, \\ \text{ii)} & V_n \to V \text{ a.e.}, \\ \text{iii)} & \|V_n\|_{L^\infty(\mathbb{R}^d)} \le \|V\|_{L^\infty(\mathbb{R}^d)}. \end{cases}$$

The construction of such a sequence can be performed as follows: one first multiplies V by the indicator function of a slightly smaller concentric box in B, and then uses convolution by a smoothing kernel to construct a sequence V_n of continuous functions vanishing on B^c, for which i) and iii) hold. One obtains ii) by extracting a suitable subsequence. Then Lemma 1.1.3 and (1.49) ensure that $\Lambda(V_n) \underset{n}{\longrightarrow} \Lambda(V)$, and moreover $\int V_n h\, dm_B \underset{n}{\longrightarrow} \int V h\, dm_B$, by dominated convergence. The claim (1.79) follows, and Corollary 1.2.5 is proved. □

1.3 Large deviations for occupation-time profiles of Brownian interlacements

In this section, we derive a large deviation principle for the occupation-time profile of Brownian interlacements at level $\alpha > 0$ in a box LB, as $L \to \infty$, cf. Corollary 1.3.3. Due to the scaling property (1.45) of $\mathcal{L}_\alpha$, this fixed level, large space problem (i.e. $\alpha > 0$ fixed, and $L \to \infty$) is converted into a fixed space, large level problem (i.e. L fixed, and $\alpha \to \infty$), which can be handled via general Cramér theory, see Chapter 6 §1 of [20], or Chapter 3 of [22], making use of subadditivity, see Theorem 1.3.2. The identification of the rate function relies heavily on the results of the previous section, in particular on Corollary 1.2.5.

Given a closed box B in $\mathbb{R}^d$ (see above (1.1)), we endow the space $M(B)$ of finite signed measures on B with the weak topology generated by $C(B)$ (the space of continuous functions on B). The set $M_+(B)$ (of positive measures on B) is a closed convex subset of $M(B)$. We introduce the function on $M_+(B)$ (see (1.50) for notation):

(1.81)
$$I_B(\mu) = \sup_{V \in C(B)} \Big\{ \int V\, d\mu - \Lambda(V) \Big\} \in [0,\infty], \text{ for } \mu \in M_+(B),$$

and when there is no ambiguity, we simply write $I(\cdot)$ in place of $I_B(\cdot)$.

As we now see, $I_B(\cdot)$ is closely related to $I_v(\cdot)$, $v > 0$, in (1.3).

Lemma 1.3.1. *($B \subseteq B'$ closed boxes in $\mathbb{R}^d$)*

(1.82) *$I_B(\cdot)$ is a convex, good rate function (i.e., it is convex, lower semi-continuous, and has compact level sets).*

If μ is the restriction to B of $\mu' \in M_+(B')$, one has

(1.83)
$$I_B(\mu) \le I_{B'}(\mu')\, .$$

Moreover,

(1.84)
$$\begin{aligned} &I_B(\mu) \\ &= +\infty, \textit{ if } \mu \in M_+(B) \textit{ is not absolutely continuous w.r.t. } m_B \\ &= \inf\Big\{\frac{1}{2} \int_{\mathbb{R}^d} |\nabla\varphi|^2 dy;\ \varphi \in H^1(\mathbb{R}^d),\ \varphi = \Big(\frac{d\mu}{dm_B}\Big)^{\frac{1}{2}} - 1 \textit{ a.e. on } B\Big\}, \\ &\quad \textit{if } \mu \textit{ is absolutely continuous with respect to } m_B, \end{aligned}$$

and

(1.85)
$$I_v(\cdot) = v\, I_B\Big(\frac{\cdot}{v}\Big), \textit{ for } v > 0 \textit{ (see (1.3) for notation).}$$

Proof. We begin with (1.82). The convexity and lower semi-continuity are direct consequences of (1.81). Moreover, when $\lambda > 0$ is small enough so that $\lambda \|G 1_B\|_{L^\infty(\mathbb{R}^d)} < 1$, we know that $\Lambda(\lambda 1_B) < \infty$, see

(1.46), and hence,

$$(1.86) \qquad I(\mu) \geq \lambda\langle \mu, 1_B\rangle - \Lambda(\lambda 1_B), \text{ for } \mu \in M_+(B).$$

Therefore, for any $M \geq 0$, $\langle \mu, 1_B\rangle$ remains bounded on $\{\mu \in M_+(B); I(\mu) \leq M\}$, which is therefore a compact subset of $M_+(B)$ (we already know it is closed by lower semi-continuity of I). This proves (1.82).

We now turn to (1.84). The second line of (1.84) is an immediate consequence of Corollary 1.2.5 and (1.73). To complete the proof of (1.84), it thus suffices to show that $I(\mu) = \infty$ when μ is not absolutely continuous with respect to m_B.

Indeed, for such a measure μ, we can find a compact subset K of B such that $\mu(K) > 0$, but $m(K) = 0$. Then, consider for $n \geq 1$, $W_n(\cdot) = (1 - nd(\cdot, K))_+ \in C(B)$ (we set W_n equal to zero on B^c), so that W_n is $[0,1]$-valued, takes the values 1 on K and vanishes outside the open $\frac{1}{n}$-neighborhood of K in B. Thus, choosing $r > \frac{d}{2}$, we see that $\delta_n = \|W_n\|_{L^r(\mathbb{R}^d)} \underset{n}{\longrightarrow} 0$, and setting $V_n = \delta_n^{-\frac{1}{2}} W_n$, it follows by Lemma 1.1.3 and (1.49) that $\Lambda(V_n) \underset{n}{\longrightarrow} 0$. As a consequence, we see that

$$(1.87) \quad I(\mu) \overset{(1.81)}{\geq} \int V_n d\mu - \Lambda(V_n) \geq \delta_n^{-\frac{1}{2}} \mu(K) - \Lambda(V_n) \underset{n}{\longrightarrow} +\infty.$$

This shows that $I(\mu) = \infty$ and completes the proof of (1.84).

We now prove (1.83). Without loss of generality, we can assume that $I_{B'}(\mu') < \infty$. By (1.84) this implies that μ' is absolutely continuous with respect to $m_{B'}$, and hence μ is absolutely continuous with respect to m_B. The inequality $I_B(\mu) \leq I_B(\mu')$ is now a direct consequence of (1.84). This proves (1.83). Finally, (1.85) is immediate by comparison of (1.84) and (1.3). □

We denote by $\mathcal{L}_{\alpha,B}$ the restriction of the (random) Radon measure $\mathcal{L}_\alpha$ on $\mathbb{R}^d$ to the Borel subsets of B. The main step towards the large deviation principle for $\nu_{L,\alpha}$, with $\alpha > 0$ fixed and $L \to \infty$, see (1.5), is a large deviation principle for $\frac{1}{\alpha}\mathcal{L}_{\alpha,B}$, as $\alpha \to \infty$, which we derive in Theorem 1.3.2 below. We use general Cramér theory, see

Theorem 6.1.3, p. 252 of [20] (with $\chi = M(B)$ and $\mathcal{E} = M_+(B)$, in the notation of [20]).

Theorem 1.3.2. *(B a closed box)*

(1.88) *As $\alpha \to \infty$, the laws under $\mathbb{P}$ of $\frac{1}{\alpha}\,\mathcal{L}_{\alpha,B}$ on $M_+(B)$ satisfy a large deviation principle at speed α, with the convex good rate function I_B from (1.81).*

Moreover, for any open convex subset O of $M(B)$,

(1.89)
$$\lim_{\alpha\to\infty} \frac{1}{\alpha}\, \log \mathbb{P}\Big[\frac{1}{\alpha}\,\mathcal{L}_{\alpha,B} \in O\Big] = -\inf\{I_B(\mu);\ \mu \in O \cap M_+(B)\}$$

(with $\inf \phi = \infty$, by convention).

Proof. For any open convex subset O of $M(B)$, the function

(1.90)
$$f_O(\alpha) = -\log \mathbb{P}\Big[\frac{1}{\alpha}\,\mathcal{L}_{\alpha,B} \in O\Big] \in [0,+\infty], \text{ for } \alpha > 0,$$

is subadditive (since $\mathcal{L}_{\alpha,B}$ is an $M_+(B)$-valued Lévy-process). When $O \cap M_+(B) = \phi$, then f_O is identically infinite. Otherwise, when $O \cap M_+(B) \neq \phi$, then, as we now explain, for $\varepsilon \in (0,1]$ small,

(1.91)
$$\sup_{1\le\alpha\le 1+\varepsilon} f_O(\alpha) < \infty.$$

Indeed, for small $\varepsilon > 0$, there is a positive $\mathbb{P}$-probability that exactly one trajectory of the interlacement at level 1 enters B (i.e. $\mu_{B,\alpha=1} = 1$, in the notation of (1.40)) and that $\mathcal{L}_{1,B} \in \alpha O$, for all $1 \le \alpha \le 1+\varepsilon$ (this can be arranged, using the fact that O is open, $O\cap M_+(B) \neq \phi$, with the help of the support theorem for the Wiener measure, and the observation that once reaching distance 1 from B, a Brownian trajectory has a non-degenerate probability of never returning to B). Since $\mathcal{L}_{1+\varepsilon,B} - \mathcal{L}_{1,B}$ is independent from $\mathcal{L}_{1,B}$, and vanishes with positive probability, we see that on an event of positive $\mathbb{P}$-probability $\frac{1}{\alpha}\,\mathcal{L}_{\alpha,B} \in O$, for $1 \le \alpha \le 1+\varepsilon$. This is more than enough to prove (1.91).

Hence, by Lemma 4.2.5, p. 112 of [22], we see that when $O \cap$

$M_+(B) \neq \phi$,

$$\begin{aligned} &\lim_{\alpha\to\infty} \frac{1}{\alpha}\, f_O(\alpha) \text{ exists and is finite} \\ &\Big(\text{and equals } \inf_{\alpha\geq\alpha_0} \frac{f_O(\alpha)}{\alpha} \text{ for large } \alpha_0\Big). \end{aligned} \tag{1.92}$$

Moreover, choosing λ as above (1.86), the Chebyshev Inequality and (1.46) imply that

$$\mathbb{P}[\mathcal{L}_\alpha(B) \geq \alpha M] \leq \exp\big\{ -\alpha\big(\lambda M - \Lambda(\lambda 1_B)\big)\big\}, \text{ for } M > 0,\, \alpha > 0.$$

Since $\{\mu \in M_+(B);\, \mu(B) \leq M\}$ is compact for each $M > 0$, and $\lambda M - \Lambda(\lambda 1_B) \underset{M}{\longrightarrow} \infty$, the exponential tightness of the laws of $\frac{1}{\alpha}\, \mathcal{L}_{\alpha,B}$ follows. Thus, by Theorem 4.1.11, p. 120 of [20], and p. 8 of the same reference, we see that, as $\alpha \to \infty$, the laws of $\frac{1}{\alpha}\, \mathcal{L}_{\alpha,B}$ satisfy a large deviation principle at speed α. Restricting α to integer values, it follows from Theorem 6.1.3, p. 252 of [20] (Assumption 6.1.2 is straightforward to check in our set-up) that the rate function of the above large deviation principle coincides with I_B in (1.81) and that the limit in (1.92) coincides with the right-hand side of (1.89) (when $O \cap M_+(B) = \phi$, both members of (1.92) equal $+\infty$). This concludes the proof of Theorem 1.3.2. □

The large deviation principle for the profile $\nu_{L,\alpha}$ of the occupation-time measure of Brownian interlacements at level α in a large box LB is now a direct consequence of Theorem 1.3.2 and the scaling property (1.45). We refer to (1.3), (1.5), and above (1.1) for notation.

Corollary 1.3.3. *(Large deviation principle for the profile, B a closed box, $\alpha > 0$)*

(1.93) *As $L \to \infty$, the laws under $\mathbb{P}$ of $\nu_{L,\alpha}$ on $M_+(B)$ satisfy a large deviation principle with speed L^{d-2} and rate function I_α.*

Moreover, for any open convex subset O of $M(B)$,

$$(1.94)\quad \lim_{L\to\infty} \frac{1}{L^{d-2}} \log \mathbb{P}[\nu_{L,\alpha} \in O] = -\inf\{I_\alpha(\mu);\ \mu \in O \cap M_+(B)\}$$

(with $\inf \phi = +\infty$ *by convention).*

Proof. By the scaling property (1.45), we see that $\nu_{L,\alpha}$ has the same law as $\widetilde{\nu}_L = \frac{1}{L^{d-2}} \mathcal{L}_{\alpha\, L^{d-2},B}$, for any $L \geq 1$. Our claims are now direct consequences of Theorem 1.3.2. □

1.4 The discrete space set-up

In this section, we introduce some additional notation concerning continuous time random interlacements on $\mathbb{Z}^d$, $d \geq 3$. We recall the scaling limit relating the discrete space occupation-times to the occupation-time measure of Brownian interlacements established in Theorem 3.2 of [60], see (1.98) below. Further, as a preparation to the large deviation principle for $\rho_{N,u}$ (see (1.1)), which we derive in the next section, we collect here the statements corresponding to Theorem 1.2.2' and Corollaries 1.2.4, 1.2.5 in the present set-up.

Given $N \geq 1$, we introduce the scaled lattice

$$(1.95)\quad \mathbb{L}_N = \frac{1}{N}\, \mathbb{Z}^d (\subseteq \mathbb{R}^d),$$

and B being a closed box (see above (1.1)), we set

$$(1.96)\quad B_N = B \cap \mathbb{L}_N.$$

For functions f, h on $\mathbb{L}_N$ such that $\sum_{y\in\mathbb{L}_N} |f(y)\, h(y)| < \infty$, we write

$$(1.97)\quad \langle f, h\rangle_{\mathbb{L}_N} = \frac{1}{N^d} \sum_{y\in\mathbb{L}_N} f(y)\, h(y),$$

and we introduce the spaces $L^p(\mathbb{L}_N)$, $1 \leq p < \infty$, and their corresponding norms in a similar manner.

We refer to [59] for the precise construction of continuous time random interlacements on $\mathbb{Z}^d$ (with $d \geq 3$). As mentioned in the introduction, we denote by $(\overline{\Omega}, \overline{\mathcal{A}}, \overline{\mathbb{P}})$ the canonical space on which they are constructed, and by $(L_{x,u})_{x \in \mathbb{Z}^d}$ the field of occupation-times of random interlacements at level $u \geq 0$. Fixing $u \geq 0$, this is a stationary field on $\mathbb{Z}^d$, and by the same proof as for (2.7) of [56], it is ergodic. One knows that $\overline{\mathbb{E}}[L_{x,u}] = u$, for $x \in \mathbb{Z}^d$, $u \geq 0$, so by the ergodic theorem, see Theorem 2.8, p. 205 of [36], one sees that

$$\text{(1.98)} \qquad \begin{array}{l} \text{for } u \geq 0,\ \overline{\mathbb{P}}\text{-a.s., } \frac{1}{N^d} \sum\limits_{y \in \mathbb{L}_N} L_{Ny,u}\, \delta_y \\ \text{converges vaguely to } u\, dy, \text{ as } N \to \infty \end{array}$$

(and $\rho_{N,u}$ converges weakly to $u\, m_B$, in the notation of (1.1)).

The occupation-time measure of Brownian interlacements can be expressed as a scaling limit of the occupation times of random interlacements on $\mathbb{Z}^d$. Namely, one know by Theorem 3.2 of [60] that for $\alpha \geq 0$, as $N \to \infty$,

(1.99)

$$\mathcal{L}_{N,\alpha} \overset{\text{def}}{=} \frac{1}{dN^2} \sum_{y \in \mathbb{L}_N} L_{Ny, d\alpha N^{2-d}}\, \delta_y \text{ converges in distribution to } \mathcal{L}_\alpha$$

(we endow the set of Radon measures on $\mathbb{R}^d$ with the topology of vague convergence).

For $N \geq 1$ and $V \colon \mathbb{L}_N \to \mathbb{R}$ with finite support, we introduce

$$\text{(1.100)} \qquad \Lambda_N(V) = \log \overline{\mathbb{E}}[e^{\langle \mathcal{L}_{N,1}, V \rangle}] \in (-\infty, +\infty],$$

and note that

(1.101)

$$\Lambda_N(V) = \frac{d}{N^{d-2}} \log \overline{\mathbb{E}}[e^{\frac{N^{d-2}}{d} \langle \rho_{N,u=1}, V \rangle}], \text{ when } V \text{ vanishes outside } B_N$$

(where we used (1.1) and the fact that the right-hand side of (1.69) does not depend on u).

To state the identity corresponding to Theorem 1.2.2' in the present set-up, we define, for $N \geq 1$ and $V \colon \mathbb{L}_N \to \mathbb{R}$ finitely supported,

(1.102)

$$\Gamma_N(V) = \langle V, 1 \rangle_{\mathbb{L}_N} + \sup_{\varphi \in L^2(\mathbb{L}_N)} \{2 \langle V, \varphi \rangle_{\mathbb{L}_N} + \langle V\varphi, \varphi \rangle_{\mathbb{L}_N} - \mathcal{E}_N(\varphi, \varphi)\},$$

where for $\varphi \in L^2(\mathbb{L}_N)$

$$(1.103)\quad \mathcal{E}_N(\varphi,\varphi) = \frac{1}{2N^{d-2}} \sum_{y\sim y' \text{ in } \mathbb{L}_N} \frac{1}{2}\,\big(\varphi(y') - \varphi(y)\big)^2 \in [0,\infty]$$

($y \sim y'$ means that y and y' are neighbors in $\mathbb{L}_N$, i.e. $|y-y'| = \frac{1}{N}$).

We also keep the notation $\mathcal{E}_N(\varphi,\varphi)$ when φ belongs to the extended space of the Dirichlet form $\mathcal{E}_N$ (corresponding to functions on $\mathbb{L}_N$ that are pointwise limits of an $\mathcal{E}_N$-Cauchy sequence of finitely supported functions on $\mathbb{L}_N$). After proper normalization (the V in Theorem 1.2.2' corresponds to $\frac{1}{dN^2}\,V(\frac{\cdot}{N})$), Theorem 1.2.2' now yields

Theorem 1.4.1. $(N \ge 1)$

(1.104) $\Lambda_N(V) = \Gamma_N(V)$, *for all* $V\colon \mathbb{L}_N \to \mathbb{R}$ *with finite support.*

Before stating the corollary corresponding to Corollaries 1.2.4, 1.2.5 in the present set-up, we define for ψ: $B_N \to \mathbb{R}$,

$$(1.105)\quad \widetilde{\mathcal{E}}_N(\psi,\psi) = \inf\{\mathcal{E}_N(\varphi,\varphi);\ \varphi \in L^2(L_N),\ \varphi = \psi \text{ on } B_N\}.$$

Denoting by P_y^N, for $N \ge 1$, $y \in \mathbb{L}_N$, the canonical law of the simple random walk on $\mathbb{L}_N$ with exponential holding times of parameter N^2, starting at $y \in \mathbb{L}_N$, and using similar notation for the canonical process, the entrance times, and the exit times, as described at the beginning of Section 1, one knows, as below (1.73), that

$$(1.106)\quad \widetilde{\mathcal{E}}_N(\psi,\psi) = \mathcal{E}_N(\varphi,\varphi) \in [0,\infty),$$

where $\varphi(y) = E_y^N[\psi(X_{H_{B_N}}), H_{B_N} < \infty]$, for $y \in \mathbb{L}_N$, is harmonic outside B_N, tends to zero at infinity, and belongs to the extended Dirichlet space of $\mathcal{E}_N$ (with H_{B_N} the entrance time in B_N).

Similarly to Corollaries 1.2.4 and 1.2.5 (but in a much simpler fashion), we have

Corollary 1.4.2. $(N > 1)$

When $V\colon \mathbb{L}_N \to \mathbb{R}$ *vanishes outside* B_N

$$(1.107)\quad \Lambda_N(V) = \sup_{h \ge 0 \text{ on } B_N} \{\langle V, h\rangle_{\mathbb{L}_N} - \widetilde{\mathcal{E}}_N(\sqrt{h}-1, \sqrt{h}-1)\} \in [0,\infty],$$

and for $h\colon B_N \to \mathbb{R}_+$ *(extended to be equal to* 0 *outside* B_N*)*

(1.108) $$\widetilde{\mathcal{E}}_N(\sqrt{h}-1,\sqrt{h}-1) = \sup_{V=0 \text{ on } \mathbb{L}_N \setminus B_N} \{\langle V, h\rangle_{\mathbb{L}_N} - \Lambda_N(V)\}.$$

1.5 Large deviations for occupation-time profiles of random interlacements

The main object of this section is to prove a large deviation principle for the occupation-time profile $\rho_{N,u}$, see (1.1), of continuous time random interlacements on $\mathbb{Z}^d$ at level u, when $N \to \infty$, cf. Theorem 1.116. Subadditivity is an important ingredient in our proof, see Proposition 1.5.2 and Corollary 1.5.3. The lower bound in the large deviation principle appears in Theorem 1.5.4. It relies on Corollary 1.5.3, and on the combination of the large deviation principle for the occupation-time profile of Brownian interlacements proved in Section 3, and the scaling limit result (1.99) proved in [60], see also Lemma 1.5.1 below. The upper bound appears in Theorem 1.113. It relies on Corollary 1.5.3 and on Proposition 1.5.6, which provides an asymptotic lower bound for a sequence of discrete variational problems, in the spirit of Γ-convergence, see Proposition 7.2, p. 68 of [19].

We pick $u > 0$ and a closed box B (see above (1.1)). The space $M(B)$ is equipped with the weak topology generated by $C(B)$, as explained in Section 3. We consider $\nu \in M_+(B)$, a finite collection $f_\ell \in C(B)$, $1 \le \ell \le K$, with $f_1 = 1_B$, and a number $\delta > 0$. We define the convex open subset A of $M_+(B)$ consisting of positive measures on B with integrals with respect to f_ℓ, $1 \le \ell \le K$, δ-close to the corresponding integrals with respect to ν, and we denote by O the homothetic image of A with ratio u:

(1.109)
$$A = \{\rho \in M_+(B); |\langle \rho, f_\ell\rangle - \langle \nu, f_\ell\rangle| < \delta, \text{ for } 1 \le \ell \le K\}, \text{ and } O = uA.$$

We use the shorthand notation $c(A)$ to denote a positive constant, which depends on $d, B, \nu, K, (f_\ell)_{1\le \ell\le K}, \delta$. The collection of sets A (or O) above constitutes a base for the relative topology on $M_+(B)$ (viewed as a subset of $M(B)$).

We also define for $N \geq 1$, $t \geq 0$ (see (1.99) for notation)

$$\widetilde{\mathcal{L}}_{N,t} = \text{the restriction of } \mathcal{L}_{N,t} \text{ to Borel subsets of } B \tag{1.110}$$

(a random element of $M_+(B)$). We now state a consequence of (1.99).

Lemma 1.5.1. ($\alpha \geq 0$)

(1.111) *As $N \to \infty$, $\widetilde{\mathcal{L}}_{N,\alpha}$ converges in distribution to $\mathcal{L}_{\alpha,B}$ (see above (1.88) for notation).*

Proof. By (1.99) and the continuous mapping theorem, see Theorem 5.1, p. 30 of [10], it suffices to show that (denoting the set of Radon measures on $\mathbb{R}^d$ by $\mathcal{M}_+(\mathbb{R}^d)$):

(1.112) the set of continuity points of the map $\rho \in \mathcal{M}_+(\mathbb{R}^d) \to \widetilde{\rho} \in M_+(B)$, where $\widetilde{\rho}$ is the restriction of ρ to Borel subsets of B, has full measure under the law of $\mathcal{L}_\alpha$.

To see this point, note that by (1.44), for a.e. ρ under the law of $\mathcal{L}_\alpha$, one has $\rho(\partial B) = 0$. Hence, for any such ρ, for any $V \in C(B)$ (extended as 0 outside B), and any sequence ρ_n converging vaguely to ρ, one has $\langle \rho_n, V\rangle \underset{n}{\longrightarrow} \langle \rho, V\rangle$. This proves that $\widetilde{\rho}_n$ converges weakly to $\widetilde{\rho}$ in $M_+(B)$ and (1.112) follows. □

We then introduce

$$f_{N,A}(t) = -\log \overline{\mathbb{P}}\Big[\frac{1}{t}\, \widetilde{\mathcal{L}}_{N,t} \in A\Big], \text{ for } t > 0, N \geq 1. \tag{1.113}$$

Since $t \to \widetilde{\mathcal{L}}_{N,t}$ has independent, stationary, increments, and A is convex,

$$\text{for each } N \geq 1,\ f_{N,A}(\cdot) \text{ is subadditive.} \tag{1.114}$$

The next proposition collects some important bounds on $\overline{\mathbb{P}}[\frac{1}{t}\mathcal{L}_{N,t} \in A]$, which exploit subadditivity. It comes as a step towards Corollary

1.5.3 below. We recall the convention on constants stated below (1.109).

Proposition 1.5.2. *When $N \geq c_0(A)$, then for $t_1 \geq c(A)$ and $t \geq 2t_1$, one has*

$$\tag{1.115} \begin{array}{l} \exp\Big\{-t\,\dfrac{f_{N,A}(t_1)}{t_1} - 2t_1\,c'(A)\Big\} \\ \leq \overline{\mathbb{P}}\Big[\frac{1}{t}\,\widetilde{\mathcal{L}}_{N,t} \in A\Big] \leq \exp\Big\{-t \lim\limits_{s\to\infty} \dfrac{f_{N,A}(s)}{s}\Big\} \end{array}$$

(and the limit in the rightmost term of (1.115) exists and is finite).

Proof. We first show that for some $\varepsilon_A = 1/q_A$, where q_A is some positive integer depending on A (with a similar meaning as below (1.109)) and for some $N_0(A)$

$$\tag{1.116} \sup_{N\geq N_0(A)}\ \sup_{t\in[1,1+\varepsilon_A]} f_{N,A}(t) = M < \infty.$$

To see this point, we introduce A' defined as A in (1.109) with δ replaced by $\frac{\delta}{2}$. By (1.111) and (1.91), we know that $\liminf_N \overline{\mathbb{P}}[\widetilde{\mathcal{L}}_{N,1} \in A'] \geq \mathbb{P}[\mathcal{L}_{1,B} \in A'] > 0$. Hence, for $N \geq N_0(A)$, $\overline{\mathbb{P}}[\widetilde{\mathcal{L}}_{N,1} \in A'] \geq c(A)$. Then, the probability that no trajectory of the interlacement with label in $(dN^{2-d}, d(1+\varepsilon)N^{2-d}]$ enters NB_N is equal to $e^{-d\varepsilon N^{2-d}\mathrm{cap}_{\mathbb{Z}^d}(NB_N)} \geq e^{-c(B)\varepsilon}$ (with $\mathrm{cap}_{\mathbb{Z}^d}(\cdot)$ the capacity on $\mathbb{Z}^d$, see for instance (1.57) of [56]). Such an event is independent under $\overline{\mathbb{P}}$ of $\{\widetilde{\mathcal{L}}_{N,1} \in A'\}$. When $\varepsilon = \varepsilon_A = 1/q_A$ with q_A a large enough integer, so that the set of multiples by a scalar in $[(1+\varepsilon_A)^{-1}, 1]$ of a measure in A' is contained in A, on the intersection of these two events, one has $\frac{1}{t}\,\widetilde{\mathcal{L}}_{N,t} = \frac{1}{t}\,\widetilde{\mathcal{L}}_{N,1} \in A$, for all $1 \leq t \leq 1+\varepsilon_A$. The claim (1.116) follows.

We can now apply Lemma 4.2.5, p. 112 of [22], and find that when $t_1 \geq q_A$, then for $t \geq 2t_1$ and $N \geq N_0(A)$, one has (with $[\cdot]$ denoting the integer part)

$$\tag{1.117} f_{N,A}(t) \leq \Big(\Big[\frac{t}{t_1}\Big] - 1\Big)\, f_{N,A}(t_1) + 2t_1\,M \leq t\,\frac{f_{N,A}(t_1)}{t_1} + 2t_1 M.$$

This proves the first inequality in (1.115).

As for the second inequality, we know by the same Lemma 4.2.5 of [22], that the limit in the rightmost term of (1.115) exists, is finite, and that moreover, for $t \geq q_A$,

$$\lim_{s \to \infty} \frac{f_{N,A}(s)}{s} \leq \frac{f_{N,A}(t)}{t}.$$

This implies the second inequality in (1.115) (and one can choose $c(A) = q_A$ in (1.115)). □

For fixed $N \geq 1$, the finite dimensional space of signed measures spanned by the basis $\frac{1}{N^d}\, \delta_y, y \in B_N$, is a closed subspace of $M(B)$, and the corresponding coordinates yield an homeomorphism with $\mathbb{R}^{B_N}$. The intersection of this space with $M_+(B)$ consists of linear combinations of $\frac{1}{N^d}\, \delta_y, y \in B_N$, with non-negative coefficients, and defines a closed convex subset of $M_+(B)$. The finite convex lower semi-continuous (it is actually continuous) function on $\mathbb{R}_+^{B_N}$ in (1.108), extended to be equal to $+\infty$ on the complement in $M_+(B)$ of its domain of definition, yields a convex rate function (i.e. a convex, lower semi-continuous, $[0, +\infty]$-valued function) denoted by

$$(1.118) \qquad \begin{aligned} I_N(\mu) &= \widetilde{\mathcal{E}}_N(\sqrt{h} - 1, \sqrt{h} - 1), \text{ if } \mu = \frac{1}{N^d} \sum_{y \in B_N} h(y)\, \delta_y, \\ &\qquad\qquad \text{with } h\colon B_N \to \mathbb{R}_+, \\ &= +\infty, \text{ otherwise.} \end{aligned}$$

We can now state a consequence of Proposition 1.5.2. The difference lies in the rightmost inequality, which now involves the functional I_N. The next Corollary 1.5.3 encapsulates the sub-additivity lower and upper bounds, which we will respectively use in the proofs of Theorem 1.112 and of Theorem 1.5.5.

Corollary 1.5.3. *(with A as in (1.109))*

When $N \ge N_0(A)$, for $t_1 \ge c(A)$ and $t \ge 2t_1$, one has

$$\begin{aligned}(1.119)\qquad &\exp\Big\{-t\,\frac{f_{N,A}(t_1)}{t_1} - 2t_1\,c'(A)\Big\} \\ \le\ &\overline{\mathbb{P}}\Big[\frac{1}{t}\,\widetilde{\mathcal{L}}_{N,t} \in A\Big] \le \exp\big\{-t\,\inf_{\rho\in A}\,I_N(\rho)\big\}.\end{aligned}$$

Proof. Recall that $t \to \widetilde{\mathcal{L}}_{N,t}$ has independent, stationary increments. By Theorem 6.1.3, p. 252 of [20], and (1.100), (1.108), we know that for each fixed $N \ge 1$, the laws of $\frac{1}{n}\,\widetilde{\mathcal{L}}_{N,n}$ (on $M_+(B)$), as $n \to \infty$, satisfy a weak large deviation principle at speed n, with convex rate function I_N, see (1.118). Moreover, by the same reference, $\lim_n -\frac{f_{N,A}(n)}{n} = -\inf_A I_N$ ($\in [-\infty, 0]$). The comparison with (1.115) yields (1.119). □

We can now derive the asymptotic lower bound.

Theorem 1.5.4. *(Large deviation lower bound, $u > 0$, O as in (1.109), see (1.3) for notation)*

$$(1.120)\qquad \underline{\lim}_N\,\frac{1}{N^{d-2}}\,\log\overline{\mathbb{P}}[\rho_{N,u} \in O] \ge -\inf_{\mu\in O}\,\frac{1}{d}\,I_u(\mu).$$

Proof. We first use sub-additivity. We set $t = \frac{u}{d}\,N^{d-2}$, so that $\rho_{N,u} = \frac{u}{t}\,\widetilde{\mathcal{L}}_{N,t}$ (see (1.1) and (1.99), (1.110)), we see by the first inequality of (1.119) that for any $t = \frac{u}{d}\,N^{d-2} \ge 2t_1 \ge c(A)$ and $N \ge N_0(A)$ (recall $O = uA$),

$$\overline{\mathbb{P}}[\rho_{N,u} \in O] \overset{(1.109)}{=} \overline{\mathbb{P}}\Big[\frac{1}{t}\,\widetilde{\mathcal{L}}_{N,t} \in A\Big] \ge \exp\Big\{-t\,\frac{f_{N,A}(t_1)}{t_1} - 2t_1\,c'(A)\Big\}.$$

Hence, for any $t_1 \ge c(A)$,

$$(1.121)\qquad \underline{\lim}_N\,\frac{1}{N^{d-2}}\,\log\overline{\mathbb{P}}[\rho_{N,u} \in O] \ge -\frac{u}{d}\,\overline{\lim}_N\,\frac{f_{N,A}(t_1)}{t_1}\,.$$

We now use convergence in law, and note that by Lemma 1.5.1

$$\underline{\lim}_N\,\overline{\mathbb{P}}[\widetilde{\mathcal{L}}_{N,t_1} \in t_1 A] \ge \mathbb{P}[\mathcal{L}_{t_1,B} \in t_1 A].$$

Taking logarithms (recall the notation from (1.113) and (1.90)), we find that

$$-\overline{\lim_N} f_{N,A}(t_1) \ge -f_A(t_1).$$

We are now in position to use the key large deviation result from Section 3. Specifically, coming back to (1.121) we see that for $t_1 \ge c(A)$

$$\begin{aligned} &\underline{\lim_N} \, \frac{1}{N^{d-2}} \log \overline{\mathbb{P}}[\rho_{N,u} \in O] \\ &\ge -\frac{u}{d} \, \frac{f_A(t_1)}{t_1} \underset{t_1 \to \infty}{\overset{(1.89)}{\longrightarrow}} -\frac{u}{d} \inf_{\rho \in A} I_B(\rho) \overset{(1.85)}{=} -\inf_{\mu \in O} \frac{1}{d} \, I_u(\mu)\,. \end{aligned} \tag{1.122}$$

This proves Theorem 1.5.4. □

We now turn to the proof of the asymptotic upper bound. An argument based on the convergence of $\Lambda_N(V)$ to $\Lambda(V)$ (or on the asymptotic domination of $\Lambda_N(V)$ by $\Lambda(V)$), for all V in $C(B)$ does not seem straightforward. Instead, we use a strategy, which is in the spirit of Γ-convergence for the functions I_N, see Proposition 1.5.6 below, and Proposition 7.2, p. 68 of [19]. Another possible, although somewhat indirect, route might be to use the isomorphism theorem of [59] and the large deviation principles on the empirical distribution functional of the Gaussian free field on $\mathbb{Z}^d$, proved in [11].

Theorem 1.5.5. *(Large deviation upper bound, $u > 0$, O as in (1.109))*

$$\overline{\lim_N} \, \frac{1}{N^{d-2}} \, \log \overline{\mathbb{P}}[\rho_{N,u} \in O] \le -\inf_{\mu \in \overline{O}} \frac{1}{d} \, I_u(\mu) \tag{1.123}$$

(with $\overline{O}$ the closure of O in $M_+(B)$).

Proof. We first exploit sub-additivity. Setting $t = \frac{u}{d} \, N^{d-2}$, we know by Corollary 1.5.3 that for $t \ge 2c(A)$, $N \ge N_0(A)$ (recall $O = uA$),

$$\overline{\mathbb{P}}[\rho_{N,u} \in O] \overset{(1.109)}{=} \overline{\mathbb{P}}\Big[\, \frac{1}{t} \, \widetilde{\mathcal{L}}_{N,t} \in A\Big] \overset{(1.119)}{\le} \exp\big\{-t \inf_{\rho \in A} \; I_N(\rho)\big\}.$$

Hence, we find that

$$\overline{\lim_N} \frac{1}{N^{d-2}} \log \overline{\mathbb{P}}[\rho_{N,u} \in O] \le -\frac{u}{d} \lim_N \inf_{\rho \in A} I_N(\mu). \tag{1.124}$$

The proof of (1.123) relies on the crucial next proposition.

Proposition 1.5.6. *(K compact subset of $M_+(B)$, see (1.81), (1.118) for notation)*

$$\underline{\lim_N} \inf_{\rho \in K} I_N(\rho) \ge \inf_{\rho \in K} I(\rho). \tag{1.125}$$

Let us admit Proposition 1.5.6 for the time being, and first complete the proof of Theorem 1.5.5. By (1.124) and (1.125) (with $K = \overline{A}$), we see that

$$\overline{\lim_N} \frac{1}{N^{d-2}} \log \overline{\mathbb{P}}[\rho_{N,u} \in O] \le -\frac{u}{d} \inf_{\overline{A}} I \underset{(1.109)}{\overset{(1.85)}{=}} -\inf_{\overline{O}} \frac{1}{d} I_u.$$

This proves Theorem 1.5.5 (conditionally on Proposition 1.5.6). □

There remains to prove Proposition 1.5.6.

It may be useful at this point to provide an outline of its proof. In essence, assuming that the left-hand side of (1.125) is finite (otherwise (1.125) is obvious), we will consider a minimizing subsequence μ_ℓ in $K(\subseteq M_+(B))$ such that $I_{N_\ell}(\mu_\ell)$ tends to the left-hand side of (1.125) and N_ℓ tends to infinity. By the relationship between I_{N_ℓ} and $\mathcal{E}_{N_\ell}$ from (1.118) and (1.106), we will recast $I_{N_\ell}(\mu_\ell)$ as $\mathcal{E}_{N_\ell}(\varphi_\ell, \varphi_\ell)$, where φ_ℓ are functions on $\mathbb{L}_{N_\ell}$, harmonic outside B_{N_ℓ}, tending to zero at infinity, with value at least -1 in B_{N_ℓ}, and such that μ_ℓ has density $1_{B_{N_\ell}}(1+\varphi_\ell)^2 N_\ell^{-d}$ with respect to the counting measure on the scaled lattice $\mathbb{L}_{N_\ell}$. With the help of a cut-off lemma (see Lemma 1.5.7 below), we will replace the sequence (φ_ℓ) by a sequence $(\overline{\varphi}_\ell)$. In particular, $\overline{\varphi}_\ell$ will coincide with φ_ℓ on B_{N_ℓ}, but will vanish outside a fixed compact set, independent of ℓ, and $\mathcal{E}_{N_\ell}(\overline{\varphi}_\ell, \overline{\varphi}_\ell)$ will not be substantially bigger than $\mathcal{E}_{N_\ell}(\varphi_\ell, \varphi_\ell)$. We will then introduce a step function Φ_ℓ on $\mathbb{R}^d$, constant on cubes of side-length $\frac{1}{N_\ell}$, which coincides with $\overline{\varphi}_\ell$ on $\mathbb{L}_{N_\ell}$. Making use of the controls on $\overline{\varphi}_\ell$ stemming from the Dirichlet form and the compact support of the functions

Φ_ℓ, we will show that Φ_ℓ, $\ell \geq 1$, is relatively compact in $L^2(\mathbb{R}^d)$. We will extract a convergent subsequence to a compactly supported Φ in $L^2(\mathbb{R}^d)$, having value at least -1 on B, such that the measure $\mu = (1+\Phi)^2 m_B$ belongs to K (the compact subset of $M_+(B)$ in the statement of Proposition 1.5.6), and $I(\mu)$ is not substantially bigger than the left-hand side of (1.125). This will show that the infimum of I over K is smaller or equal to $\underline{\lim}_N \, \underline{\inf}_K I_N$ and conclude the proof of Proposition 1.5.6.

Proof of Proposition 1.5.6. We begin with a cut-off lemma for functions on $\mathbb{L}_N$, which are harmonic off B_N and tend to 0 at infinity. This lemma, as mentioned above, will be an important ingredient when proving the relative compactness of a suitable nearly minimizing sequence we later construct.

We introduce $L_B \geq 1$ the smallest positive integer such that (see the beginning of Section 1 for notation)

$$B \subseteq B_\infty(0, L_B), \tag{1.126}$$

and for $R \geq 0$ integer, the closed box

$$C_R = B_\infty\big(0, L_B(R+2)\big) \supseteq B, \tag{1.127}$$

so that $d_\infty\big(B_\infty(0, L_B), \partial C_R\big) \geq L_B(R+1)$. For functions φ defined on $\mathbb{L}_N$, we use, as a shorthand, the notation $\sup_B \varphi$, $\inf_{C_R} \varphi, \dots$, in place of $\sup_{B \cap \mathbb{L}_N} \varphi$, $\inf_{C_R \cap \mathbb{L}_N} \varphi, \dots$.

We are now ready to state and prove the cut-off lemma. The reader may choose to first skip its proof.

Lemma 1.5.7. *(Cut-off lemma, $R \geq 1$ integer)*

Let $N \geq 1$, and φ on $\mathbb{L}_N$ be harmonic outside B_N and tend to 0 *at infinity. There exists $\overline{\varphi}$ on $\mathbb{L}_N$ such that*

$$\begin{cases} \text{i)} & \overline{\varphi} = \varphi, \ \text{on } C_R, \\ \text{ii)} & \overline{\varphi} = 0 \ \text{outside } C_{100R}, \\ \text{iii)} & \mathcal{E}_N(\overline{\varphi}, \overline{\varphi}) \leq \mathcal{E}_N(\varphi, \varphi)\Big(1 + \frac{c(B)}{R^{d-2}}\Big). \end{cases} \tag{1.128}$$

Proof. To simplify notation, all constants in the proof implicitly depend on d and B.

We let ψ and γ stand for the harmonic extensions of $\varphi_+ = \max\{\varphi, 0\}$ and $\varphi_- = \max\{-\varphi, 0\}$ outside B_N (on $\mathbb{L}_N \backslash B_N$), which tend to 0 at infinity. Since $\widetilde{\mathcal{E}}_N$ in (1.105) is a Dirichlet form, we know that the restrictions $\widetilde{\varphi}, \widetilde{\varphi}_+, \widetilde{\varphi}_-$ of $\varphi, \varphi_+, \varphi_-$ to B_N satisfy

(1.129)
$$\begin{aligned} \mathcal{E}_N(\psi,\psi) &\overset{(1.106)}{=} \widetilde{\mathcal{E}}_N(\widetilde{\varphi}_+,\widetilde{\varphi}_+) \le \widetilde{\mathcal{E}}_N(\widetilde{\varphi},\widetilde{\varphi}) \overset{(1.106)}{=} \mathcal{E}_N(\varphi,\varphi) \text{ and} \\ \mathcal{E}_N(\gamma,\gamma) &\le \mathcal{E}_N(\varphi,\varphi) \text{ (in a similar fashion).} \end{aligned}$$

By the Harnack inequality and chaining (see Theorem 1.7.2, p. 42 of [38]), we have

(1.130)
$$\max_{\partial C_0} \psi \le c \, \min_{\partial C_0} \psi \text{ and } \max_{\partial C_0} \gamma \le c \, \min_{\partial C_0} \gamma$$

(we recall that all constants in the proof of Lemma 1.5.7 implicitly depend on d and B). Since ψ is harmonic outside C_0 and tends to zero at infinity, it follows, by a stopping argument, that for y in ∂C_R, $\psi(y)$ is smaller than the product of $\max_{\partial C_0} \psi$ with the probability for the walk on $\mathbb{L}_N$ starting at y to reach ∂C_0. A similar bound holds for γ. By (1.130) and classical random walk estimates, we obtain

(1.131)
$$\max_{\partial C_R} \psi \le \frac{c}{R^{d-2}} \, \min_{\partial C_0} \psi \overset{\text{def}}{=} a_R, \; \max_{\partial C_R} \gamma \le \frac{c}{R^{d-2}} \, \min_{\partial C_0} \gamma \overset{\text{def}}{=} b_R, \text{ for } R \ge 1.$$

By estimates on the discrete gradient of harmonic functions in large balls, see Theorem 1.7.1, p. 42 of [38], we have for $R \ge 1$, $y \in \mathbb{L}_N \backslash C_R$ and $y' \sim y$ in $\mathbb{L}_N$,

(1.132)
$$|\psi(y') - \psi(y)| \le \frac{c}{R^{d-2}} \, \frac{1}{RN} \, \min_{\partial C_0} \psi, \; |\gamma(y') - \gamma(y)| \le \frac{c}{R^{d-2}} \, \frac{1}{RN} \, \min_{\partial C_0} \gamma.$$

With a similar notation as below (1.105), we define for $R \ge 1$

$$q_R(y) = P_y^N[H_{C_R \cap \mathbb{L}_N} > T_{C_{100R} \cap \mathbb{L}_N}], \text{ for } y \in \mathbb{L}_N,$$

the probability for the walk on $\mathbb{L}_N$ starting at y to exit C_{100R} before entering C_R. We then define

$$
\begin{aligned}
\overline{\varphi}(y) &= \varphi(y) \text{ on } C_R \cap \mathbb{L}_N, \\
&= \psi(y) \wedge \big(a_R\, q_R(y)\big) - \gamma(y) \wedge \big(b_R\, q_R(y)\big), \text{ on } \mathbb{L}_N \backslash C_R
\end{aligned}
\tag{1.133}
$$

(note that the expressions on both lines are equal for $y \in \partial C_R \cap \mathbb{L}_N$).

For simple random walk on $\mathbb{Z}^d$, the capacity of $(NC_R) \cap \mathbb{Z}^d$ relative to $(NC_{100R}) \cap \mathbb{Z}^d$ is at most $c(NR)^{d-2}$, see for instance (1.134) of [55]. Moreover, it is equal to the Dirichlet form of the function $q_R(\frac{\cdot}{N})$. One thus has the bound

$$
\mathcal{E}_N(q_R, q_R) \le c\, R^{d-2}. \tag{1.134}
$$

Further, note that when $y' \sim y$ in $\mathbb{L}_N$ (see below (1.103) for notation) are not both in C_R,

$$
\begin{aligned}
|\overline{\varphi}(y') - \overline{\varphi}(y)|^2 \le\ & 2(|\psi(y') - \psi(y)|^2 + a_R^2 |q_R(y') - q_R(y)|^2 \\
+\ & |\gamma(y') - \gamma(y)|^2 + b_R^2 |q_R(y') - q_R(y)|^2).
\end{aligned}
$$

Thus, coming back to (1.133), we see that, denoting by $\sum'$ the summation over $y' \sim y$ in $\mathbb{L}_N$, not both in C_R, one has

$$
\begin{aligned}
\mathcal{E}_N(\overline{\varphi}, \overline{\varphi}) \le\ & \mathcal{E}_N(\varphi, \varphi) + \tfrac{c}{N^{d-2}} \ {\textstyle\sum}' \big(|\psi(y') - \psi(y)|^2 + |\gamma(y') - \gamma(y)|^2\big) \\
& + 2(a_R^2 + b_R^2)\, \mathcal{E}_N(q_R, q_R) \\
\overset{(1.132)}{\le}\ & \mathcal{E}_N(\varphi, \varphi) + \Big(\tfrac{c}{N^{d-2}} \sum_{\ell \ge NR} \tfrac{\ell^{d-1}}{(\frac{\ell}{N})^{2(d-2)}}\ \tfrac{1}{\ell^2} + \tfrac{c}{R^{d-2}}\Big) \times \\
& \big(\min_{\partial C_0} \psi^2 + \min_{\partial C_0} \gamma^2\big) \\
\le\ & \mathcal{E}_N(\varphi, \varphi) + \tfrac{c}{R^{d-2}} \big(\min_{\partial C_0} \psi^2 + \min_{\partial C_0} \gamma^2\big).
\end{aligned}
\tag{1.135}
$$

Since the capacity of a finite subset of $\mathbb{Z}^d$ is smaller than the Dirichlet form of any function in the extended Dirichlet space equal to 1 on the set, and ψ, resp. γ, is bigger or equal to $\min_{\partial C_0} \psi$, resp. $\min_{\partial C_0} \gamma$, on $(\partial C_0) \cap \mathbb{L}_N$, we find that

$$\mathcal{E}_N(\psi,\psi) \ge \frac{d}{N^{d-2}} \, \text{cap}_{\mathbb{Z}^d}(N\partial C_0)(\min_{\partial C_0} \psi)^2 \ge c(\min_{\partial C_0} \psi)^2$$
$$\text{and likewise } \mathcal{E}_N(\gamma,\gamma) \ge c(\min_{\partial C_0} \gamma)^2.$$

Inserting these bounds in the last line of (1.135) and using (1.129), we find

$$\mathcal{E}_N(\overline{\varphi},\overline{\varphi}) \le \mathcal{E}_N(\varphi,\varphi)\Big(1 + \frac{c}{R^{d-2}}\Big).$$

Moreover, by (1.133), $\overline{\varphi} = \varphi$ on C_R and 0 on $\mathbb{L}_N \backslash C_{100R}$. We have proved Lemma 1.5.7. □

We resume the proof of (1.125). We denote by $\alpha \in [0,\infty]$ the left-hand side of (1.125). Without loss of generality, we can assume $\alpha < \infty$, otherwise (1.125) is immediate. We consider a subsequence N_ℓ, $\ell \ge 1$, as well as sequences $\mu_\ell \in M_+(B)$, and $\varphi_\ell \ge -1$ on B_{N_ℓ}, of functions on $\mathbb{L}_{N_\ell}$ harmonic outside B_{N_ℓ} and tending to 0 at infinity, such that using (1.118) and (1.106)

(1.136) $$\mu_\ell \in K, \text{ and } I_{N_\ell}(\mu_\ell) \to \alpha \;\; \big(\overset{\text{def}}{=} \varliminf_N \inf_{\rho \in K} I_N(\rho)\big),$$

(1.137) $$\begin{cases} \text{i)} & \mu_\ell = \dfrac{1}{N_\ell^d} \displaystyle\sum_{y \in B_{N_\ell}} \big(1 + \varphi_\ell(y)\big)^2 \delta_y, \\ \text{ii)} & I_{N_\ell}(\mu_\ell) = \mathcal{E}_{N_\ell}(\varphi_\ell,\varphi_\ell). \end{cases}$$

We choose $R \ge 1$, and construct with the help of Lemma 1.5.7 a sequence $\overline{\varphi}_\ell, \ell \ge 1$, of functions on $\mathbb{L}_{N_\ell}$ for which (1.128) holds (with φ replaced by φ_ℓ and N by N_ℓ). The functions $\overline{\varphi}_\ell$ vanish outside $C_{100R} \cap \mathbb{L}_{N_\ell}$. Since the principal Dirichlet eigenvalue of the discrete Laplacian in a box of side-length $L \ge 1$ in $\mathbb{Z}^d$ is at least $c\,L^{-2}$, see for instance [32], p. 185, we find that

(1.138) $$\begin{aligned} &\|\overline{\varphi}_\ell\|_{L^2(\mathbb{L}_{N_\ell})} \le c(B)\, R^2\, \mathcal{E}_{N_\ell}(\overline{\varphi}_\ell,\overline{\varphi}_\ell) \\ &(\text{and the limsup in } \ell \text{ is at most } c'(B)\, R^2 \alpha). \end{aligned}$$

The functions $\overline{\varphi}_\ell$ are defined on the different lattices $\mathbb{L}_{N_\ell}$, and it is convenient to introduce the functions Φ_ℓ, $\ell \geq 1$, on $\mathbb{R}^d$, which take the value $\overline{\varphi}_\ell(y)$ on $y + \frac{1}{N^\ell}\,[0,1)^d$, i.e.

$$\Phi_\ell(z) = \sum_{y \in \mathbb{L}_{N_\ell}} \overline{\varphi}_\ell(y)\; 1_{y+\frac{1}{N_\ell}[0,1)^d}(z),\ z \in \mathbb{R}^d,\ \ell \geq 1. \tag{1.139}$$

Note that, by construction, for all $\ell \geq 1$,

$$1 + \Phi_\ell(z) \geq 0,\ \text{if } z \in B \text{ and } d_\infty(z, \partial B) \geq \frac{1}{N_\ell}, \tag{1.140}$$

$$\Phi_\ell = 0,\ \text{on } \mathbb{R}^d \backslash C_{200R}, \tag{1.141}$$

$$\begin{aligned} &\|\Phi_\ell\|_{L^2(\mathbb{R}^d)} = \|\overline{\varphi}_\ell\|_{L^2(\mathbb{L}_{N_\ell})} \\ &(\text{and } \sup_\ell \|\Phi_\ell\|_{L^2(\mathbb{R}^d)} < \infty, \text{ by (1.138)}). \end{aligned} \tag{1.142}$$

We will now prove that the functions Φ_ℓ, $\ell \geq 1$, are equicontinuous in $L^2(\mathbb{R}^d)$ with respect to translations:

$$\lim_{h \to 0} \sup_{\ell \geq 1} \|\Phi_\ell(\cdot + h) - \Phi_\ell(\cdot)\|_{L^2(\mathbb{R}^d)} = 0. \tag{1.143}$$

Using the triangle inequality and translation invariance, we can assume, without loss of generality, that h is parallel to and pointing in the direction of e_i, the i-th vector of the canonical basis of $\mathbb{R}^d$. For notational simplicity, we treat the case $i = 1$ (the other cases are handled similarly).

We write $N_\ell h = (k + r)e_1$, where $k \geq 0$, is an integer and $0 \leq r < 1$ (both depend on ℓ). We see that

$$\begin{aligned} &\|\Phi_\ell(\cdot + h) - \Phi_\ell(\cdot)\|^2_{L^2(\mathbb{R}^d)} \leq 2(a + b), \\ \text{where}\quad & a = \Big\|\Phi_\ell\Big(\cdot + \frac{k}{N_\ell}\, e_1\Big) - \Phi_\ell(\cdot)\Big\|^2_{L^2(\mathbb{R}^d)} \\ \text{and}\quad & b = \Big\|\Phi_\ell\Big(\cdot + \frac{r}{N_\ell}\, e_1\Big) - \Phi_\ell(\cdot)\Big\|^2_{L^2(\mathbb{R}^d)}. \end{aligned}$$

Expressing b in terms of $\overline{\varphi}_\ell$, we find that

$$\begin{aligned} b \; &= r \Big\| \overline{\varphi}_\ell\Big(\,\cdot + \frac{e_1}{N_\ell}\Big) - \overline{\varphi}_\ell(\cdot)\Big\|^2_{L^2(\mathbb{L}_{N_\ell})} \overset{(1.103)}{\leq} c\, \frac{r}{N_\ell^2}\, \mathcal{E}_{N_\ell}(\overline{\varphi}_\ell, \overline{\varphi}_\ell) \\ &\leq \frac{c}{N_\ell^2}\, |h|\, \mathcal{E}_{N_\ell}(\overline{\varphi}_\ell, \overline{\varphi}_\ell). \end{aligned}$$

On the other hand, by the triangle inequality and translation invariance, we have

$$\begin{aligned} a &\leq k^2 \,\Big\| \Phi_\ell\Big(\,\cdot + \frac{e_1}{N_\ell}\Big) - \Phi_\ell(\cdot)\Big\|^2_{L^2(\mathbb{R}^d)} \\ &= k^2 \Big\| \overline{\varphi}_\ell\Big(\,\cdot + \frac{e_1}{N_\ell}\Big) - \overline{\varphi}_\ell(\cdot)\Big\|^2_{L^2(\mathbb{L}_{N_\ell})} \\ &\overset{(1.104)}{\leq} c\, \frac{k^2}{N_\ell^2}\, \mathcal{E}_{N_\ell}(\overline{\varphi}_\ell, \overline{\varphi}_\ell) \leq c\, |h|^2\, \mathcal{E}_{N_\ell}(\overline{\varphi}_\ell, \overline{\varphi}_\ell). \end{aligned}$$

Combining the bounds on a and b, we see that for $h = |h|\, e_1$,

(1.144)
$$\|\Phi_\ell(\cdot + h) - \Phi_\ell(\cdot)\|^2_{L^2(\mathbb{R}^d)} \leq c\, |h| \Big(|h| \vee \, \frac{1}{N_\ell}\Big)\, \mathcal{E}_{N_\ell}(\overline{\varphi}_\ell, \overline{\varphi}_\ell), \quad \text{for } \ell \geq 1.$$

The claim (1.143) now follows since $\sup_\ell \mathcal{E}_{N_\ell}(\overline{\varphi}_\ell, \overline{\varphi}_\ell) < \infty$.

By (1.138), (1.141), (1.143), and Theorem 2.21, p. 31 of [1], we find that Φ_ℓ, $\ell \geq 1$, is a relatively compact subset of $L^2(\mathbb{R}^d)$.

Hence, up to extraction of a subsequence (which we still denote by Φ_ℓ), we can assume that

(1.145) $$\Phi_\ell \underset{\ell}{\longrightarrow} \Phi \text{ in } L^2(\mathbb{R}^d),$$

and by (1.140), (1.141), we see that

(1.146) $$1 + \Phi \geq 0 \;\text{ a.e. on } B \text{ and } \Phi = 0 \;\text{ a.e. on } \mathbb{R}^d \backslash C_{200R}.$$

By (1.145), we see that

(1.147) $$(1 + \Phi_\ell)^2 \underset{\ell}{\longrightarrow} (1 + \Phi)^2 \text{ in } L^1_{\text{loc}}(\mathbb{R}^d).$$

By uniform integrability, it follows that the integral of $(1+\Phi_\ell)^2$ over $\{z \in \mathbb{R}^d;\ d_\infty(z, \partial B) \le \frac{1}{N_\ell}\}$ tends to 0 with ℓ. In addition, since $\overline{\varphi}_\ell = \varphi_\ell$ on $C_R \supseteq \{z \in \mathbb{R}^d;\ d_\infty(z, B) \le 1\}$, cf. (1.127), we see by (1.137) i) and (1.139) that $\mu_\ell(B \cap \{z \in \mathbb{R}^d;\ d_\infty(z, \partial B) \le \frac{1}{N_\ell}\}) \underset{\ell}{\longrightarrow} 0$. Thus, letting B'_N stand for the $y \in \mathbb{L}_N$ such that $y + \frac{1}{N}\,[0,1)^d \subseteq B$, we find that for $V \in C(B)$,

$$\begin{aligned}
&\langle \mu_\ell, V\rangle - \int_B (1+\Phi_\ell)^2 V\, dz \\
&= \sum_{y \in B'_{N_\ell}} \int_{y + \frac{1}{N_\ell}[0,1)^d} \big(V(y) - V(z)\big)\big(1 + \Phi_\ell(z)\big)^2 dz + o(1), \text{ as } \ell \to \infty.
\end{aligned}$$

Moreover, the sum in the right-hand side tends to 0 with ℓ, by uniform continuity of V and (1.147). Hence, by (1.147), we see that μ_ℓ ($\in K$, by (1.136)) converges in $M_+(B)$ to $(1+\Phi)^2 m_B$. This shows that

$$\mu = (1+\Phi)^2 m_B \in K. \tag{1.148}$$

Moreover, by (1.128) iii), (1.136), (1.137) ii), we find that

$$\begin{aligned}
\alpha\Big(1 + \frac{c(B)}{R^{d-2}}\Big) &\ge \overline{\lim_\ell}\ \mathcal{E}_{N_\ell}(\overline{\varphi}_\ell, \overline{\varphi}_\ell) \\
&= \overline{\lim_\ell}\ \frac{1}{2} \sum_{j=1}^d N_\ell^2 \Big\| \overline{\varphi}_\ell\Big(\cdot + \frac{e_j}{N_\ell}\Big) - \overline{\varphi}_\ell(\cdot)\Big\|^2_{L^2(\mathbb{L}_{N_\ell})}
\end{aligned} \tag{1.149}$$

and expressing this last quantity in terms of Φ_ℓ and using the Fourier transform

$$\begin{aligned}
&= \overline{\lim_\ell}\ \frac{1}{2} \sum_{j=1}^d \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} N_\ell^2 \left| e^{i\frac{\xi_j}{N_\ell}} - 1\right|^2 |\widehat{\Phi}_\ell(\xi)|^2 d\xi \\
&\overset{\text{Fatou}}{\ge} \frac{1}{2(2\pi)^d} \int_{\mathbb{R}^d} |\xi|^2\, |\widehat{\Phi}(\xi)|^2 d\xi = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \Phi(z)|^2 dz
\end{aligned}$$

(extracting some subsequence along which $\widehat{\Phi}_\ell$ converges a.e. to $\widehat{\Phi}$, before the last inequality). We have thus shown that $\Phi \in H^1(\mathbb{R}^d)$

and

$$(1.150)\qquad \mathcal{E}(\Phi,\Phi) \le \alpha\Big(1 + \frac{c(B)}{R^{d-2}}\Big).$$

Combined with (1.146), (1.148), this implies, see (1.84), that

$$(1.151)\qquad \inf_K\, I \le \alpha\Big(1 + \frac{c(B)}{R^{d-2}}\Big), \text{ for any } R > 1.$$

Letting $R \to \infty$, we obtain Proposition 1.5.6. □

We now come to the main theorem of this section, see (1.1), (1.3) for notation.

Theorem 1.5.8. *(The large deviation principle, $u > 0$)*
(1.152)
As $N \to \infty$, the laws of $\rho_{N,u}$ on $M_+(B)$ satisfy a large deviation principle at speed N^{d-2}, with convex, good rate function $\frac{1}{d}\, I_u$.

Proof. Combining Theorems 1.5.4 and 1.5.5, it follows from Theorem 4.1.11, p. 120 of [20], that $\rho_{N,u}$ satisfies a weak large deviation principle at speed N^{d-2}, with rate function $\frac{1}{d}\, I_u$ (which is good and convex by (1.82), (1.85)).

In addition, one has exponential tightness for the laws of the $\rho_{N,u}$, due to the fact that (see (1.83) and (1.91) of [60]) for small $\lambda > 0$,

$$\overline{\lim_N}\ \frac{1}{N^{d-2}}\ \log \overline{\mathbb{E}}\big[\exp\big\{N^{d-2}\langle \rho_{N,u}, \lambda 1_B\rangle\big\}\big] \overset{(1.100)}{\underset{(1.101)}{=}} \overline{\lim_N}\ \frac{1}{d}\ \log \overline{\mathbb{E}}[e^{\langle \mathcal{L}_{N,u}, d\lambda 1_B\rangle}] = u\, c(B,\lambda) < \infty,$$

and the Chebyshev Inequality. The claim (1.152) now follows (see p. 8 of [20]). □

1.6 An application

In this section, we apply the large deviation principle proved in the last section, see Theorem 1.5.8, and control the probability of existence of "high local density" regions insulating a given compact subset of $\mathbb{R}^d$. Our main results appear in Theorems 1.6.2 and 1.6.4. Extensions are discussed in Remark 1.6.5. We begin with some definitions and preliminary remarks.

We consider a compact set K and a closed box B_0 in $\mathbb{R}^d$ such that

$$\phi \neq K \subset B_0. \tag{1.153}$$

Given $a \in \mathbb{R}$ and a continuous function f on $\mathbb{R}^d$, we say that $\{f \geq a\}$ *disconnects* K *from* ∂B_0, if

(1.154) for any continuous function ψ: $[0,1] \to B_0$, such that $\psi(0) \in K$ and $\psi(1) \in \partial B_0$, one has $\sup\limits_{0 \leq t \leq 1} f\big(\psi(t)\big) \geq a$.

Note that the collection of bounded continuous functions f for which $\{f \geq a\}$ disconnects K from ∂B_0 is closed for the sup-norm topology.

Further, we consider $\delta \in (0,1)$, and a closed box B, so that

$$K \subset B_0 \subset B, \text{ and } d(\partial B_0, \partial B) > \delta, \tag{1.155}$$

as well as a continuous probability density φ_δ (with respect to Lebesgue measure), which is supported in $B(0,\delta)$. Denoting by $C_0(\mathbb{R}^d)$ the set of continuous functions on $\mathbb{R}^d$ that tend to 0 at infinity (endowed with the sup-norm topology), we consider the regularization map r_δ from $M_+(B)$ into $C_0(\mathbb{R}^d)$

$$\mu \in M_+(B) \to r_\delta(\mu)(\cdot) = \int_B \varphi_\delta(\cdot - y)\, \mu(dy) \in C_0(\mathbb{R}^d), \tag{1.156}$$

and introduce for $a \geq 0$, $\delta \in (0,1)$, the subset of $M_+(B)$

$$\mathcal{D}_{a,\delta} = \{\mu \in M_+(B);\ \{r_\delta(\mu) \geq a\} \text{ disconnects } K \text{ from } \partial B_0\}. \tag{1.157}$$

The next lemma collects some useful properties of the above objects.

Lemma 1.6.1. ($a \geq 0$, $0 < \delta < 1$, $u > 0$)

(1.158) $\quad r_\delta$ *is continuous.*

(1.159) $\quad \mathcal{D}_{a,\delta}$ *is a closed subset of* $M_+(B)$.

(1.160) $\quad$ *The event* $\{\rho_{N,u} \in \mathcal{D}_{a,\delta}\}$ *does not depend on the choice of* B *(satisfying (1.155)).*

Proof. We start with the proof of (1.158). The functions $\varphi_\delta(z - \cdot)$ on B, as z varies in $\mathbb{R}^d$, are equicontinuous at each point of B and uniformly bounded. It now follows from Theorem 6.8, p. 51 of [44], that when μ_n converges weakly to μ in $M_+(B)$, $r_\delta(\mu_n)$ converges uniformly to $r_\delta(\mu)$. This proves (1.158). Then, (1.159) is an immediate consequence of (1.158) and the observation below (1.154). As for (1.160), it suffices to notice that the restriction of $r_\delta(\rho_{N,u})$ to B_0 does not depend on the choice of B when (1.155) holds. $\square$

By the observation below (1.98) and (1.158), we know that for $u > 0$, $\overline{\mathbb{P}}$-a.s., $r_\delta(\rho_{N,u})$ converges uniformly to $r_\delta(um_B)(\cdot) = u \int_B \varphi_\delta(\cdot - y)dy$, and this function equals u on B_0 by (1.155). We will now consider the case where $a > u$ and study the large N behavior of the probability of occurence of high values of $r_\delta(\rho_{N,u})$ insulating K from ∂B_0.

Theorem 1.6.2. *(Insulation upper bound, $a > u > 0$)*

$$\overline{\lim_N} \frac{1}{N^{d-2}} \log \overline{\mathbb{P}}[\rho_{N,u} \in \mathcal{D}_{a,\delta}] \leq -\frac{1}{d} \left(\sqrt{a} - \sqrt{u}\right)^2 \mathrm{cap}(K) \tag{1.161}$$

Proof. By (1.159) and Theorem 1.5.8 we have

$$\overline{\lim_N} \frac{1}{N^{d-2}} \log \overline{\mathbb{P}}[\rho_{N,u} \in \mathcal{D}_{a,\delta}] \leq - \inf_{\mathcal{D}_{a,\delta}} \frac{1}{d} I_u. \tag{1.162}$$

We will use the notation I_{u,B_0} or $I_{u,B}$ to highlight the dependence on the closed box at hand in the definition (1.3). We have the following control.

Lemma 1.6.3.

$$I_{u,B_0}\big(r_\delta(\mu) m_{B_0}\big) \le I_{u,B}(\mu), \text{ for any } \mu \in M_+(B) \tag{1.163}$$

(we view $r_\delta(\mu)\, m_{B_0}$ as an element of $M_+(B_0)$).

Proof. Without loss of generality, we can assume that $I_{u,B}(\mu) < \infty$, so that $\mu = h m_B$ with $h \in L^1_+(m_B)$. Note that $y \in \mathbb{R}^d \to 1_{B_0}(\cdot)\, h(\cdot - y) \in L^1(m_{B_0})$ is a continuous map (we extend h outside B as being equal to 0). Moreover, by (1.156)

$$\big(1_{B_0} r_\delta(\mu)\big)(\cdot) = \int 1_{B_0}(\cdot) h(\cdot - y)\, \varphi_\delta(y) dy.$$

We also know that $f \in L^1_+(m_{B_0}) \to I_{u,B_0}(f m_{B_0})$ is a convex, lower semi-continuous map, see (1.78), (1.81), (1.85). Hence, we have

$$\begin{aligned} I_{u,B_0}(r_\delta(\mu) m_{B_0}) &\le \sup_{|y|\le\delta} I_{u,B_0}(1_{B_0}(\cdot)\, h(\cdot - y) m_{B_0}) \\ &\overset{(1.81)}{\underset{(1.85)}{=}} \sup_{|y|\le\delta} I_{u,B_0-y}(1_{B_0-y}\, h m_{B_0-y}) \\ &\overset{(1.83)}{\underset{(1.155)}{\le}} I_{u,B}(h m_B) = I_{u,B}(\mu). \end{aligned} \tag{1.164}$$

This proves (1.163). □

We will now bound the right-hand side of (1.162). Given $\mu \in \mathcal{D}_{a,\delta}$, we define

$$\varphi(z) = E_z\big[\big(\sqrt{r_\delta(\mu)} - \sqrt{u}\big)(X_{H_{B_0}}),\ H_{B_0} < \infty\big],\ z \in \mathbb{R}^d, \tag{1.165}$$

so that $\varphi \in C_0(\mathbb{R}^d)$. If $I_{u,B_0}(r_\delta(\mu) m_{B_0}) < \infty$, then by (1.81), (1.78), and the explanation below (1.73), we have $\varphi \in \mathcal{F}_e$ and

$$(1.166) \qquad \begin{aligned} \mathcal{E}(\varphi,\varphi) &= \widetilde{\mathcal{E}}_{B_0}\big(\sqrt{r_\delta(\mu)} - \sqrt{u}, \sqrt{r_\delta(\mu)} - \sqrt{u}\big) \\ &\underset{(1.85),(1.78)}{\overset{(1.81)}{=}} I_{u,B_0}(r_\delta(\mu) m_{B_0}). \end{aligned}$$

In addition, $\varphi = \sqrt{r_\delta(\mu)} - \sqrt{u}$ on B_0, and since $\mu \in \mathcal{D}_{a,\delta}$, we see that $\{\varphi \geq \sqrt{a} - \sqrt{u}\}$ disconnects K from ∂B_0. Setting $\widetilde{K} = \{y \in B_0, \varphi(y) \geq \sqrt{a} - \sqrt{u}\}$, it follows from Theorem 1.10, p. 58 of [47], that $\mathrm{cap}(\widetilde{K}) \geq \mathrm{cap}(K)$, and from p. 71 of [29], that $\mathcal{E}(\varphi,\varphi) \geq \big(\sqrt{a} - \sqrt{u}\big)^2 \mathrm{cap}(\widetilde{K})$. Hence, for $\mu \in \mathcal{D}_{a,\delta}$, we have

$$(1.167) \qquad \begin{aligned} I_u(\mu) &\underset{(1.166)}{\overset{(1.163)}{\geq}} \mathcal{E}(\varphi,\varphi) \geq \big(\sqrt{a} - \sqrt{u}\big)^2 \mathrm{cap}(\widetilde{K}) \\ &\geq \big(\sqrt{a} - \sqrt{u}\big)^2 \mathrm{cap}(K). \end{aligned}$$

Inserting this bound in the right-hand side of (1.162), we obtain (1.161). □

We now complement the asymptotic upper bound from Theorem 1.6.2 with an asymptotic lower bound. We denote by $K^\delta = \{z \in \mathbb{R}^d;\, d(z,K) \leq \delta\}$ the closed δ-neighborhood of K for the Euclidean distance. One knows that $\mathrm{cap}(K^\delta) \downarrow \mathrm{cap}(K)$ as $\delta \to 0$, see Remark 1.6.5 1) below.

Theorem 1.6.4. *(Insulation lower bound, $a > u > 0$)*

$$(1.168) \qquad \varliminf_N \frac{1}{N^{d-2}} \log \overline{\mathbb{P}}[\rho_{N,u} \in \mathcal{D}_{a,\delta}] \geq -\frac{1}{d} \big(\sqrt{a} - \sqrt{u}\big)^2 \mathrm{cap}(K^\delta).$$

Proof. We consider $\varepsilon > 0$ and define

$$(1.169) \qquad \mu_\varepsilon = \big(\sqrt{u} + \big(\sqrt{a+\varepsilon} - \sqrt{u}\big)\, h\big)^2\, m_B,$$

where $h(y) = P_y[\widetilde{H}_{K^\delta} < \infty] = P_y[H_{K^\delta} < \infty]$, for $y \in \mathbb{R}^d$, is the equilibrium potential of K^δ (every point of K^δ is regular for K^δ so

$h \in C_0(\mathbb{R}^d)$ and $h = 1$ on K^δ). Observe that

$$
\begin{aligned}
& r_\delta(\mu_\varepsilon)(y) \\
(1.170) \qquad & \stackrel{(1.156)}{=} \int_B \big(\sqrt{u} + \big(\sqrt{a+\varepsilon} - \sqrt{u}\big)\, h(z)\big)^2 \varphi_\delta(y-z)dz \\
& \geq \; a + \varepsilon \text{ , when } y \in K \text{ (since } h = 1 \text{ on } K^\delta, \\
& \quad \text{and this is actually an equality).}
\end{aligned}
$$

By (1.158), we can find an open neighborhood O of μ_ε in $M_+(B)$ so that for all $\mu \in O$, $r_\delta(\mu) \geq a + \frac{\varepsilon}{2}$ on K. It follows that $\{r_\delta(\mu) \geq a\}$ separates K from ∂B_0, for all $\mu \in O$, i.e. $O \subseteq \mathcal{D}_{a,\delta}$. As a result of Theorem 1.5.8 we find that

$$(1.171) \quad \varliminf_N \frac{1}{N^{d-2}} \log \overline{\mathbb{P}}[\rho_{N,u} \in \mathcal{D}_{a,\delta}] \geq - \inf_O \frac{1}{d} I_u \geq -\frac{1}{d} I_u(\mu_\varepsilon).$$

Now h belongs to $\mathcal{F}_e$ and $\mathcal{E}(h,h) = \mathrm{cap}(K^\delta)$, see [29], p. 71. Therefore, by (1.84), (1.85) and below (1.73),

(1.172)

$$I_u(\mu_\varepsilon) = \big(\sqrt{a+\varepsilon} - \sqrt{u}\big)^2 \mathcal{E}(h,h) = \big(\sqrt{a+\varepsilon} - \sqrt{u}\big)^2 \mathrm{cap}(K^\delta).$$

Inserting this identity in (1.171) and letting $\varepsilon \to 0$, we obtain (1.168).

□

Remark 1.6.5.

1) One knows from Proposition 1.13, p. 60 of [47], that $\mathrm{cap}(K^\delta) \downarrow \mathrm{cap}(K)$, as $\delta \to 0$. So, when $\delta \to 0$, the constant in the right-hand side of the lower bound (1.168) tends to the constant in the right-hand side of the upper bound (1.161).

2) Actually, one can let δ slowly tend to 0 in (1.161) and (1.168). More precisely, given a choice of φ_δ for each $\delta \in (0,1)$, $a > u > 0$ and K, B_0, B satisfying (1.155) when $\delta = \frac{1}{2}$, one can, using a diagonal type procedure, the remark above, and Theorems 1.6.2, 1.6.4, to construct a sequence δ_N slowly tending to zero so that

$$(1.173) \quad \lim_N \frac{1}{N^{d-2}} \log \overline{\mathbb{P}}[\rho_{N,u} \in \mathcal{D}_{a,\delta_N}] = -\frac{1}{d}\big(\sqrt{a} - \sqrt{u}\big)^2 \mathrm{cap}(K).$$

3) We can, in place of $\mathcal{D}_{a,\delta}$ in (1.157), instead consider "disconnection by sub-level sets", i.e.

(1.174)
$$\mathcal{D}'_{a,\delta} = \{\mu \in M_+(B);\ \{r_\delta(\mu) \le a\} \text{ disconnects } K \text{ from } \partial B_0\}$$

(replacing $\sup_{0\le t\le 1} f(\psi(t)) \ge a$ by $\inf_{0\le t\le 1} f(\psi(t)) \le a$ in (1.154) to define the above event).

The same arguments of Theorems 1.6.2 and 1.6.4 apply (one replaces $a+\varepsilon$ by $a-\varepsilon$, in (1.169), with $0<\varepsilon<a$) and we obtain that for $0<a<u$, $0<\delta<1$, under (1.155)

(1.175)
$$\begin{aligned}
&\overline{\lim_N}\ \frac{1}{N^{d-2}}\ \log \overline{\mathbb{P}}\big[\rho_{N,u} \in \mathcal{D}'_{a,\delta}\big] \le -\frac{1}{d}\big(\sqrt{a}-\sqrt{u}\big)^2\ \mathrm{cap}(K),\\
&\underline{\lim_N}\ \frac{1}{N^{d-2}}\ \log \overline{\mathbb{P}}\big[\rho_{N,u} \in \mathcal{D}'_{a,\delta}\big] \ge -\frac{1}{d}\big(\sqrt{a}-\sqrt{u}\big)^2\ \mathrm{cap}(K^\delta),
\end{aligned}$$

4) It is instructive to compare (1.168) in Theorem 1.6.4 with the lower bound one obtains by the following intuitive "change of measure" strategy. Namely, assume $0<u<a$ and introduce the new probability measure

$$\widetilde{\mathbb{P}}_N = e^{\lambda\eta - u\,\mathrm{cap}_{\mathbb{Z}^d}(K^{\delta,N})(e^\lambda-1)}\ \overline{\mathbb{P}},$$

where η stands for the total number of bilateral trajectories modulo time-shift with label at most u, which enter $K^{\delta,N} = (NK^\delta)\cap\mathbb{Z}^d$, $\lambda = \log(\frac{a+\varepsilon}{u})$, and we recall that for A finite in $\mathbb{Z}^d$, $\mathrm{cap}_{\mathbb{Z}^d}(A)$ stands for the capacity of A (attached to the simple random walk on $\mathbb{Z}^d$).

Under $\widetilde{\mathbb{P}}_N$, the variable η has Poisson distribution with parameter $(a+\varepsilon)\,\mathrm{cap}_{\mathbb{Z}^d}(K^{\delta,N})$ (instead of $u\,\mathrm{cap}_{\mathbb{Z}^d}(K^{\delta,N})$ under $\overline{\mathbb{P}}$), and $(L_{x,u})_{x\in K^{\delta,N}}$ has the same distribution as $(L_{x,a+\varepsilon})_{x\in K^{\delta,N}}$ under $\overline{\mathbb{P}}$. As we now explain,

(1.176)
$$\lim_N\ \widetilde{\mathbb{P}}_N(\rho_{N,u} \in \mathcal{D}_{a,\delta}) = 1.$$

Indeed, by (1.98), $\overline{\mathbb{P}}$-a.s., $\frac{1}{N^d}\sum_{y\in\mathbb{L}_N} L_{Ny,a+\varepsilon}\,\delta_y$ converges vaguely to $(a+\varepsilon)dy$, as $N\to\infty$, and the restriction to K of $r_\delta(\rho_{N,u})$ under $\widetilde{\mathbb{P}}_N$ has the same distribution as the restriction to K of $r_\delta(\rho_{N,a+\varepsilon})$

under $\overline{\mathbb{P}}$, which $\overline{\mathbb{P}}$-a.s. converges uniformly on K to $a+\varepsilon$. However, as soon as the restriction to K of $r_\delta(\rho_{N,u})$ exceeds a everywhere, the event $\{\rho_{N,u} \in \mathcal{D}_{a,\delta}\}$ occurs, and (1.176) follows.

By the classical relative entropy estimate, see for instance [22], p. 76, and (1.176),

(1.177)
$$\liminf_N \frac{1}{N^{d-2}} \log \overline{\mathbb{P}}[\rho_{N,u} \in \mathcal{D}_{a,\delta}] \geq -\limsup_N \frac{1}{N^{d-2}} H(\widetilde{\mathbb{P}}_N \,|\, \overline{\mathbb{P}}),$$

where $H(\widetilde{\mathbb{P}}_N \,|\, \overline{\mathbb{P}}) = \mathbb{E}^{\widetilde{\mathbb{P}}_N}[\log \frac{d\widetilde{\mathbb{P}}_N}{d\overline{\mathbb{P}}}]$ stands for the relative entropy of $\widetilde{\mathbb{P}}_N$ with respect to $\overline{\mathbb{P}}$. As we now explain,

$$\text{(1.178)} \quad \liminf_N \frac{1}{N^{d-2}} H(\widetilde{\mathbb{P}}_N \,|\, \overline{\mathbb{P}}) \geq \Big(a \log \frac{a}{u} - a + u\Big) \frac{1}{d} \operatorname{cap}(K^\delta),$$

and since (as can be checked directly)

$$\text{(1.179)} \quad v \log \frac{v}{u} - v + u > \big(\sqrt{v} - \sqrt{u}\big)^2, \text{ for any } v, u > 0, \text{ with } v \neq u,$$

the lower bound (1.177) is worse than (1.168).

To prove (1.178), one notes that by our choice of $\widetilde{\mathbb{P}}_N$,

$$\text{(1.180)} \quad \begin{aligned} &H(\widetilde{\mathbb{P}}_N \,|\, \overline{\mathbb{P}}) \\ &= E^{\widetilde{\mathbb{P}}_N}[\lambda\eta] - (a+\varepsilon-u) \operatorname{cap}_{\mathbb{Z}^d}(K^{\delta,N}) \\ &= \Big((a+\varepsilon) \log\big(\frac{a+\varepsilon}{u}\big) - (a+\varepsilon) + u\Big) \operatorname{cap}_{\mathbb{Z}^d}(K^{\delta,N}). \end{aligned}$$

The same argument leading to (1.145), (1.150) shows that for each $R \geq 1$ and $\delta' \in (0,\delta)$ one can construct $\Phi \in H^1(\mathbb{R}^d)$ with compact support, which is a.e. equal to 1 on $K^{\delta'}$, and such that $\mathcal{E}(\Phi,\Phi) \leq d(1+\frac{c(B)}{R^{d-2}})\,\beta$ with $\beta = \underline{\lim}_N \frac{1}{N^{d-2}} \operatorname{cap}_{\mathbb{Z}^d}(K^{\delta,N})$. It thus follows (see [29], p. 71) that

$$\beta \geq \frac{1}{d} \Big(1 + \frac{c(B)}{R^{d-2}}\Big)^{-1} \operatorname{cap}(K^{\delta'}).$$

Letting $R \to \infty$, $\delta' \uparrow \delta$, and noting that when $y \in K^\delta$, P_y-a.s.

Brownian motion immediately hits $\bigcup_{\delta'<\delta} K^{\delta'}$, so that by Theorem 1.10 and Proposition 1.13, p. 58 and 60 of [47], $\mathrm{cap}(K^{\delta'}) \uparrow \mathrm{cap}(K^\delta)$ as $\delta' \uparrow \delta$, we obtain that

$$\liminf_N \frac{1}{N^{d-2}} \,\mathrm{cap}_{\mathbb{Z}^d}(K^{\delta,N}) \geq \frac{1}{d}\,\mathrm{cap}(K^\delta). \tag{1.181}$$

Since $v \to v \log \frac{v}{u} - v + u$ is increasing for $v \in (u,\infty)$, the combination of (1.180) and (1.181) readily yields (1.178).

So, the intuitive lower bound we just described does not capture (1.168).

5) It is an important feature of random interlacements that the vacant set $\mathcal{V}^u$ of random interlacements at level u on $\mathbb{Z}^d, d \geq 3$, undergoes a phase transition from a percolative regime, when $u < u_*$, to a non-percolative regime, when $u > u_*$, with u_* a certain non-degenerate critical value, which is positive and finite (see [56], [52], and also [24], [46] for recent developments). Given a smooth compact subset K of $\mathbb{R}^d$ and its discrete blow-up $K^N = (NK) \cap \mathbb{Z}^d$, one can consider the disconnection event $\{K^N \overset{\mathcal{V}^u}{\not\longleftrightarrow} \infty\}$, for which the connected components of $K^N \cap \mathcal{V}^u$ in $\mathcal{V}^u$ are finite (possibly empty). Looking at a small interior ball in K and its discrete blow-up, it is straightforward to argue that for $u < u_*$, $\lim_N \overline{\mathbb{P}}[K^N \overset{\mathcal{V}^u}{\not\longleftrightarrow} \infty] = 0$. One can wonder whether the main effect in realizing this atypical disconnection event stems from a large deviation of the density profile of occupation-times, for which, roughly speaking, values of the profile exceeding u_* would insulate K from infinity, and whether one has the asymptotics

$$\lim_N \frac{1}{N^{d-2}} \log \overline{\mathbb{P}}[K^N \overset{\mathcal{V}^u}{\not\longleftrightarrow} \infty] = -\frac{1}{d} \left(\sqrt{u}_* - \sqrt{u}\right)^2 \mathrm{cap}(K), \text{ for } u < u_*.$$

We refer to [43], for a lower bound on the left-hand side of a similar nature.

□

Chapter 2

A lower bound for disconnection by random interlacements

We consider the vacant set of random interlacements on $\mathbb{Z}^d$, $d \geq 3$, in the percolative regime. Motivated by the large deviation principles recently obtained in [42], we investigate the asymptotic behavior of the probability that a large body gets disconnected from infinity by the random interlacements. We derive an asymptotic lower bound, which brings into play tilted interlacements, and relates the problem to some of the large deviations of the occupation-time profile considered in [42].

2.1 Introduction

Random interlacements constitute a percolation model with long-range dependence, and the percolative properties of their vacant set play an important role in the investigation of several questions of disconnection or fragmentation created by random walks, see [14], [55], [64]. Here, we consider random interlacements on $\mathbb{Z}^d$, $d \geq 3$.

It is by now well-known that as one increases the level u of the interlacements, the percolative properties of the vacant set undergo a phase transition, and the model evolves from a percolative phase to a non-percolative phase, see [56] and [52]. In the present work, we are mainly interested in the percolative phase of the model, and we derive an asymptotic lower bound on the probability that a macroscopic body has no connection to infinity in the vacant set. Strikingly, this lower bound corresponds to certain large deviations of the occupation-time profile of random interlacements investigated in our previous work [42], where we analyzed the exponential decay of the probability that a macroscopic body gets insulated by high values of the (regularized) occupation-time profile.

We now describe the model and our results in a more precise fashion. We refer to Section 1 for precise definitions. We consider continuous-time random interlacements on $\mathbb{Z}^d$, $d \geq 3$. We denote by $\mathbb{P}_u$ the canonical law of random interlacements at level $u > 0$, and by $\mathcal{I}^u$ and $\mathcal{V}^u = \mathbb{Z}^d \backslash \mathcal{I}^u$ the corresponding interlacement set and vacant set. It is known that there is a critical value $u_{**} \in (0, \infty)$, which can be characterized as the infimum of the levels $u > 0$ for which the probability that the vacant cluster at the origin reaches distance N from the origin has a stretched exponential decay in N, see [53]. It is an important open question whether u_{**} actually coincides with the critical level u_* for the percolation of the vacant set (but it is a simple fact that $u_* \leq u_{**}$).

In this work, we are primarily interested in the percolative regime of the vacant set, but, specifically, we assume that $0 < u \leq u_{**}$ (because our lower bound on disconnection actually provides information in this possibly wider range of levels).

We consider a compact subset K of $\mathbb{R}^d$, and its discrete blow-up:

$$K_N = \{x \in \mathbb{Z}^d;\ d_\infty(x, NK) \leq 1\}, \tag{2.1}$$

where NK denotes the homothetic of ratio N of the set K, and $d_\infty(z, NK) = \inf_{y \in NK} |z - y|_\infty$ stands for the sup-norm distance of z to NK. Of central interest for us is the event stating that K_N is not connected to infinity in $\mathcal{V}^u$, which we denote by

$$A_N = \{K_N \overset{\mathcal{V}^u}{\nleftrightarrow} \infty\}. \tag{2.2}$$

The main result of this article is the following asymptotic lower bound.

Theorem 2.1.1. *For* $u \in (0, u_{**}]$ *one has*

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}} \log(\mathbb{P}_u[A_N]) \geq -\frac{1}{d}(\sqrt{u_{**}} - \sqrt{u})^2 \mathrm{cap}_{\mathbb{R}^d}(K), \tag{2.3}$$

where $\mathrm{cap}_{\mathbb{R}^d}(K)$ *stands for the Brownian capacity of* K.

In essence, the lower bound (2.3) replicates the asymptotic behavior of the probability that the regularized occupation-time profile of random interlacements insulates K by values exceeding u_{**}, see Theorems 6.2 and 6.4, as well as Remarks 6.5 2) and 6.5 5) of [42]. It is a remarkable feature that such large deviations of the occupation-time profile induce a “thickening” of the interlacement surrounding K_N, rather than a mere change of the clocks governing the time spent by the trajectories defining the interlacement. This thickening is potent enough to typically disconnect K_N from infinity. We refer to Remark 2.3.5 for more on this topic. It is of course an important question, whether there is a matching upper bound to (2.3), when K is a smooth compact, and whether the large deviations of the occupation-time profile capture the main mechanism through which $\mathcal{I}^u$ disconnects a macroscopic body from infinity.

Incidentally, the tilted interlacements, which we heavily use in this work, come up as a kind of slowly space-modulated random interlacements. Possibly, they offer, in a discrete set-up, a microscopic model for the type of “Swiss cheese” picture advocated in [9], when studying the moderate deviations of the volume of the Wiener sausage (however the relevant modulating functions in [9] and in the present work correspond to distinct variational problems and are different).

One may also compare Theorem 2.1.1 to corresponding results for supercritical Bernoulli percolation. Unlike what happens in the present set-up, disconnecting a large macroscopic body in the percolative phase (when K is a smooth compact) would involve an exponential cost proportional to N^{d-1}, in the spirit of the study of the existence of a large finite cluster at the origin, see p. 216 of [33], or Theorem 2.5, p. 16 of [12].

Further, it is interesting to note that when $u \to 0$, the right-hand side of (2.3) has a finite limit. One may wonder about the relation of this limit to what happens in our original problem when one replaces $\mathcal{I}^u$ by a single random walk trajectory (starting for instance at the origin), that is, when we consider the probability that K_N is disconnected from infinity by the trajectory of one single random walk starting at the origin. We refer to Remark 2.6.1 2) for more on this question.

We briefly comment on the proofs. The main strategy is to use a change of probability and an entropy bound. We construct through fine-tuned Radon-Nikodym derivatives new measures $\widetilde{\mathbb{P}}_N$ corresponding to "tilted random interlacements", which have the crucial property that under $\widetilde{\mathbb{P}}_N$ the disconnection probability tends to 1 as N goes to infinity:

$$\widetilde{\mathbb{P}}_N[A_N] \to 1. \tag{2.4}$$

Then, by a classical inequality (see (2.68)), one has a lower bound for the disconnection probability in terms of the relative entropy:

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}} \log(\mathbb{P}_u[A_N]) \geq -\limsup_{N\to\infty} \frac{1}{N^{d-2}} H(\widetilde{\mathbb{P}}_N|\mathbb{P}_u). \tag{2.5}$$

We relate the relative entropy of $\widetilde{\mathbb{P}}_N$ with respect to $\mathbb{P}_u$, to the Brownian capacity of K, and show in Propositions 2.3.3 and 2.3.4 that

$$\widetilde{\lim} \frac{1}{N^{d-2}} H(\widetilde{\mathbb{P}}_N|\mathbb{P}_u) = -\frac{1}{d}(\sqrt{u_{**}} - \sqrt{u})^2 \mathrm{cap}_{\mathbb{R}^d}(K) \tag{2.6}$$

(where $\widetilde{\lim}$ refers to certain successive limiting procedures involving N first, and then various auxiliary parameters entering the construction of $\widetilde{\mathbb{P}}_N$).

The measure $\widetilde{\mathbb{P}}_N$ governing the tilted interlacements is constructed in Section 2. Intuitively, it forces a "local level" of interlacements corresponding to $u_{**} + \epsilon$, in a "fence" surrounding K_N. This creates a strongly non-percolative region surrounding K_N and leads to (2.4). Of course, a substantial part of the work is to make sense of the above heuristics. This goes through a local comparison at a mesoscopic scale between the occupied set of tilted interlacements and standard interlacements at a level exceeding u_{**}.

In particular, we show in Proposition 2.5.1 that for all mesoscopic boxes B_1, with size N^{r_1} (with r_1 small) and center in Γ^N, a "fence" around K_N, one has a coupling $\overline{Q}$ between $\mathcal{I}_1$, distributed as $\mathcal{I}^{u_{**}+\epsilon/8} \cap B_1$, and $\widetilde{\mathcal{I}}$, distributed as the intersection of the tilted interlacement set with B_1, so that

(2.7) $$\overline{Q}[\widetilde{\mathcal{I}} \supset \mathcal{I}_1] \geq 1 - ce^{-c' N^{c''}}.$$

The proof of this key stochastic domination bound relies on two main ingredients. On the one hand, it involves a comparison of equilibrium measures, see Proposition 2.4.5, which itself relies on a comparison of capacities on a slightly larger mesoscopic scale, see Proposition 2.4.1. On the other hand, it involves a domination of $\mathcal{I}^{u_{**}+\epsilon/8} \cap B_1$ by the trace on B_1 of a suitable Poisson point process of excursions of the simple random walk starting on the boundary of B_1 up to their exit from a larger box B_2. For this last step we can rely on results of [7].

We will now explain how this article is organized. In Section 1 we introduce notation and make a brief review of results concerning continuous-time random walk, Green function, continuous-time random interlacements, as well as other useful facts and tools. Section 2 is devoted to the construction of the probability measure governing the tilted random interlacements. We also compute and obtain asymptotic estimates on the relative entropy, see Propositions 2.3.3 and 2.3.4. In Section 3 we derive a comparison of capacities in Proposition 2.4.1, and, subsequently, of equilibrium measures in Proposition 2.4.4. The latter proposition plays a crucial role in the construction of the coupling in the next section. In Section 4 we prove (2.7) in Proposition 2.5.1, and the crucial statement (2.4) in Theorem 2.5.3. In the short Section 5 we assemble the various pieces and prove the main theorem.

Finally, we explain the convention we use concerning constants. We denote by $c, c', \bar{c}, \widetilde{c} \dots$ positive constants with values changing from place to place, and by $c_0, c_1, \dots$ positive constants which are fixed and refer to the value as they first appear. Throughout the article the constants depend on the dimension d. Dependence on additional constants are stated explicitly in the notation.

2.2 Some useful facts

Throughout the article we assume $d \geq 3$. In this section we introduce further notation and useful facts, in particular concerning continuous time random walk on $\mathbb{Z}^d$ and its potential theory. The Lemma 2.2.1 concerns the occupation-times of balls and will be used in Section 3. Moreover, we introduce another continuous-time reversible Markov chain on $\mathbb{Z}^d$, which will play a crucial role in the upcoming sections, and we state some useful results regarding its potential theory. We also recall the definition and basic facts concerning continuous time random interlacements. We end this section by stating some results about relative entropy and Poisson point processes.

We start with some notation. We let $\mathbb{N} = \{0, 1, \dots\}$ stand for the set of natural numbers. We write $|\cdot|$ and $|\cdot|_\infty$ for the Euclidean and l^∞-norms on $\mathbb{R}^d$. We denote by $B(x, r) = \{y \in \mathbb{Z}^d;\ |x - y| \leq r\}$ the closed Euclidean ball of radius $r \geq 0$ intersected with $\mathbb{Z}^d$, and respectively by $B_\infty(x, r) = \{y \in \mathbb{Z}^d,\ |x - y|_\infty \leq r\}$ the closed l^∞-ball of radius r intersected with $\mathbb{Z}^d$. When U is a subset of $\mathbb{Z}^d$, we write $|U|$ for the cardinality of U, and $U \subset\subset \mathbb{Z}^d$ means that U is a finite subset of $\mathbb{Z}^d$. We denote by ∂U (resp. $\partial_i U$) the boundary (resp. internal boundary) of U, and by $\overline{U}$ its "closure":

$$
\begin{aligned}
\partial U &= \{x \in U^c;\ \exists y \in U,\ |x - y| = 1\}, \\
\partial_i U &= \{x \in U;\ \exists y \in U^c,\ |x - y| = 1\}, \text{ and } \overline{U} = U \cup \partial U\,.
\end{aligned}
\tag{2.8}
$$

When $U \subset \mathbb{R}^d$, and $\delta > 0$, we write $U^\delta = \{z \in \mathbb{R}^d;\ d(z, U) \leq \delta\}$ for the closed δ-neighborhood of U, where $d(x, A) = \inf_{y \in A} |x - y|$ is the distance function. We define $d_\infty(x, A)$ in a similar fashion, with

$|\cdot|_\infty$ in place of $|\cdot|$. To distinguish balls in $\mathbb{R}^d$ from balls in $\mathbb{Z}^d$, we write $B_{\mathbb{R}^d}(x,r) = \{z \in \mathbb{R}^d;\ |x-z| \le r\}$ for the (closed) Euclidean ball of radius r in $\mathbb{R}^d$. We also introduce the N-discrete blow-up of U as

$$U_N = \{x \in \mathbb{Z}^d;\ d_\infty(x, NU) \le 1\}, \tag{2.9}$$

where $NU = \{Nz;\ z \in U\}$ denotes the homothetic of U.

We will now collect some notation concerning connectivity properties. We call $\pi : \{1,\dots n\} \to \mathbb{Z}^d$, with $n \ge 1$, a nearest-neighbor path, when $|\pi(i) - \pi(i-1)| = 1$, for $1 < i \le n$. Given K, L, U subsets of $\mathbb{Z}^d$, we say that K and L are connected by U and write $K \overset{U}{\leftrightarrow} L$, if there exists a finite nearest-neighbor path π in $\mathbb{Z}^d$ such that $\pi(1)$ belongs to K and $\pi(n)$ belongs to L, and for all k in $\{1,\cdots,n\}$, $\pi(k)$ belongs to U. Otherwise, we say that K and L are not connected by U, and write $K \overset{U}{\nleftrightarrow} L$. Similarly, for $K, U \subset \mathbb{Z}^d$, we say that K is connected to infinity by U, if $K \overset{U}{\leftrightarrow} B(0,N)^c$ for all N, and write $K \overset{U}{\leftrightarrow} \infty$. Otherwise, we say that K is not connected to infinity by U, and denote it by $K \overset{U}{\nleftrightarrow} \infty$.

We now turn to the definition of some path spaces, and of the continuous-time simple random walk. We consider $\widehat{W}_+$ and $\widehat{W}$ the spaces of infinite (resp. doubly-infinite) $(\mathbb{Z}^d) \times (0,\infty)$-valued sequences such that the first coordinate of the sequence forms an infinite (resp. doubly-infinite) nearest-neighbor path in $\mathbb{Z}^d$, spending finite time in any finite subset of $\mathbb{Z}^d$, and the sequence of the second coordinate has an infinite sum (resp. infinite "forward" and "backward" sums). The second coordinate describes the duration at each step corresponding to the first coordinate. We denote by $\widehat{\mathcal{W}}_+$ and $\widehat{\mathcal{W}}$ the respective σ-algebras generated by the coordinate maps. We denote by P_x the law on $\widehat{W}_+$ under which Z_n, $n \ge 0$, has the law of the simple random walk on $\mathbb{Z}^d$, starting from x, and ζ_n, $n \ge 0$, are i.i.d. exponential variables with parameter 1, independent from Z_n, $n \ge 0$. We denote by E_x the corresponding expectation. Moreover, if α is a measure on $\mathbb{Z}^d$, we denote by P_α and E_α the measure $\sum_{x\in\mathbb{Z}^d} \alpha(x) P_x$ (not necessarily a probability measure) and its corresponding "expectation" (i.e. the integral with respect to the measure P_α).

We attach to $\widehat{w} \in \widehat{W}_+$ a continuous-time process $(X_t)_{t\geq 0}$, and call it the random walk on $\mathbb{Z}^d$ with constant jump rate 1 under P_x, through the following relations

$$(2.10) \qquad X_t(\widehat{w}) = Z_k(\widehat{w}), \text{ for } t \geq 0, \text{ when } \sum_{i=0}^{k-1} \zeta_i \leq t < \sum_{i=0}^{k} \zeta_i$$

(if $k = 0$, the left sum term is understood as 0). We also introduce the filtration

$$(2.11) \qquad \mathcal{F}_t = \sigma(X_s,\ s \leq t),\ t \geq 0.$$

Given $U \subseteq \mathbb{Z}^d$, and $\widehat{w} \in \widehat{W}_+$, we write $H_U(\widehat{w}) = \inf\{t \geq 0;\ X_t(\widehat{w}) \in U\}$ and $T_U = \inf\{t \geq 0;\ X_t(\widehat{w}) \notin U\}$ for the entrance time in U and exit time from U. Moreover, we write $\widetilde{H}_U = \inf\{s \geq \zeta_1; X_s \in U\}$ for the hitting time of U.

For $U \subset \mathbb{Z}^d$, we write $\Gamma(U)$ for the space of all right-continuous, piecewise constant functions from $[0,\infty)$ to U, with finitely many jumps on any compact interval. We will also denote by $(X_t)_{t\geq 0}$ the canonical coordinate process on $\Gamma(U)$, and whenever an ambiguity arises, we will specify on which space we are working.

We denote by $g(\cdot,\cdot)$ and $g_U(\cdot,\cdot)$ the Green function of the walk, and the killed Green function of the walk upon leaving U,

(2.12)
$$g(x,y) = E_x\Big[\int_0^\infty 1_{\{X_s=y\}} ds\Big], \quad g_U(x,y) = E_x\Big[\int_0^{T_U} 1_{\{X_s=y\}} ds\Big].$$

It is known that g is translation invariant. Moreover, both g and g_U are symmetric and finite, that is,

$$(2.13) \qquad g(x,y) = g(y,x), \quad g_U(x,y) = g_U(y,x) \text{ for all } x,y \in \mathbb{Z}^d.$$

When x tends to infinity, one knows that (see, e.g. p. 153, Proposition 6.3.1 of [39])

$$g(0,x) = dG(x) + O(|x|^{1-d}), \tag{2.14}$$

where for $y \in \mathbb{R}^d$

$$G(y) = c_0|y|^{2-d} \tag{2.15}$$

is the Green function with a pole at the origin, attached to Brownian motion, and

$$c_0 = \frac{\bar{c}_0}{d} = \frac{1}{2\pi^{d/2}}\,\Gamma\Big(\frac{d}{2} - 1\Big). \tag{2.16}$$

We also have the following estimate on the killed Green function (see p. 157, Proposition 6.3.5 of [39]): for $x \in B(0,N)$,

$$\begin{aligned} g_{B(0,N)}(0,x) &= g(0,x) - E_x\big[g(0, X_{T_{B(0,N)}})\big] \\ &= \overline{c}_0(|x|^{2-d} - N^{2-d}) + O(|x|^{1-d}). \end{aligned} \tag{2.17}$$

We further recall the definitions of equilibrium measure and capacity, and refer to Section 2, Chapter 2 of [38] for more details. Given $M \subset\subset \mathbb{Z}^d$, and we write e_M for the equilibrium measure of M:

$$e_M(x) = P_x[\widetilde{H}_M = \infty]1_M(x),\ x \in \mathbb{Z}^d, \tag{2.18}$$

and $\mathrm{cap}(M)$ for the capacity of M, which is the total mass of e_M:

$$\mathrm{cap}(M) = \sum_{x\in K} e_M(x). \tag{2.19}$$

There is also an equivalent definition of capacity through the Dirichlet form:

$$\mathrm{cap}(M) = \inf_f \mathcal{E}_{\mathbb{Z}^d}(f,f) \tag{2.20}$$

where $f : \mathbb{Z}^d \to \mathbb{R}$ is finitely supported, $f \geq 1$ on M, and

$$\mathcal{E}_{\mathbb{Z}^d}(f,f) = \frac{1}{2} \sum_{|x-y|=1} \frac{1}{2d} \big(f(y) - f(x)\big)^2 \tag{2.21}$$

is the discrete Dirichlet form for simple random walk.

Moreover, the probability of entering M can be expressed as

$$P_x[H_M < \infty] = \sum_{y \in M} g(x,y) e_M(y), \tag{2.22}$$

and in particular, when $x \in M$, we have

$$\sum_{y \in M} g(x,y) e_M(y) = 1. \tag{2.23}$$

We now introduce some notation for (killed) entrance measures. Given $A \subseteq B$ subsets of $\mathbb{Z}^d$, with A finite, we define for $x \in \mathbb{Z}^d$, $y \in A$,

$$h_{A,B}(x,y) = P_x(H_A < T_B,\ X_{H_A} = y)\,. \tag{2.24}$$

When $B = \mathbb{Z}^d$, we simply write $h_A(x,z)$.

The equilibrium measure also satisfies the sweeping identity (for instance, seen as a consequence of (1.46) in [56]), namely, for $M \subset M' \subset\subset \mathbb{Z}^d$, $y \in M$, using the notation from above (2.10),

$$P_{e_{M'}}[H_M < \infty, X_{H_M} = y] = \sum_{x \in \partial_i M'} e_{M'}(x) h_M(x,y) = e_M(y). \tag{2.25}$$

The next lemma will be useful in Section 3, see Proposition 2.4.1. It provides an asymptotic estimate on the expected time a random walk starting at the boundary of a ball of large radius spends in this ball. We recall the convention on constants stated at the end of the Introduction.

Lemma 2.2.1. *As N tends to infinity,*

$$(2.26)\qquad \alpha(N) \overset{\text{def}}{=} \sup_{x\in\partial_i B(0,N)} \left| \frac{E_x\big[\int_0^\infty 1_{B(0,N)}(X_s)ds\big]}{c_1 N^2} - 1 \right|$$

tends to 0.

Proof. For simplicity, we fix x in this proof and write $B(0,N) = B$. We set

$$(2.27)\qquad \epsilon_N = N^{-1/2},\ r_N = \epsilon_N N.$$

We split B into two parts: $B_I = B \cap \widetilde{B}$ and $B_J = B\backslash\widetilde{B}$, where $\widetilde{B} = B(x, r_N)$.

In B_I, we use a crude upper bound for $g(x,\cdot)$, derived from (2.14),

$$(2.28)\qquad g(x,y) \leq \frac{c}{(\max\{|x-y|_\infty, 1\})^{d-2}}.$$

As a result, we find that

$$(2.29)\qquad \sum_{y\in B_I} g(x,y) \leq \sum_{l=0}^{\lceil r_N\rceil} \sum_{y:|y-x|_\infty = l} \frac{c}{(\max\{l,1\})^{d-2}} \leq c' r_N^2.$$

Let $\overline{x} = \frac{N}{|x|}x$ denote the projection of x onto the Euclidean sphere of radius N centered at 0. It is straightforward to see that

$$(2.30)\qquad \int_{B_{\mathbb{R}^d}(\overline{x}, r_N)} G(y-\overline{x})dy \leq c r_N^2.$$

By the asymptotic approximation of discrete Green function (see (2.14) and (2.15)), writing $\widehat{B} = B_{\mathbb{R}^d}(0,N)\backslash B_{\mathbb{R}^d}(\overline{x}, r_N)$, we obtain with a Riemann sum approximation argument that

$$
\begin{aligned}
&\Big|\sum_{y\in B_J} g(x,y) - d\int_{\widehat{B}} G(\overline{y}-\overline{x})d\overline{y}\Big| \\
&\leq \Big|\sum_{y\in B_J} g(x,y) - d\int_{\widehat{B}} G(\overline{y}-x)d\overline{y}\Big| \\
&+ d\Big|\int_{\widehat{B}} (G(\overline{y}-x) - G(\overline{y}-\overline{x}))d\overline{y}\Big| \\
&\leq cN.
\end{aligned} \tag{2.31}
$$

Thanks to the scaling property and rotation invariance of Brownian motion, writing

$$
c_1 = d\int_{B_{\mathbb{R}^d}(0,1)} G(\overline{y}-\overline{z})d\overline{y}, \text{ where } \overline{z}\in\mathbb{R}^d \text{ with } |z|=1 \text{ is arbitrary} \tag{2.32}
$$

(c_1/d is the expected time spent by Brownian motion in a ball of radius 1 when starting from its boundary), and putting (2.29), (2.30) and (2.31) together, we see that

$$
\Big|E_x\Big[\int_0^\infty 1_{B(0,N)}(X_s)ds\Big] - c_1N^2\Big| \leq cr_N^2 + c'N. \tag{2.33}
$$

By the definition of r_N in (2.27), we obtain (2.26) as desired. □

We now introduce a positive martingale, which plays an important role in the definition of the tilted interlacements in the next section. We will show in the lemma below that this martingale is uniformly integrable, and we will use its limiting value as a probability density.

Given a real-valued function h on $\mathbb{Z}^d$, we denote its discrete Laplacian by

$$
\Delta_{dis}h(x) = \frac{1}{2d}\sum_{|e|=1} h(x+e) - h(x). \tag{2.34}
$$

We consider a positive function f on $\mathbb{Z}^d$, which is equal to 1 outside a finite set, and we write

$$V = -\frac{\Delta_{dis} f}{f}. \tag{2.35}$$

We also introduce the stochastic process

$$M_t = \frac{f(X_t)}{f(X_0)} \exp\Big\{ \int_0^t V(X_s) ds \Big\}, t \geq 0, \tag{2.36}$$

and define for all $x \in \mathbb{Z}^d$, $T > 0$ the positive measure $\widetilde{P}_{x,T}$ (on $\widehat{W}_+$ with density M_T with respect to P_x):

$$\widetilde{P}_{x,T} = M_T P_x. \tag{2.37}$$

The next lemma plays an important role in the construction of the tilted interlacements.

Lemma 2.2.2. *For all* $x \in \mathbb{Z}^d$*,*

$$(M_t)_{t \geq 0} \text{ is an } (\mathcal{F}_t)\text{-martingale under } P_x, \tag{2.38}$$

and

$$(M_t)_{t \geq 0} \text{ is uniformly integrable under } P_x. \tag{2.39}$$

Moreover,

$$1 = E_x[M_\infty] = \frac{1}{f(x)} E_x[e^{\int_0^\infty V(X_s)ds}]. \tag{2.40}$$

Proof. The first claim (2.38) is classical. It follows for instance from Lemma 3.2, p. 174 in Chapter 4 of [27]. Note that $E_x[M_0] = 1$, so $\widetilde{P}_{x,T}$ is a probability measure for each T. Using the Markov property of X under P_x and (2.38), it readily follows that $(X_t)_{0 \leq t \leq T}$ under $\widetilde{P}_{x,T}$ is a Markov chain. By Theorem 2.5, p. 61 of [21], its semi-group (acting on the Banach space of functions on $\mathbb{Z}^d$ tending to zero at infinity) has a generator given by the bounded operator:

$$
(2.41) \qquad \begin{aligned} \widetilde{L}h &= \frac{1}{f}\Delta_{dis}(fh) - \frac{\Delta_{dis} f}{f} h, \text{ so that} \\ \widetilde{L}h(x) &= \frac{1}{2d} \sum_{|e|=1} \frac{f(x+e)}{f(x)} \big(h(x+e) - h(x)\big). \end{aligned}
$$

We introduce the law $\widetilde{Q}_x$ on $\Gamma(\mathbb{Z}^d)$ of the jump process starting from x, corresponding to the generator $\widetilde{L}$ defined as in (2.41). Outside some finite set $f = 1$, and by (2.41), outside the (discrete) closure of this finite set, this process jumps as a simple random walk. As a result, the canonical jump process attached to $\widetilde{Q}_x$ is transient. In addition, up to time T, it has the same law as $(X_t)_{0 \le t \le T}$ under $\widetilde{P}_{x,T}$. Therefore, the claim (2.39) will follow once we show that

$$
(2.42) \qquad \sup_{t \ge 0} E_x[M_t \log M_t] = \sup_{T \ge t \ge 0} \widetilde{E}_{x,T}[\log M_t] = \sup_{t \ge 0} E^{\widetilde{Q}_x}[\log M_t] < \infty.
$$

Now, setting $g = \log f$, we split $E^{\widetilde{Q}_x}[\log M_t]$ into two parts

$$
(2.43) \qquad \begin{aligned} E^{\widetilde{Q}_x}[\log M_t] &= E^{\widetilde{Q}_x}\Big[g(X_t) - g(X_0) + \int_0^t V(X_s) ds\Big] \\ &= E^{\widetilde{Q}_x}\Big[g(X_t) - g(X_0) - \int_0^t \widetilde{L}g(X_s) ds\Big] \\ &\quad + E^{\widetilde{Q}_x}\Big[\int_0^t (\widetilde{L}g + V)(X_s) ds\Big]. \end{aligned}
$$

The first term after the second equality of (2.43) is zero since $g(X_t) - g(X_0) - \int_0^t \widetilde{L}g(X_s)ds$ is a martingale under $\widetilde{Q}_x$ (see Proposition 1.7, p. 162 of [27]). As for the second term, we write

$$
(2.44) \qquad \psi = \widetilde{L}g + V.
$$

By (2.41) we see that

$$\widetilde{L}g(x) = \frac{1}{2d}\sum_{|e|=1}\frac{f(x+e)}{f(x)}(g(x+e)-g(x)). \tag{2.45}$$

Hence, with a straightforward calculation and the fact that

$$(1+u)\log(1+u) - u \geq 0, \text{ for } u > -1, \tag{2.46}$$

we see that

$$\psi(x) = \frac{1}{2d}\sum_{|e|=1}\Big(\frac{f(x+e)}{f(x)}\log\frac{f(x+e)}{f(x)} - \frac{f(x+e)-f(x)}{f(x)}\Big) \geq 0, \tag{2.47}$$

and that $\psi(x)$ is finitely supported.

Therefore, due to the transience of the canonical process under $\widetilde{Q}_x$,

$$\sup_{t\geq 0} E^{\widetilde{Q}_x}\Big[\int_0^t \psi(X_s)ds\Big] \overset{(2.47)}{\leq} E^{\widetilde{Q}_x}\Big[\int_0^\infty \psi(X_s)ds\Big] < \infty, \tag{2.48}$$

whence (2.42).

The last claim (2.40) follows by uniform integrability. Indeed, the martingale converges P_x-a.s. and in $L^1(P_x)$ towards

$$M_\infty = \frac{1}{f(X_0)}\exp\Big\{\int_0^\infty V(X_s)ds\Big\}, \tag{2.49}$$

so we have,

$$E_x[M_\infty] = E_x[M_0] = 1. \tag{2.50}$$

This finishes the proof. □

We thus define for all x in $\mathbb{Z}^d$ the positive measure on $\widehat{W}_+$:

$$\widetilde{P}_x \overset{\text{def}}{=} M_\infty P_x = \frac{1}{f(x)}\exp\Big\{\int_0^\infty V(X_s)ds\Big\}P_x. \tag{2.51}$$

The following corollary is a consequence of Lemma 2.2.2 and its proof.

Corollary 2.2.3. *For all x in $\mathbb{Z}^d$,*

$$\widetilde{P}_x \text{ is a probability measure.} \tag{2.52}$$

Moreover, $(X_t)_{t\geq 0}$ under $\widetilde{P}_x$, $x \in \mathbb{Z}^d$, is a reversible Markov chain on $\mathbb{Z}^d$ with reversible measure

$$\widetilde{\lambda}(x) = f^2(x),\ x \in \mathbb{Z}^d, \tag{2.53}$$

and its semi-group in $L^2(\widetilde{\lambda})$ has the bounded generator

$$\widetilde{L}h(x) = (\frac{1}{f}\,\Delta_{dis}(fh) + Vh)(x) = \frac{1}{2d}\sum_{|e|=1}\frac{f(x+e)}{f(x)}(h(x+e)-h(x)), \tag{2.54}$$

for all h in $L^2(\widetilde{\lambda})$ and x in $\mathbb{Z}^d$. (Note that X has variable jump rate under $\widetilde{P}_x$, unless f is constant.)

Similar to the results in potential theory for the continuous-time simple random walk earlier in this section, we can also define for $(X_t)_{t\geq 0}$ under $\{\widetilde{P}_x\}_{x\in\mathbb{Z}^d}$ the corresponding notions such as (killed) Green function, equilibrium measure, and capacity. We also refer to Section 2.1 and 2.2 of Chapter 2 and Section 4.2 of Chapter 4 of [30] for more details. We denote the corresponding objects with a tilde, and refer to them as tilted objects.

Specifically, we write $\widetilde{g}$ and $\widetilde{g}_U$ for the tilted Green function and killed Green function (outside $U \subseteq \mathbb{Z}^d$):

$$\begin{aligned} \widetilde{g}(x,y) &= \frac{1}{\widetilde{\lambda}(y)}\widetilde{E}_x\Big[\int_0^\infty 1_{\{X_s=y\}}ds\Big], \\ \widetilde{g}_U(x,y) &= \frac{1}{\widetilde{\lambda}(y)}\widetilde{E}_x\Big[\int_0^{T_U} 1_{\{X_s=y\}}ds\Big]. \end{aligned} \tag{2.55}$$

One knows that $\widetilde{g}$ and $\widetilde{g}_U$ are symmetric and finite. Given $M \subset\subset \mathbb{Z}^d$, the tilted equilibrium measure and tilted capacity of M are defined

as:
(2.56)
$$\widetilde{e}_M(x) = \widetilde{P}_x[\widetilde{H}_M = \infty] 1_M(x) f(x) \Big(\frac{1}{2d} \sum_{|e|=1} f(x+e) \Big), \ \ \text{for } x \in \mathbb{Z}^d$$

(the expression after the indicator function of M is a reversibility measure of the discrete skeleton of the continuous-time chain, which can be viewed as a random walk among the conductances $\frac{1}{2d} f(x) f(y)$, for x, y neighbors in $\mathbb{Z}^d$, and $\widetilde{g}(\cdot,\cdot)$ is also the corresponding Green density of this discrete-time walk). Then (see (2.2.13), p. 79 of [30])

$$\widetilde{\text{cap}}(M) = \sum_{x \in M} \widetilde{e}_M(x). \tag{2.57}$$

Moreover, the following identities, analogues of (2.23) and (2.25), are valid:

$$\sum_{y \in M} \widetilde{g}(x,y) \widetilde{e}_M(y) = 1, \ \ \text{for all } x \in M, \tag{2.58}$$

and for $M \subset M' \subset\subset \mathbb{Z}^d$,

$$\widetilde{P}_{\widetilde{e}_{M'}}[H_M < \infty, X_{H_M} = y] = \sum_{x \in M'} \widetilde{e}_{M'}(x) \widetilde{h}_M(x,y) = \widetilde{e}_M(y) \tag{2.59}$$

for all $y \in M$, where for $A \subseteq B \subseteq \mathbb{Z}^d$, $x \in \mathbb{Z}^d$, $y \in A$,

$$\begin{aligned} \widetilde{h}_A(x,y) &= \widetilde{P}_x[H_A < \infty, X_{H_A} = y] \\ \widetilde{h}_{A,B}(x,y) &= \widetilde{P}_x[H_A < T_B, X_{H_A} = y] \end{aligned} \tag{2.60}$$

are the respective tilted entrance measure in A and tilted entrance measure in A relative to B, when starting at x.

We now turn to continuous-time random interlacements. We refer to [59] for more details. We define $\widehat{W}^* = \widehat{W}/\sim$, where $\widehat{w} \sim \widehat{w}'$ is defined as $\widehat{w}(\cdot) = \widehat{w}'(\cdot + k)$ for some $k \in \mathbb{Z}$, for $\widehat{w}, \widehat{w}' \in \widehat{W}$. We also define the canonical map as $\pi^* : \widehat{W} \to \widehat{W}^*$. We write $\widehat{W}^*_M$ for the subset of $\widehat{W}^*$ of trajectories modulo time-shift that intersect

$M \subset\subset \mathbb{Z}^d$. For $\widehat{w}^* \in \widehat{W}^*_M$, we write $\widehat{w}^*_{M,+}$ for the unique element of $\widehat{W}^+$, which follows $\widehat{w}^*$ step by step from the first time it enters M.

The continuous-time random interlacement can be seen as a Poisson point process on the space $\widehat{W}^*$, with intensity measure $u\,\widehat{\nu}$, where $u > 0$ and $\widehat{\nu}$ is a σ-finite measure on $\widehat{W}$ such that its restriction to $\widehat{W}^*_M$ (denoted by $\widehat{\nu}_M$), is equal to $\pi^* \circ \widehat{Q}_M$, where $\widehat{Q}_M$ is a finite measure on $\widehat{W}$ such that (see (1.7) in [59]) if $(X_t)_{t\in\mathbb{R}}$, is the continuous-time process attached to $\widehat{w} \in \widehat{W}$, then

$$\widehat{Q}_M[X_0 = x] = e_M(x), \tag{2.61}$$

and when $e_M(x) > 0$,

(2.62) under $\widehat{Q}_M$ conditioned on $X_0 = x$, $(X_t)_{t\geq 0}$ and the right-continuous regularization of $(X_{-t})_{t>0}$ are independent and have same respective distribution as $(X_t)_{t\geq 0}$ under P_x and X after its first jump under $P_x[\cdot|\widetilde{H}_M = \infty]$

We define the space Ω of point measures on $\widehat{W}^*$ as

$$\Omega = \left\{ \begin{array}{l} \widehat{\omega} = \sum_{i\geq 0} \delta_{\widehat{w}^*_i}; \widehat{w}^*_i \in \widehat{W}^* \text{ for all } i \geq 0, \\ \widehat{\omega}(\widehat{W}^*_M) < \infty \text{ for all } M \subset\subset \mathbb{Z}^d \end{array} \right\}. \tag{2.63}$$

If $F : \widehat{W}^* \to \mathbb{R}$ and $\widehat{\omega} = \sum_i \delta_{\widehat{w}^*_i}$, we write $< \widehat{\omega}, F >= \sum_i F(\widehat{w}^*_i)$ for the integral of F with respect to $\widehat{\omega}$. Given $M \subset\subset \mathbb{Z}^d$ and $\widehat{\omega} = \sum_{i\geq 0} \delta_{\widehat{w}^*_i}$ in Ω, we let $\mu_M(\widehat{\omega})$ stand for the point measure on $\widehat{W}^+$, $\mu_M(\widehat{\omega}) = \sum_{i\geq 0} 1_{\widehat{w}^*_i \in \widehat{W}^*_M} \delta_{(\widehat{w}^*_i)_{M,+}}$, which collects the cloud of onward trajectories after the first entrance in M (see below (2.60) for notation).

We write $\mathbb{P}_u$ for the probability measure governing random interlacements at level u, that is the canonical law on Ω of the Poisson point process on $\widehat{W}^*$ with intensity measure $u\,\widehat{\nu}$. We write $\mathbb{E}_u$ for its expectation. Given $\widehat{\omega} = \sum_i \delta_{\widehat{w}^*_i}$, we define the interlacement set and vacant set at level u respectively as the random subsets of $\mathbb{Z}^d$:

$$\mathcal{I}^u(\widehat{\omega}) = \{\cup_i \text{Range}(\widehat{w}_i^*)\} \tag{2.64}$$

where for $\widehat{w}^*$ in $\widehat{W}^*$, Range$(\widehat{w}^*)$ stands for the set of points in $\mathbb{Z}^d$ visited by any $\widehat{w}$ in $\widehat{W}$ with $\pi^*(\widehat{w}) = \widehat{w}^*$, and

$$\mathcal{V}^u = \mathbb{Z}^d \backslash (\mathcal{I}^u(\widehat{\omega})). \tag{2.65}$$

The above random sets have the same law as $\mathcal{I}^u$ or $\mathcal{V}^u$ in [56].

The connectivity function of the vacant set of random interlacements is known to have a stretched-exponential decay when the level exceeds a certain critical value (see Theorem 4.1 of [58], or Theorem 0.1 of [53], and Theorem 3.1 of [46] for recent developments). Namely, there exists a $u_{**} \in (0,\infty)$, which, for our purpose in this article, can be characterized as the smallest positive number such that for all $u > u_{**}$,

$$\mathbb{P}_u[0 \overset{\mathcal{V}^u}{\leftrightarrow} \partial B_\infty(0,N)] \leq c_2(u) e^{-c_3(u) N^{c_4(u)}}, \text{ for all } N \geq 0. \tag{2.66}$$

(actually, by Theorem 3.1 of [46], one can choose $c_4 = 1$, when $d \geq 4$, and $c_4 = \frac{1}{2}$ or any other value in $(0,1)$, when $d = 3$).

We also wish to recall a classical result on relative entropy which will be helpful in Section 2. For $\widetilde{\mathbb{P}}$ absolutely continuous with respect to $\mathbb{P}$, the relative entropy of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$ is defined as

$$H(\widetilde{\mathbb{P}}|\mathbb{P}) = \widetilde{\mathbb{E}}\Big[\log \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\Big] = \mathbb{E}\Big[\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} \log \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\Big] \in [0,\infty]. \tag{2.67}$$

For an event A with positive $\widetilde{\mathbb{P}}$-probability, we have the following inequality (see p. 76 of [22]):

$$\mathbb{P}[A] \geq \widetilde{\mathbb{P}}[A] e^{-\frac{1}{\widetilde{\mathbb{P}}[A]}(H(\widetilde{\mathbb{P}}|\mathbb{P}) + \frac{1}{e})}. \tag{2.68}$$

We end this section by recalling one property of the Poisson point process on general spaces. It rephrases Lemma 1.4 of [42]. Let μ be a Poisson point process on E with finite intensity measure η (i.e.

$\eta(E) < \infty$), and let $\Phi : E \to \mathbb{R}$ be a measurable function. Then, one has

$$E[e^{<\omega,\Phi>}] = e^{\int_E e^{\Phi}-1 d\mu} \tag{2.69}$$

(this is an identity in $(0, +\infty]$).

2.3 The tilted interlacements

In this section, we define a new probability measure $\widetilde{\mathbb{P}}_N$ on $\widehat{W}^*$, which is absolutely continuous with respect to $\mathbb{P}_u$, see Proposition 2.3.1. It governs a Poisson point process on $\widehat{W}^*$, which corresponds to the "tilted random interlacements". Intuitively, these tilted interlacements describe a kind of slowly space-modulated random interlacements. The motivation for the exponential tilt entering the definition of $\widetilde{\mathbb{P}}_N$ actually stems from the analysis of certain large deviations of the occupation-time profile of random interlacements considered in [42], see Remark 2.3.5 below. In Proposition 2.3.1 we compute the relative entropy of $\widetilde{\mathbb{P}}_N$ with respect to $\mathbb{P}_u$, and we then relate this result to the capacity of K after a suitable limiting procedure, see Proposition 2.3.4.

We begin with the construction of the new measure $\widetilde{\mathbb{P}}_N$, which will correspond to an exponential tilt of $\mathbb{P}_u$, see (2.76).

We recall that K is a compact subset of $\mathbb{R}^d$ as above (2.1). We consider δ, ϵ in $(0,1)$, and let U and $\widetilde{U}$ be the open Euclidean balls centered at 0 with respective radii r_U and $r_{\widetilde{U}}$, where $r_U > 0$ and $r_{\widetilde{U}} = r_U + 4$. We assume that r_U is sufficiently large such that $K^{2\delta} \subset U \subset \widetilde{U} \subset \mathbb{R}^d$ (recall that $K^{2\delta}$ stands for the closed 2δ-neighborhood of K, see below (2.8)). By the end of this section we will eventually let $r_U, r_{\widetilde{U}}$ tend to infinity and then let δ tend to 0. We denote by W_z the Wiener measure starting from z and by H_F, for F a closed subset of $\mathbb{R}^d$, the entrance time of the canonical Brownian motion in F. We write

$$h(z) = W_z[H_{K^{2\delta}} < T_U], \ z \in \mathbb{R}^d, \tag{2.70}$$

for the equilibrium potential of $K^{2\delta}$ relative to U. For $\eta \in (0, \delta)$ and ϕ^η a non-negative smooth function supported in $B_{\mathbb{R}^d}(0, \eta)$ such that $\int \phi^\eta(z)dz = 1$, we write

$$h^\eta = h * \phi^\eta \tag{2.71}$$

for the convolution of h and ϕ^η.

We then define the restriction to $\mathbb{Z}^d$ of the blow-up of h as

$$h_N(x) = h^\eta\Big(\frac{x}{N}\Big), \text{ for } x \in \mathbb{Z}^d. \tag{2.72}$$

We now specify our choice of f in (2.36) as

$$f(x) = \Big(\sqrt{\frac{u_{**} + \epsilon}{u}} - 1\Big)\, h_N(x) + 1, \tag{2.73}$$

and recall that

$$V = -\frac{\Delta_{dis} f}{f}.$$

f and V tacitly depend upon $\epsilon, \delta, \eta, N$. We drop this dependence from the notation for the sake of simplicity. We denote by $\widetilde{U}_N$ the discrete blow-up of $\widetilde{U}$ (as in (2.1) or (2.9)). We also note that

$$f = 1 \text{ on } (\mathbb{Z}^d \backslash \widetilde{U}_N) \cup \overline{\partial_i \widetilde{U}_N}, \text{ and for large } N,\ f = \sqrt{\tfrac{u_{**}+\epsilon}{u}} \text{ on } K_N^\delta. \tag{2.74}$$

From now on, we will denote by $\widetilde{P}_x$ the probability measure defined in (2.51), with f as in (2.73).

We define a function F on $\widehat{W}^*$ through

$$F(\widehat{w}^*) = \begin{cases} \displaystyle\int_0^\infty V(X_s)(\widehat{w}_{\widetilde{U}_N})ds, \text{ for } \widehat{w}^* \in \widehat{W}^*_{\widetilde{U}_N}, \text{ with } \pi^*(\widehat{w}) = \widehat{w}^*, \\ \text{and } \widehat{w}_{\widetilde{U}_N} \text{ the time-shift of } \widehat{w} \text{ at its first entrance in } \widetilde{U}_N, \\ 0, \qquad \text{otherwise.} \end{cases} \tag{2.75}$$

We refer to (2.63) for the definition of Ω.

Proposition 2.3.1.

$$\widetilde{\mathbb{P}}_N = e^{<\widehat{\omega},F>}\mathbb{P}_u \text{ defines a probability measure on } \Omega. \tag{2.76}$$

Moreover, under $\widetilde{\mathbb{P}}_N$

$$\begin{array}{c}\text{the canonical point measure } \widehat{\omega} \text{ is a Poisson point process} \\ \text{on } \widehat{W}^* \text{ with intensity measure } u\widetilde{\nu}, \text{ where } \widetilde{\nu} = e^F\widehat{\nu},\end{array} \tag{2.77}$$

and for $M \subset\subset \mathbb{Z}^d$ *(see below (2.63) for notation),*

$$\begin{array}{c}\mu_M \text{ is a Poisson point process on } \widehat{W}^+ \\ \text{with intensity measure } u\widetilde{P}_{\widetilde{e}_M}.\end{array} \tag{2.78}$$

Proof. We begin with the proof of (2.76). By the first equality of (2.74) and using (2.40) of Lemma 2.2.2, we see that for all $x \in \partial_i\widetilde{U}_N$,

$$E_x[e^{\int_0^\infty V(X_s)ds}] = 1. \tag{2.79}$$

Since F vanishes outside $\widehat{W}^*_{\widetilde{U}_N}$, it follows that

$$\int_{\widehat{W}^*}(e^F-1)d\widehat{\nu} = \int_{\widehat{W}^*_{\widetilde{U}_N}}(e^F-1)d\widehat{\nu} \overset{(2.62)}{=} E_{e_{\widetilde{U}_N}}[e^{\int_0^\infty V(X_s)ds}-1] \overset{(2.79)}{=} 0, \tag{2.80}$$

and by (2.69),

$$\mathbb{E}_u[e^{<\widehat{\omega},F>}] = 1, \tag{2.81}$$

whence (2.76). We now turn to the proof of (2.77).

Writing $\widetilde{\mathbb{E}}_N$ as the expectation under $\widetilde{\mathbb{P}}_N$, taking G a non-negative, measurable function on $\widehat{W}^*$, we have

$$\begin{aligned}
\widetilde{\mathbb{E}}_N[e^{-<\widehat{\omega},G>}] &\overset{(2.76)}{=} \mathbb{E}_u[e^{<\widehat{\omega},F-G>}] \\
&\overset{(2.80)}{=} \mathbb{E}_u[e^{<\widehat{\omega},F-G>}]e^{-u\int(e^F-1)d\widehat{\nu}} \\
&\underset{\text{on } \widehat{W}_{\widetilde{U}_N}}{\overset{(2.69)}{=}} e^{u\int(e^{F-G}-1)d\widehat{\nu}-u\int(e^F-1)d\widehat{\nu}} \\
&= e^{u\int(e^{-G}-1)e^F d\widehat{\nu}}.
\end{aligned} \tag{2.82}$$

This identifies the Laplace transform of $\widehat{\omega}$ under $\widetilde{\mathbb{P}}_N$ and (2.77) follows by Proposition 36, p. 130 of [48].

There remains to prove (2.78). By (2.77) and the definition of μ_M (below (2.63)), we see that μ_M is a Poisson point process on $\widehat{W}^+$ with intensity measure $u\gamma_M$, where γ_M is the image of $1_{\widehat{W}^*_M}\widetilde{\nu}$ under the map $\widehat{w}^* \to \widehat{w}^*_{M,+}$ (see above (2.61) for notation). The claim (2.78) will thus follow once we show that

$$\gamma_M = \widetilde{P}_{\widetilde{e}_M}. \tag{2.83}$$

We introduce $\widetilde{M} = M \cup \widetilde{U}_N$. We observe that

$$\widetilde{e}_{\widetilde{M}} = e_{\widetilde{M}}. \tag{2.84}$$

Indeed, this follows by (2.18) and (2.56), together with the first equality in (2.74). We also note that in (2.75) the function F does not change if we replace $\widetilde{U}_N$ in the definition by $\widetilde{M}$, since $\widetilde{U}_N \subset \widetilde{M}$, and V vanishes outside $\widetilde{U}_N$. Therefore, in order to prove (2.83), it suffices to verify that for any bounded measurable function $g : \widehat{W}^+ \to \mathbb{R}$, its integral with respect to γ_M coincides with that with respect to $\widetilde{P}_{\widetilde{e}_M}$. We begin with $\langle \gamma_M, g\rangle$. By the definition of γ_M:

$$\begin{aligned}
<\gamma_M, g> &= \int_{\widehat{W}^*_{\widetilde{M}}} e^F 1_{\{\widehat{w}^*\in\widehat{W}^*_M\}} g(\widehat{w}^*_{M,+}) d\widehat{\nu}(\widehat{w}^*) \\
&\underset{(2.62)}{\overset{(2.61)}{=}} E_{e_{\widetilde{M}}}[e^{\int_0^\infty V(X_s)ds} g(\widehat{w}_M) 1_{\{H_M<\infty\}}],
\end{aligned} \tag{2.85}$$

where for $\widehat{w} \in \widehat{W}^+$, we let $\widehat{w}_M \in \widehat{W}^+$ stand for the time-shift of $\widehat{w}$ starting at its first entrance in M. We then apply the strong Markov property at H_M, and decompose according to where the walks enter M,

$$(2.86)\quad \begin{aligned} &< \gamma_M, g > \\ \overset{\text{Markov}}{=}\; & E_{e_{\widetilde{M}}}[e^{\int_0^{H_M} V(X_s)ds} 1_{\{H_M<\infty\}} E_{X_{H_M}}[e^{\int_0^\infty V(X_s)ds} g]] \\ =\; & E_{e_{\widetilde{M}}}\Big[f(X_{H_M}) e^{\int_0^{H_M} V(X_s)ds} 1_{\{H_M<\infty\}} \times \\ & \qquad E_{X_{H_M}}\big[\tfrac{1}{f(X_0)} e^{\int_0^\infty V(X_s)ds} g\big]\Big] \\ =\; & \sum_{y\in\partial_i M} E_{e_{\widetilde{M}}}\Big[f(y) e^{\int_0^{H_M} V(X_s)ds} 1_{\{H_M<\infty, X_{H_M}=y\}}\Big] \times \\ & \qquad E_y\Big[\tfrac{1}{f(y)} e^{\int_0^\infty V(X_s)ds} g\Big] \\ \underset{\text{Markov}}{\overset{(2.51)}{=}}\; & \sum_{y\in\partial_i M} \widetilde{P}_{e_{\widetilde{M}}}[H_M<\infty, X_{H_M}=y]\widetilde{E}_y[g]. \end{aligned}$$

On the other hand, we can express $\widetilde{P}_{\widetilde{e}_M}$ in terms of the tilted entrance measure by the sweeping identity (see (2.59)) and incorporate the fact that the tilted equilibrium measure of $\widetilde{M}$ coincides with the standard equilibrium measure of $\widetilde{M}$:

$$(2.87)\quad \begin{aligned} \widetilde{E}_{\widetilde{e}_M}[g] &\overset{(2.59)}{=} \textstyle\sum_{y\in\partial_i M} \widetilde{P}_{\widetilde{e}_{\widetilde{M}}}[H_M<\infty, X_{H_M}=y]\widetilde{E}_y[g] \\ &\overset{(2.84)}{=} \textstyle\sum_{y\in\partial_i M} \widetilde{P}_{e_{\widetilde{M}}}[H_M<\infty, X_{H_M}=y]\widetilde{E}_y[g]. \end{aligned}$$

Comparing (2.86) and (2.87), we obtain (2.83). □

We will call the canonical Poisson point process under $\widetilde{\mathbb{P}}_N$ the tilted random interlacements.

Remark 2.3.2. The tilted interlacements do retain an interlacement-like character because $\widetilde{\nu} = e^F \widehat{\nu}$ is a measure on $\widehat{W}^*$, which has the following property. Its restriction to $\widehat{W}^*_M$, for $M \subset\subset \mathbb{Z}^d$, is equal to $\pi^* \circ \widetilde{Q}_M$, where

$$(2.88)\quad \widetilde{Q}_M[X_0 = x] = \widetilde{e}_M(x),$$

and when $\widetilde{e}_M(x) > 0$,

(2.89) under $\widetilde{Q}_M$ conditioned on $X_0 = x$, $(X_t)_{t\geq 0}$ and the right-continuous regularization of $(X_{-t})_{t>0}$ are independent and with same respective distribution as $(X_t)_{t\geq 0}$ under $\widetilde{P}_x$ and X after its first jump under $\widetilde{P}_x[\cdot|\widetilde{H}_M = \infty]$.

We do not need the above fact, but mention it because it states the property analogous to (2.61) and (2.62) satisfied by $\tilde{\nu}$. □

We will now calculate the relative entropy of $\widetilde{\mathbb{P}}_N$ with regard to $\mathbb{P}_u$ and relate it to the Dirichlet form of h_N (see (2.21) for notation).

Proposition 2.3.3.

$$H(\widetilde{\mathbb{P}}_N|\mathbb{P}_u) = (\sqrt{u_{**}+\epsilon} - \sqrt{u})^2 \mathcal{E}_{\mathbb{Z}^d}(h_N, h_N). \tag{2.90}$$

Proof. By the definition of relative entropy (see (2.67)),

$$H(\widetilde{\mathbb{P}}_N|\mathbb{P}_u) = \widetilde{\mathbb{E}}_N[\log \frac{d\widetilde{\mathbb{P}}_N}{d\mathbb{P}_u}] \overset{(2.76)}{=} \widetilde{\mathbb{E}}_N[< \widehat{\omega}, F >], \tag{2.91}$$

and

$$\begin{aligned} & \widetilde{\mathbb{E}}_N[< \widehat{\omega}, F >] \\ = \; & u < \widetilde{\nu}, F > \\ \underset{(2.78)}{\overset{(2.75)}{=}} \; & u\widetilde{E}_{\widetilde{e}_{\widetilde{U}_N}}\Big[\int_0^\infty V(X_s)ds\Big] \\ \overset{(2.55)}{=} \; & u \textstyle\sum_{x\in\widetilde{U}_N,\, x'\in\mathbb{Z}^d} \widetilde{e}_{\widetilde{U}_N}(x)\widetilde{g}(x,x')V(x')\widetilde{\lambda}(x') \\ \underset{\text{supp } V\subseteq\widetilde{U}_N}{\overset{(2.58)}{=}} \; & u \textstyle\sum_{x'\in\mathbb{Z}^d} V(x')\widetilde{\lambda}(x') \\ \underset{(2.35)}{\overset{(2.53)}{=}} \; & -u \textstyle\sum_{x\in\mathbb{Z}^d} f(x)\Delta_{dis} f(x). \end{aligned} \tag{2.92}$$

We also have, by the definition of f in (2.73), that
(2.93)

$$-u \sum_{x\in\mathbb{Z}^d} f(x)\Delta_{dis} f(x) = u \sum_{x\in\mathbb{Z}^d} \Big(\sqrt{\frac{u_{**}+\epsilon}{u}} - 1\Big) f(x)\Delta_{dis} h_N(x)$$

and since h_N is finitely supported, by the Green-Gauss theorem, the left-hand side of (2.93) equals

(2.94)

$$= \; u\Big(\sqrt{\frac{u_{**}+\epsilon}{u}} - 1\Big)\frac{1}{2} \sum_{|x-x'|=1} \frac{1}{2d}(f(x')-f(x))(h_N(x')-h_N(x))$$

$$\stackrel{(2.73)}{=} u \sum_{x'\in\mathbb{Z}^d} \Big(\sqrt{\frac{u_{**}+\epsilon}{u}} - 1\Big)^2 \mathcal{E}_{\mathbb{Z}^d}(h_N, h_N),$$

and (2.90) follows. □

We will now successively let $N \to \infty$, $\eta \to 0$, $r_U \to \infty$, and $\delta \to 0$. The capacity of K will appear in the limit (in the above sense) of the properly scaled Dirichlet form of h_N.

Proposition 2.3.4.

$$\text{(2.95)} \qquad \lim_{\delta\to 0} \lim_{r_U\to\infty} \lim_{\eta\to 0} \lim_{N\to\infty} \frac{1}{N^{d-2}} \mathcal{E}_{\mathbb{Z}^d}(h_N, h_N) = \frac{1}{d}\mathrm{cap}_{\mathbb{R}^d}(K).$$

Proof. First, by the definition of h_N and (2.21) we have

$$\frac{1}{N^{d-2}}\mathcal{E}_{\mathbb{Z}^d}(h_N,h_N) = \frac{1}{N^{d-2}} \sum_{x\in\mathbb{Z}^d} \sum_{|e|=1} \frac{1}{4d}(h_N(x+e)-h_N(x))^2$$

$$\text{(2.96)} \qquad \underset{(2.72)}{\stackrel{(2.74)}{=}} \frac{1}{4dN^d} \sum_{x\in\widetilde{U}_N} \sum_{|e|=1} N^2\Big(h^\eta\Big(\frac{x+e}{N}\Big) - h^\eta\Big(\frac{x}{N}\Big)\Big)^2.$$

Then, we take the limit of both sides. By the smoothness of h^η and

a Riemann sum argument we have:
(2.97)
$$\lim_{N\to\infty} \frac{1}{N^{d-2}} \mathcal{E}_{\mathbb{Z}^d}(h_N, h_N) = \frac{1}{2d} \int |\nabla h^\eta(y)|^2 dy = \frac{1}{d} \mathcal{E}_{\mathbb{R}^d}(h^\eta, h^\eta),$$

where $\mathcal{E}_{\mathbb{R}^d}(\cdot,\cdot)$ denotes the usual Dirichlet form on $\mathbb{R}^d$.

Since h in (2.70) belongs to $H^1(\mathbb{R}^d)$, see Theorem 4.3.3, p. 152 of [30] (due to the killing outside of U, the extended Dirichlet space is contained in $H^1(\mathbb{R}^d)$), $h^\eta \to h$ in $H^1(\mathbb{R}^d)$, as $\eta \to 0$. We thus find that

$$\text{(2.98)} \qquad \lim_{\eta\to 0} \mathcal{E}_{\mathbb{R}^d}(h^\eta, h^\eta) = \mathcal{E}_{\mathbb{R}^d}(h, h) = \mathrm{cap}_{\mathbb{R}^d, U}(K^{2\delta}),$$

where $\mathrm{cap}_{\mathbb{R}^d,U}(K^{2\delta})$ is the relative capacity of $K^{2\delta}$ with respect to U, and the last equality follows from [30], pp. 152 and 71.

Letting $r_U \to \infty$, the relative capacity converges to the usual Brownian capacity (this follows for instance from the variational characterization of the capacity in Theorem 2.1.5 on pp. 70 and 71 of [30]):

$$\text{(2.99)} \qquad \mathrm{cap}_{\mathbb{R}^d,U}(K^{2\delta}) \to \mathrm{cap}_{\mathbb{R}^d}(K^{2\delta}), \text{ as } r_U \to \infty.$$

Then, letting $\delta \to 0$, by Proposition 1.13, p. 60 of [47], we find that

$$\text{(2.100)} \qquad \mathrm{cap}_{\mathbb{R}^d}(K^{2\delta}) \to \mathrm{cap}_{\mathbb{R}^d}(K), \text{ as } \delta \to 0.$$

The claim (2.95) follows. □

Remark 2.3.5. Our main objective in the next two sections is to prove (2.4), i.e. $\widetilde{\mathbb{P}}_N[A_N] \to 1$. Actually, we could also use the above $\widetilde{\mathbb{P}}_N$ (with $a > u$ in place of u_{**} in the definition of f in (2.73)) and the change of probability method to provide an alternative proof of Theorem 6.4 of [42] (it derives the asymptotic lower bound for the probability that the regularized occupation-time profile of random interlacements insulates K by values exceeding a). It is a remarkable feature that such a bulge of the occupation-time profile is constructed in the tilted interlacements by mostly steering the tilted walk towards

K_N, and not by seriously tinkering the jump rates, see for instance (2.54), as well as Propositions 2.4.1 and 2.4.4 in the next section. □

2.4 Domination of equilibrium measures

In this section, our main goal is Proposition 2.4.4, where we prove that on a mesoscopic box inside K_N^δ, the tilted equilibrium measure dominates $(u_{**} + \epsilon/4)/u$ times the corresponding standard equilibrium measure. It is the key ingredient for constructing the coupling in Proposition 2.5.1 in the next section. A major step is achieved in Proposition 2.4.1, where we prove that the tilted capacity of a mesoscopic ball (larger than the above mentioned box) inside K_N^δ is at least $(u_{**} + \epsilon/2)/u$ times its corresponding standard capacity.

We start with the precise definition of the objects of interest in this and the next section. We denote by $\Gamma^N = \partial K_N^{\delta/2}$ the boundary in $\mathbb{Z}^d$ of the discrete blow-up of $K^{\frac{\delta}{2}}$ (we recall (2.8) and (2.9) for the definitions of the boundary and of the discrete blow-up). The above Γ^N will serve as a set "surrounding" K_N. We fix numbers r_i, $i = 1, \ldots, 4$ such that

$$0 < 2r_1 < r_2 < r_3 < r_4 < 1 \tag{2.101}$$

We define for x in Γ^N two boxes centered at x (when there is ambiguity we add a superscript for its center x, and B_2 will only be used in Section 4):

$$B_1 = B_\infty(x, N^{r_1}), \qquad B_2 = B_\infty(x, N^{r_2}); \tag{2.102}$$

and three balls also centered at x:

$$B_3 = B(x, N^{r_3}), \quad B_4 = B(x, N^{r_4}), \quad B_5 = B(x, 2N^{r_4}), \tag{2.103}$$

so that (in the notation of (2.8)) one has

$$B_1 \subset B_2 \subset B_3 \subset B_4 \subset B_5 \subset \overline{B_5} \subseteq K_N^\delta \subset\subset \mathbb{Z}^d. \tag{2.104}$$

(we now tacitly assume that N is sufficiently large so that for all

$x \in \Gamma^N$, $\overline{B_5^x} \subset K_N^\delta$, and the second equality of (2.74) holds).

We start with the domination of capacities. To prove the next Proposition 2.4.1, we calculate the time spent by the random walk in the mesoscopic body B_3 in two different ways (see Lemma 2.4.2), and relate these expressions to the capacity and to the tilted capacity.

Proposition 2.4.1. *When N is large, we have for all $x \in \Gamma^N$*

$$u\widetilde{\mathrm{cap}}(B_3) \geq \Big(u_{**} + \frac{\epsilon}{2}\Big)\mathrm{cap}(B_3). \tag{2.105}$$

The proof of this proposition relies on Lemmas 2.4.2 and 2.4.3.

Lemma 2.4.2.

$$\widetilde{E}_{\widetilde{e}_{B_3}}\Big[\int_0^\infty 1_{B_3}(X_s)ds\Big] = \frac{u_{**}+\epsilon}{u}\, E_{e_{B_3}}\Big[\int_0^\infty 1_{B_3}(X_s)ds\Big] \tag{2.106}$$

Proof. By the definition of the tilted Green function (see (2.55)) and by (2.58),

(2.107)

$$\begin{aligned}\widetilde{E}_{\widetilde{e}_{B_3}}\Big[\textstyle\int_0^\infty 1_{B_3}(X_s)ds\Big] &= \textstyle\sum_{v\in\partial_i B_3,\, y\in B_3} \widetilde{e}_{B_3}(v)\widetilde{g}(v,y)\widetilde{\lambda}(y)\\ &\overset{(2.58)}{=} \textstyle\sum_{y\in B_3} 1_{B_3}(y)\widetilde{\lambda}(y).\end{aligned}$$

Moreover, $\widetilde{\lambda}(y) = f^2(y) = \frac{u_{**}+\epsilon}{u}$ for $y \in B_3 \subset K_N^\delta$ (see (2.53), (2.74), (2.104)). Hence,

$$\widetilde{E}_{\widetilde{e}_{B_3}}\Big[\int_0^\infty 1_{B_3}(X_s)ds\Big] = \frac{u_{**}+\epsilon}{u}|B_3|. \tag{2.108}$$

By a similar calculation, we also find that

$$\mathbb{E}_{e_{B_3}}\Big[\int_0^\infty 1_{B_3}(X_s)ds\Big] = |B_3|. \tag{2.109}$$

Comparing (2.108) and (2.109) , we obtain (2.106) as desired. □

In the second lemma we prove that starting from the boundary of B_4, the tilted walk hits B_3 with a probability tending to 0 with N.

Lemma 2.4.3.

(2.110) $\beta(N) \stackrel{\text{def}}{=} \max_{x\in\Gamma^N, v\in\partial B_4} \widetilde{P}_v(H_{B_3} < \infty)$ *tends to 0 as* $N \to \infty$.

Proof. For v in ∂B_4, we have

$$\widetilde{P}_v(H_{B_3} < \infty) = \widetilde{P}_v(H_{B_3} < T_{B_5}) + \widetilde{P}_v(T_{B_5} < H_{B_3} < \infty), \tag{2.111}$$

By the second equality of (2.74), and in view of (2.54), (2.104), when starting in $v \in B_4$, under $\widetilde{P}_v$, $X_{\cdot \wedge T_{B_5}}$ behaves as stopped simple random walk. Thus, by classical simple random walk estimates, we have an upper bound for the probability that the tilted walk hits B_3 before exiting B_5:

$$\begin{aligned} &\max_{v\in\partial B_4} \widetilde{P}_v(H_{B_3} < T_{B_5}) \le \max_{v\in\partial B_4} P_v(H_{B_3} < \infty) \\ &\stackrel{\text{def}}{=} \beta_0(N) = O(N^{(r_3-r_4)(d-2)}), \end{aligned} \tag{2.112}$$

(note that $\beta_0(N)$ does not depend on $x \in \Gamma^N$).

By the strong Markov property successively applied at times T_{B_5} and $H_{\overline{B_4}}$, we have:

$$\widetilde{P}_v(T_{B_5} < H_{B_3} < \infty) \le \max_{y\in\partial B_5} \widetilde{P}_y(H_{\overline{B_4}} < \infty) \max_{v'\in\partial B_4} \widetilde{P}_{v'}(H_{B_3} < \infty). \tag{2.113}$$

Taking the maximum over v in ∂B_4 on the left-hand side of (2.113), and inserting this bound in (2.111), we find with the help of (2.112):

$$\max_{v\in\partial B_4} \widetilde{P}_v(H_{B_3} < \infty) \le \frac{\beta_0(N)}{1 - \max_{y\in\partial B_5} \widetilde{P}_y(H_{\overline{B_4}} < \infty)}. \tag{2.114}$$

To prove (2.110), it now suffices to show that

$$\liminf_N \min_{x\in\Gamma^N, y\in\partial B_5} \widetilde{P}_y(H_{\overline{B_4}} = \infty) > 0. \tag{2.115}$$

As a result of (2.14) and the stopping theorem, for large N, and any

$x \in \Gamma^N$,

(2.116) $$\min_{y \in \partial B_5} P_y(H_{\overline{B_4}} = \infty) > c.$$

By a similar argument as in Lemma 2.2.1,

(2.117) $$E_z\Big[\int_0^\infty 1_{\widetilde{U}_N}(X_s)ds\Big] \leq c(\widetilde{U})N^2, \text{ for } z \in \mathbb{Z}^d \text{ and } N \geq 1.$$

By the Chebyshev Inequality, writing $\widetilde{c}(\widetilde{U}) = 2c(\widetilde{U})/c$, with c as in (2.116), and $I_N = \{\int_0^\infty 1_{\widetilde{U}_N}(X_s)ds \leq \widetilde{c}(\widetilde{U})N^2\}$, we have

(2.118) $$P_z[I_N] \geq 1 - \frac{c}{2}, \text{ for all } z \in \mathbb{Z}^d.$$

With (2.116) and (2.118) put together, we obtain that for all z in ∂B_5,

(2.119) $$P_z(\{H_{\overline{B_4}} = \infty\} \cap I_N) \geq \frac{c}{2}.$$

By definition of f (see (2.73)) and since $h^\eta \in C_0^\infty$, we see that

(2.120) $$|V| = \Big|\frac{\Delta_{dis} f}{f}\Big| \leq c(u)\Big|\Delta_{dis} h_N\Big| \leq \frac{\overline{c}(h^\eta, u)}{N^2}.$$

By the first equality of (2.74), we have $\Delta_{dis} f = 0$ outside $\widetilde{U}_N$. Hence, we find that for large N, for all $x \in \Gamma^N$ and $y \in \partial B_5$, on the event I_N,

(2.121) $$\begin{aligned} \frac{d\widetilde{P}_y}{dP_y} &\geq c(u)\exp\Big\{\int_0^\infty V(X_s)ds\Big\} \\ &\underset{(2.120)}{\overset{(2.117)}{\geq}} c(u)\exp\Big\{-\widetilde{c}N^2 \cdot \frac{\overline{c}}{N^2}\Big\} = c(u)e^{-\widetilde{c}\overline{c}}. \end{aligned}$$

Therefore, by (2.119, (2.121) we find that

$$\begin{aligned}&\liminf_{N\to\infty} \min_{x\in\Gamma^N, y\in\partial B_5} \widetilde{P}_y[\{H_{\overline{B_4}} = \infty\}] \geq \\ &\liminf_{N\to\infty} \min_{x\in\Gamma^N, y\in\partial B_5} E_y\Big[\frac{d\widetilde{P}_y}{dP_y} 1_{\{H_{\overline{B_4}}=\infty\}}, I_N\Big] > 0.\end{aligned} \tag{2.122}$$

This proves (2.115) and concludes the proof of Lemma 2.4.3. □

With all ingredients prepared, we are ready to prove the domination of capacities stated in Proposition 2.4.1. In the proof we combine the estimates obtained in Lemmas 2.2.1 and 2.4.2, perform an argument similar to (2.111), (2.112) and (2.113), and employ Lemma 2.4.3 to control the tilted return probability.

Proof of Proposition 2.4.1. We will bound the left term of (2.106) from above and the right term from below. We start with the upper bound on the left-hand side of (2.106).

For all y in $\partial_i B_3$, by strong Markov property at time T_{B_4} (and then at time H_{B_3}) we have

$$\begin{aligned}&\widetilde{E}_y\Big[\int_0^\infty 1_{B_3}(X_s)ds\Big] \\ &= \widetilde{E}_y\Big[\int_0^{T_{B_4}} 1_{B_3}(X_s)ds\Big] + \widetilde{E}_y\Big[\widetilde{E}_{X_{T_{B_4}}}\Big[\int_0^\infty 1_{B_3}(X_s)ds\Big]\Big] \\ &\leq \max_{y\in\partial_i B_3}\Big\{\widetilde{E}_y\Big[\int_0^{T_{B_4}} 1_{B_3}(X_s)ds\Big]\Big\} \\ &\quad + \max_{v\in\partial B_4}\Big\{\widetilde{P}_v[H_{B_3}<\infty]\Big\}\max_{y\in\partial_i B_3}\Big\{\widetilde{E}_y\Big[\int_0^\infty 1_{B_3}(X_s)ds\Big]\Big\}.\end{aligned} \tag{2.123}$$

Taking the maximum over $y \in \partial_i B_3$ on the left-hand side of (2.123) and rearranging, we find in view of (2.110):

$$\max_{y\in\partial_i B_3} \widetilde{E}_y\Big[\int_0^\infty 1_{B_3}(X_s)ds\Big] \leq \frac{\max_{y\in\partial_i B_3}\widetilde{E}_y\big[\int_0^{T_{B_4}} 1_{B_3}(X_s)ds\big]}{1-\beta(N)}. \tag{2.124}$$

Then we notice that, since f is constant on $K_N^\delta \supseteq \overline{B_4}$, see (2.74) and (2.104),

$$\widetilde{E}_y\Big[\int_0^{T_{B_4}} 1_{B_3}(X_s)ds\Big] = E_y\Big[\int_0^{T_{B_4}} 1_{B_3}(X_s)ds\Big]. \tag{2.125}$$

We now have the following upper bound on the left-hand side of (2.106) under $\widetilde{P}_{\widetilde{e}_{B_3}}$:

$$\begin{aligned}
&\widetilde{E}_{\widetilde{e}_{B_3}}\Big[\int_0^\infty 1_{B_3}(X_s)ds\Big] \\
\leq\ & \widetilde{\mathrm{cap}}(B_3)\ \max\nolimits_{y\in\partial_i B_3} \widetilde{E}_y\Big[\int_0^\infty 1_{B_3}(X_s)ds\Big] \\
\overset{(2.124)}{\leq}\ & \widetilde{\mathrm{cap}}(B_3)\ \frac{\max_{y\in\partial_i B_3}\big\{\widetilde{E}_y\big[\int_0^{T_{B_4}} 1_{B_3}(X_s)ds\big]\big\}}{1-\beta(N)} \\
\overset{(2.125)}{=}\ & \widetilde{\mathrm{cap}}(B_3)\ \frac{\max_{y\in\partial_i B_3}\big\{E_y\big[\int_0^{T_{B_4}} 1_{B_3}(X_s)ds\big]\big\}}{1-\beta(N)} \\
\leq\ & \widetilde{\mathrm{cap}}(B_3)\ \frac{\max_{y\in\partial_i B_3}\big\{E_y\big[\int_0^{\infty} 1_{B_3}(X_s)ds\big]\big\}}{1-\beta(N)} \\
\overset{(2.26)}{\leq}\ & \widetilde{\mathrm{cap}}(B_3)c_1N^{2r_3}\ \frac{1+\alpha(N)}{1-\beta(N)}.
\end{aligned} \tag{2.126}$$

On the other hand, by (2.26) of Lemma 2.2.1, we have a lower bound on the right-hand side of (2.106):

$$\frac{u_{**}+\epsilon}{u}E_{e_{B_3}}\Big[\int_0^\infty 1_{B_3}(X_s)ds\Big] \geq \frac{u_{**}+\epsilon}{u}\mathrm{cap}(B_3)c_1N^{2r_3}(1-\alpha(N)). \tag{2.127}$$

Combining (2.126), (2.127) and Lemma 2.4.2, we find

$$\widetilde{\mathrm{cap}}(B_3)\frac{1+\alpha(N)}{1-\beta(N)} \geq \frac{u_{**}+\epsilon}{u}\mathrm{cap}(B_3)(1-\alpha(N)). \tag{2.128}$$

With the help of (2.26) and (2.110) we see that Proposition 2.4.1 readily follows. □

We now turn to the domination of the equilibrium measures at a smaller scale on B_1. In the proof of Proposition 2.4.4, thanks to the domination of capacities proved in Proposition 2.4.1, we are able to reduce the domination of equilibrium measures to the domination of (relative) entrance measures. This is performed in Lemma 2.4.5.

Proposition 2.4.4. *When N is large, for all $x \in \Gamma^N$ and $z \in \partial_i B_1$,*

$$u\widetilde{e}_{B_1}(z) \geq \Big(u_{**} + \frac{\epsilon}{4}\Big) e_{B_1}(z). \tag{2.129}$$

The proof of Proposition 2.4.4 relies on the following lemma, where we prove that the killed entrance measure of B_1 almost dominates the corresponding standard entrance measure. From now on, we fix $\epsilon' = \epsilon/(4u_{**} + 2\epsilon)$. We recall (2.24) for notation.

Lemma 2.4.5. *For sufficiently large N, for all $x \in \Gamma^N$ and $z \in \partial_i B_1$,*

$$\min_{y \in \partial_i B_3} h_{B_1,B_4}(y,z) \geq (1-\epsilon') \max_{\widetilde{y} \in \partial_i B_3} h_{B_1}(\widetilde{y}, z). \tag{2.130}$$

The proof of Lemma 2.4.5 has the same flavour as Section 3 of [8] and indeed relies on Lemma 3.3 of the same reference.

Proof. We decompose $h_{B_1, B_4}(y,z)$ according to the time and place of the last step before entering B_1 at z, and obtain for y outside B_1 and z in B_1

$$h_{B_1,B_4}(y,z) = \frac{1}{2d} \sum_{z' \sim z, z' \in \partial B_1} g_{B_4 \setminus B_1}(y, z'). \tag{2.131}$$

Similarly, we have for $\widetilde{y}$ outside B_1 and z in B_1,

$$h_{B_1}(\widetilde{y}, z) = \frac{1}{2d} \sum_{z' \sim z, z' \in \partial B_1} g_{B_1^c}(\widetilde{y}, z'). \tag{2.132}$$

Therefore, to prove (2.130), it suffices to show that for large N and for all $y, \widetilde{y} \in \partial_i B_3$ and $z' \in \partial B_1$

$$g_{B_4 \backslash B_1}(y, z') \geq (1 - \epsilon') g_{B_1^c}(\widetilde{y}, z'). \tag{2.133}$$

By an argument similar to Lemma 3.3 of [8] to B_4 and B_1, we have that

$$\begin{aligned} & g_{B_4 \backslash B_1}(y, z') \\ \overset{\text{symmetry}}{=} \; & g_{B_4 \backslash B_1}(z', y) \\ \overset{\text{Markov}}{=} \; & g_{B_4}(z', y) - E_{z'}[g_{B_4}(X_{H_{B_1}}, y),\ H_{B_1} < T_{B_4}] \\ \overset{\text{symmetry}}{=} \; & E_{z'}[g_{B_4}(y, z') - g_{B_4}(y, X_{H_{B_1}}),\ H_{B_1} < T_{B_4}] \\ & + g_{B_4}(y, z') P_{z'}[H_{B_1} > T_{B_4}] \overset{\text{def}}{=} A + B. \end{aligned} \tag{2.134}$$

Then, by the gradient estimate and the Harnack inequality in Theorems 1.7.1, and 1.7.2, p. 42 of [38],

$$|A| \leq \frac{c}{N^{r_3}} N^{r_1} g_{B_4}(y, z'), \tag{2.135}$$

and by a similar argument as below (3.30) of [8],

$$B \geq \frac{c}{N^{r_1}} g_{B_4}(y, z'). \tag{2.136}$$

Hence, collecting (2.134), (2.135), (2.136), we find that

$$g_{B_4 \backslash B_1}(y, z') \geq g_{B_4}(y, z') P_{z'}[H_{B_1} > T_{B_4}](1 - cN^{2r_1 - r_3}). \tag{2.137}$$

By analogous arguments we also obtain

$$g_{B_1^c}(\widetilde{y}, z') \leq g(\widetilde{y}, z') P_{z'}[H_{B_1} = \infty](1 + cN^{2r_1 - r_3}). \tag{2.138}$$

By the definition of r_1 and r_3 (see (2.101)), $N^{2r_1 - r_3} \ll 1$. Therefore, combining (2.137), (2.138) together with the fact that

$$P_{z'}[H_{B_1} > T_{B_4}] \geq P_{z'}[H_{B_1} = \infty], \tag{2.139}$$

the claim (2.133) will follow once we show (see above Lemma 2.4.5 for our choice of ϵ') that when N is sufficiently large, for all $x \in \Gamma^N$, all $y, \widetilde{y} \in \partial_i B_3$ and all $z' \in \partial B_1$,

$$g_{B_4}(y, z') \geq \left(1 - \frac{\epsilon'}{2}\right) g(\widetilde{y}, z'). \tag{2.140}$$

By (2.14) and (2.17), for large N, setting $\widetilde{B} = B(y, \frac{N^{r_4}}{2})$ we have the following bounds:

$$g_{B_4}(y, z') \geq g_{\widetilde{B}}(y, z') \geq \overline{c}_0 |y - z'|^{(2-d)} - cN^{r_4(2-d)} - c' N^{r_3(1-d)} \tag{2.141}$$

and

$$g(\widetilde{y}, z') \leq \overline{c}_0 |y - z'|^{(2-d)} + cN^{r_3(1-d)}. \tag{2.142}$$

Hence, we obtain (2.140) and (2.133) follows. This proves Lemma 2.4.5. □

We are now ready to prove Proposition 2.4.4. In the proof, we make use of the sweeping identity, and, in effect, reduce the comparison of the standard and tilted equilibrium measures of B_1 to the comparison on the standard and tilted capacities of B_3, and to the comparison of the (killed) entrance measures.

Proof of Proposition 2.4.4. For large N and for all $x \in \Gamma^N$ and $z \in \partial_i B_1$, we find that

$$\begin{aligned} u\widetilde{e}_{B_1}(z) &\overset{(2.59)}{=} u\,\widetilde{P}_{\widetilde{e}_{B_3}}(X_{H_{B_1}} = z,\, H_{B_1} < \infty) \\ &\geq u\,\widetilde{\mathrm{cap}}(B_3)\,\min_{y \in \partial_i B_3} \widetilde{h}_{B_1}(y, z) \\ &\overset{(2.105)}{\geq} \left(u_{**} + \frac{\epsilon}{2}\right) \mathrm{cap}(B_3) \min_{y \in \partial_i B_3} \widetilde{h}_{B_1}(y, z) \\ &\geq \left(u_{**} + \frac{\epsilon}{2}\right) \mathrm{cap}(B_3) \min_{y \in \partial_i B_3} \widetilde{h}_{B_1, B_4}(y, z). \end{aligned} \tag{2.143}$$

Since up to the exit time from B_4 the tilted and standard walk have the same law (see (2.74)), we see that for $y \in \partial_i B_3$ and $z \in \partial B_1$, we have

$$\widetilde{h}_{B_1,B_4}(y,z) = h_{B_1,B_4}(y,z). \tag{2.144}$$

Taking Lemma 2.4.5 into account, we find that for large N and for all $x \in \Gamma^N$ and $z \in \partial B_1$,

$$\min_{y \in \partial_i B_3} h_{B_1,B_4}(y,z) \overset{(2.130)}{\geq} (1-\epsilon') \max_{\widetilde{y} \in \partial_i B_3} h_{B_1}(\widetilde{y},z). \tag{2.145}$$

Thus, coming back to (2.143), we find that with our choice of ϵ' (above Lemma 2.4.5),

$$\begin{aligned} u\widetilde{e}_{B_1}(z) &\geq \Big(u_{**} + \frac{\epsilon}{4}\Big)\mathrm{cap}(B_3) \max_{\widetilde{y} \in \partial_i B_3} h_{B_1}(\widetilde{y},z) \\ &\overset{(2.24)}{\geq} \Big(u_{**} + \frac{\epsilon}{4}\Big) P_{e_{B_3}}(X_{H_{B_1}} = z,\, H_{B_1} < \infty) \\ &\overset{(2.25)}{=} \Big(u_{**} + \frac{\epsilon}{4}\Big) e_{B_1}(z). \end{aligned} \tag{2.146}$$

This completes the proof of Proposition 2.4.4. □

2.5 Coupling and Disconnection

In this section, we prove in Theorem 2.5.3 that the tilted interlacements disconnect K_N from infinity with a probability, which tends to 1 as N goes to infinity. To this end, we show that in mesoscopic boxes with centers in Γ^N (introduced above (2.101)), the tilted random interlacements locally "dominate" random interlacements with level higher than u_{**}, and thus typically disconnect in each such box the center from its boundary with very high probability. Therefore, there is a high probability as well for the tilted interlacement to disconnect the macroscopic body from infinity. The main step is Proposition 2.5.1 where we construct at each point of Γ^N a coupling so that the tilted random interlacements with high probability locally dominate some standard random interlacements with level higher than u_{**}.

We recall the definitions of B_1 and B_2 from (2.102).

Proposition 2.5.1. *When N is large, for all $x \in \Gamma^N$, there exists a probability space $(\overline{\Omega}, \overline{\mathcal{A}}, \overline{Q})$ and random sets $\widetilde{\mathcal{I}}$ and $\mathcal{I}_1$ defined on $\overline{\Omega}$, with same respective laws as $\mathcal{I}^u \cap B_1$ under $\widetilde{\mathbb{P}}_N$ and $\mathcal{I}^{u_{**}+\frac{\epsilon}{8}}$ under $\mathbb{P}_{u_{**}+\frac{\epsilon}{8}}$, so that*

$$\overline{Q}[\widetilde{\mathcal{I}} \supset \mathcal{I}_1] \geq 1 - c_5 e^{-c_6 N^{c_7}} \tag{2.147}$$

(the constants depend on r_1, r_2, ϵ).

The idea of the proof is to stochastically dominate the trace in B_1 of random interlacements with level higher than u_{**} by the "first excursions" (from some inner boundary of B_1 to ∂B_2) of the trajectories from some random interlacements with slightly higher intensity, and then, further dominate these excursions by the same kind of "first excursions" of trajectories of the tilted interlacement. The following proposition for the above mentioned first stochastic domination in essence rephrases Proposition 4.4 of [7]. We begin with some notation.

For $A \subset B \subset\subset \mathbb{Z}^d$, we write $k_{A,B}$ for the law on $\Gamma(\mathbb{Z}^d)$ (see below (2.11)) of the stopped process $X_{\cdot \wedge T_B}$ under P_{e_A}. We also denote the trace of a point process $\eta = \sum_i \delta_{w_i}$ on the space $\Gamma(Z^d)$ by

$$\mathcal{I}(\eta) = \cup_i Range(w_i). \tag{2.148}$$

Proposition 2.5.2. *When N is large, for all $x \in \Gamma^N$, there exists a probability space $(\Sigma, \mathcal{B}, Q)$ endowed with a Poisson point process η, with intensity measure $(u_{**} + \epsilon/4)k_{B_1,B_2}$, and a random set $\mathcal{I}_1 \subset \mathbb{Z}^d$ with the law of $\mathcal{I}^{u_{**}+\frac{\epsilon}{8}} \cap B_1$* under $\mathbb{P}_{u_{**}+\frac{\epsilon}{8}}$, and

$$Q[\mathcal{I}_1 \subset \mathcal{I}(\eta) \cap B_1] \geq 1 - c_5 e^{-c_6 N^{c_7}}. \tag{2.149}$$

We refer the readers to Proposition 5.4 of [7] and to Section 8 of [7] for the proof of Proposition 2.5.2.

We now construct another coupling such that the trace on B_1 of the first excursions of the tilted random interlacements dominate the trace of the corresponding excursions for random interlacements at level $u_{**} + \frac{\epsilon}{4}$. Combined with Proposition 2.5.2, this will complete the proof of Proposition 2.5.1.

Proof of Proposition 2.5.1. We keep the notation of Proposition 2.5.2. Let α be the measure on $\partial_i B_1$ such that for all $z \in \partial_i B_1$,

$$\alpha(z) = u\widetilde{e}_{B_1}(z) - \left(u_{**} + \frac{\epsilon}{4}\right) e_{B_1}(z). \tag{2.150}$$

By Proposition 2.4.4 α is a positive measure. Hence, we can construct an auxiliary probability space $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{Q})$, endowed with a Poisson point process $\widetilde{\eta}$ on $\Gamma(\mathbb{Z}^d)$ with intensity measure $k_\alpha(\cdot) = P_\alpha(X_{\cdot \wedge T_{B_2}})$. Since for all z in $\partial_i B_1$, the tilted walk coincides with the simple random walk up to the exit from B_2, we obtain that under $\widetilde{\mathbb{P}}_N$,

(2.151)
$$\widetilde{\mathcal{I}} = (\mathcal{I}(\widetilde{\eta}) \cup \mathcal{I}(\eta)) \cap B_1 \text{ is stochastically dominated by } \mathcal{I}^u \cap B_1.$$

We can thus construct on some extension $(\overline{\Omega}, \overline{\mathcal{A}}, \overline{Q})$ an $\widetilde{\mathcal{I}}$ distributed as $\mathcal{I}^u \cap B_1$ under $\widetilde{\mathbb{P}}_N$, so that $\widetilde{\mathcal{I}} \supseteq \mathcal{I}(\eta)$, $\overline{Q}$-a.s.. We then have

$$\begin{aligned} \overline{Q}[\widetilde{\mathcal{I}} \supset \mathcal{I}_1] &\geq \overline{Q}[\mathcal{I}(\eta) \cap B_1 \supset \mathcal{I}_1] \\ &= Q[\mathcal{I}(\eta) \cap B_1 \supset \mathcal{I}_1] \overset{(2.149)}{\geq} 1 - c_5 e^{-c_6 N^{c_7}}. \end{aligned} \tag{2.152}$$

□

We are now ready to derive a key step for the proof of Theorem 2.1.1. Namely, we will now show that with $\widetilde{\mathbb{P}}_N$-probability tending to 1, the event $A_N (= \{K_N \overset{\mathcal{V}^u}{\not\leftrightarrow} \infty\}$, see (2.2)) does occur.

Theorem 2.5.3.

$$\lim_{N \to \infty} \widetilde{\mathbb{P}}_N[A_N] = 1. \tag{2.153}$$

Proof. Note that for large N, when K_N is connected to infinity by a nearest-neighbor path, this path must go through the set Γ^N at some point x (see above (2.101)). Hence, this path connects x to the inner boundary of B_1^x, so that

$$A_N^c \subset \cup_{x \in \Gamma^N} \{x \stackrel{\mathcal{V}^u}{\longleftrightarrow} \partial_i B_1^x\}. \tag{2.154}$$

Thus, we find that for large N

$$\widetilde{\mathbb{P}}_N[A_N^c] \leq \sum_{x \in \Gamma^N} \widetilde{\mathbb{P}}_N[x \stackrel{\mathcal{V}^u}{\longleftrightarrow} \partial_i B_1^x]. \tag{2.155}$$

By Proposition 2.5.1, for large N, uniformly in $x \in \Gamma^N$, we can bound the probability in the right-hand side of (2.155) as follows,
(2.156)

$$\begin{aligned} \widetilde{\mathbb{P}}_N[x \stackrel{\mathcal{V}^u}{\leftrightarrow} \partial_i B_1^x] &\stackrel{(2.147)}{\leq} \mathbb{P}_{u_{**}+\frac{\epsilon}{8}}[x \stackrel{\mathcal{V}^{u_{**}+\frac{\epsilon}{8}}}{\longleftrightarrow} \partial_i B_1^x] + c_5 e^{-c_6 N^{c_7}}. \\ &\stackrel{(2.66)}{\leq} c e^{-c' N^{\widetilde{c}}}, \end{aligned}$$

where the constants depend on r_1, r_2, ϵ. Hence, we see that for large N,

$$\widetilde{\mathbb{P}}_N[A_N^c] \leq |\Gamma^N| c e^{-c' N^{\widetilde{c}}} \underset{N}{\longrightarrow} 0. \tag{2.157}$$

This concludes the proof of Theorem 2.5.3. □

2.6 Denouement

In this section we combine the various ingredients, namely Theorem 2.5.3, Propositions 2.3.3 and 2.3.4, and prove Theorem 2.1.1.

Proof of Theorem 2.1.1. We recall the entropy inequality (see (2.68)), and apply it to $\mathbb{P}_u$ and $\widetilde{\mathbb{P}}_N$ defined in Sections 1 and 2. By Theorem 2.5.3, we know that
$\lim_{N\to\infty} \widetilde{\mathbb{P}}_N[A_N] = 1$, and (2.68) yields that

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}} \log(\mathbb{P}_u[A_N]) \geq -\limsup_{N\to\infty} \frac{1}{N^{d-2}} H(\widetilde{\mathbb{P}}_N|\mathbb{P}_u). \tag{2.158}$$

By Proposition 2.3.3, we represent the right-hand side of (2.158) as

$$(2.159)\quad \begin{aligned} & - \limsup_{N\to\infty} \frac{1}{N^{d-2}} H(\widetilde{\mathbb{P}}_N|\mathbb{P}_u) \\ & = - (\sqrt{u_{**}+\epsilon} - \sqrt{u})^2 \limsup_{N\to\infty} \frac{1}{N^{d-2}} \mathcal{E}_{\mathbb{Z}^d}(h_N, h_N). \end{aligned}$$

Then, by Proposition 2.3.4, taking consecutively the limits $\eta \to 0$, $r_U \to \infty$, and $\delta \to 0$, and we obtain

(2.160)

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}} \log(\mathbb{P}_u[A_N]) \geq -\frac{1}{d} (\sqrt{u_{**}+\epsilon} - \sqrt{u})^2 \mathrm{cap}_{\mathbb{R}^d}(K).$$

Finally, by taking $\epsilon \to 0$ we obtain (2.3) as desired. □

Remark 2.6.1.

1) It is an important question whether Theorem 2.1.1 can be complemented by a matching asymptotic upper bound, say when K is a smooth compact set. In view of Theorems 6.2 and 6.4 of [42] (see also Remark 6.5 2) of [42]), this would indicate that the large deviations of the occupation-time profile of random interlacements, insulating K by values u' of the local field (with u' corresponding to a non-percolative behaviour of $\mathcal{V}^{u'}$) capture the main mechanism underlying the disconnection of a macroscopic body, in the percolative regime of the vacant set.

2) As $u \to 0$, the right-hand side of (2.3) tends to the finite limit $-\frac{u_{**}}{d}\mathrm{cap}(K)$. One may wonder whether this limiting procedure retains any pertinence for the study of the disconnection of the macroscopic body K_N by a simple random walk trajectory? For instance, does one have

(2.161)

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}} \log P_0\Big[\Big\{K_N \overset{Range\{(X_t)_{t\geq 0}\}^c}{\longleftrightarrow\!\!\!\!\!\!/\;\;} \infty\Big\}\Big] \geq -\frac{u_{**}}{d}\mathrm{cap}_{\mathbb{R}^d}(K)\ ?$$

□

Chapter 3

A lower bound for disconnection by Simple Random Walk

We consider simple random walk on $\mathbb{Z}^d$, $d \geq 3$. Motivated by Chapters 1 and 2, we investigate the asymptotic behaviour of the probability that a large body gets disconnected from infinity by the set of points visited by a simple random walk. We derive asymptotic lower bounds that bring into play random interlacements. Although open at the moment, some of the lower bounds we obtain possibly match the asymptotic upper bounds recently obtained in [62]. This potentially yields special significance to the tilted walks that we use in this work, and to the strategy that we employ to implement disconnection.

3.1 Introduction

How hard is it to disconnect a macroscopic body from infinity by the trace of a simple random walk in $\mathbb{Z}^d$, when $d \geq 3$? In this work, we partially answer this question, motivated by Chapters 1 and 2, by deriving an asymptotic lower bound on the probability of such a disconnection. Remarkably, our bounds bring into play random interlacements as well as a suitable strategy to implement disconnection. Although open at the moment, some of the lower bounds we obtain in this work may be sharp, and match the recent upper bounds from [62].

We now describe the model and our results in a more precise fashion. We refer to Section 3.2 for precise definitions. We consider the continuous-time simple random walk on $\mathbb{Z}^d$, $d \geq 3$, and we denote by P_0 the (canonical) law of the walk starting from the origin. We denote by $\mathcal{V} = \mathbb{Z}^d \setminus X_{[0,\infty)}$ the complement of the set of points visited by the walk.

We consider K, a non-empty compact subset of $\mathbb{R}^d$ and for $N \geq 1$ its discrete blow-up:

$$K_N = \big\{x \in \mathbb{Z}^d; d_\infty(x, NK) \leq 1\big\}, \tag{3.1}$$

where NK, a non-empty compact subset of $\mathbb{R}^d$, stands for the set homothetic to K with ratio N, and

$$d_\infty(z, NK) = \inf_{y \in NK} |z - y|_\infty \tag{3.2}$$

stands for the sup-norm distance of z to NK. Of central interest for us is the event specifying that K_N is not connected to infinity in $\mathcal{V}$, which we denote by

$$\{K_N \overset{\mathcal{V}}{\nleftrightarrow} \infty\}. \tag{3.3}$$

Our main result brings into play the model of random interlacements. Informally, random interlacements in $\mathbb{Z}^d$ are a Poissonian cloud of doubly-infinite nearest-neighbour paths, with a positive parameter u, which is a multiplicative factor of the intensity of the cloud (we refer to [13] and [23] for further details and references). We denote by $\mathcal{I}^u$ the trace of random interlacements of level u on $\mathbb{Z}^d$, and by $\mathcal{V}^u = \mathbb{Z}^d \setminus \mathcal{I}^u$ the corresponding vacant set. It is known that there is a critical value $u_{**} \in (0, \infty)$, which can be characterized as the infimum of the levels $u > 0$ for which the probability that the vacant cluster at the origin reaches distance N from the origin has a stretched exponential decay in N; see [53] or [23].

The main result of this article is the following asymptotic lower bound, which confirms the conjecture proposed in Remark 5.1(2) of [43].

Theorem 3.1.1.

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}} \log\big(P_0[K_N \overset{\mathcal{V}}{\nleftrightarrow} \infty]\big) \geq -\frac{u_{**}}{d}\mathrm{cap}_{\mathbb{R}^d}(K), \tag{3.4}$$

where $\mathrm{cap}_{\mathbb{R}^d}(K)$ *stands for the Brownian capacity of* K*.*

Actually, the proof of Theorem 3.1.1 (after minor changes) also shows that for any $M > 1$,

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}} \log\big(P_0[B_N \overset{\mathcal{V}}{\nleftrightarrow} S_N]\big) \geq -\frac{u_{**}}{d}\mathrm{cap}_{\mathbb{R}^d}\big([-1,1]^d\big), \tag{3.5}$$

where $B_N = \{x \in \mathbb{Z}^d; |x|_\infty \leq N\}$ and $S_N = \{x \in \mathbb{Z}^d; |x|_\infty = [MN]\}$ with $[MN]$ the integer part of MN; see Remark 3.7.1.

On the other hand, the recent article [62] improves on [61], and shows that for any $M > 1$, the following asymptotic upper bound holds:

$$\limsup_{N\to\infty} \frac{1}{N^{d-2}} \log\big(P_0[B_N \overset{\mathcal{V}}{\nleftrightarrow} S_N]\big) \leq -\frac{\overline{u}}{d}\mathrm{cap}_{\mathbb{R}^d}\big([-1,1]^d\big), \tag{3.6}$$

where $\overline{u}$ is a certain critical level introduced in [62], such that $0 < u < \overline{u}$ corresponds to the *strongly percolative* regime of $\mathcal{V}^u$. Precisely, one knows that $0 < \overline{u} \leq u_* \leq u_{**} < \infty$, where u_* stands for the critical level for the percolation of $\mathcal{V}^u$ (the positivity of $\overline{u}$, for all $d \geq 3$, actually stems from [24] as explained in Section 2 of [62]). It is plausible, but unproven at the moment, that actually $\overline{u} = u_* = u_{**}$. If this is the case, the asymptotic lower bound (3.5) from the present article matches the asymptotic upper bound (3.6) from [62].

In the case of (3.4), one can also wonder whether one actually has the following asymptotics (possibly with some regularity assumption on K)

$$\lim_{N\to\infty} \frac{1}{N^{d-2}} \log\big(P_0[K_N \overset{\mathcal{V}}{\nleftrightarrow} \infty]\big) = -\frac{u_*}{d} \mathrm{cap}_{\mathbb{R}^d}(K). \tag{3.7}$$

Our proof of Theorem 3.1.1 [and of (3.5)] relies on the change of probability method. The feature that the asymptotic lower bounds, which we derive in this article, are potentially sharp, yields special significance to the strategy that we employ to implement disconnection.

Let us give some comments about the strategy and the proof. We construct through fine-tuned Radon–Nikodym derivatives new measures $\widetilde{P}_N$, corresponding to the "tilted walks". In essence, these walks evolve as recurrent walks with generator

$$\widetilde{L}g(x) = \frac{1}{2d} \sum_{|x'-x|=1} \frac{h_N(x')}{h_N(x)} (g(x') - g(x)),$$

up to a deterministic time T_N, and then as the simple random walk afterward, with $h_N(x) = h(\frac{x}{N})$, where h is the solution of (assuming that K is regular)

$$\begin{cases} \Delta h = 0, & \text{on } \mathbb{R}^d \setminus K, \\ h = 1, & \text{on } K, \text{ and } h \text{ tends to } 0 \text{ at } \infty, \end{cases} \tag{3.8}$$

and T_N is chosen so that the expected time spent by the tilted walk up to T_N at any x in K_N is $u_{**} h_N^2(x) = u_{**}$ (by the choice of h). Informally, $\widetilde{P}_N$ achieves this at a "low entropic cost". Quite remarkably, this constraint on the *time* spent at points and low entropic cost induces a local behaviour of the trace of the tilted walk which *geometrically* behaves as random interlacements with a slowly space-modulated parameter $u_{**} h_N^2(x)$, at least close to K_N. This creates a "fence" around K_N, where the vacant set left by the tilted walk is locally in a strongly non-percolative mode, so that

$$\lim_{N\to\infty} \widetilde{P}_N[K_N \overset{\mathcal{V}}{\nleftrightarrow} \infty] = 1. \tag{3.9}$$

On the other hand, we show that

$$\widetilde{\lim} \frac{1}{N^{d-2}} H(\widetilde{P}_N | P_0) \le \frac{u_{**}}{d} \mathrm{cap}_{\mathbb{R}^d}(K), \tag{3.10}$$

where $\widetilde{\lim}$ refers to a certain limiting procedure, in which N goes first to infinity, and $H(\widetilde{P}_N|P_0)$ stands for the relative entropy of $\widetilde{P}_N$ with respect to P_0 [see (2.67)]. The main claim (3.4), or (3.5) then quickly follow by a classical inequality; see (2.68).

The above lines are of course mainly heuristic, and the actual proof involves several mollifications of the above strategy: K is slightly enlarged, h is replaced by a compactly supported function smoothed near K, we work with $u_{**}(1+\varepsilon)$ in place of u_{**}, and the tilted walk lives in a ball of radius RN up to time T_N, ... These various mollifications naturally enter the limiting procedure alluded to above in (3.10).

Clearly, a substantial part of this work is to make sense of the above heuristics. Observe that unlike what happened in [43], where an asymptotic lower bound was derived for the disconnection of a macroscopic body by random interlacements, in the present set-up, we only have one single trajectory at our disposal. So the tilted walk behaves as a recurrent walk up to time T_N in order to implement disconnection. This makes the extraction of the necessary independence implicit to comparison with random interlacements more delicate. This is achieved by several sorts of analysis on the mesoscopic level. More precisely, on all mesoscopic boxes A_1^x with the center x varying in a "fence" around K_N, we bound from above the tilted probability that there is a path in $\mathcal{V}$ that connects x to the (inner) boundary of A_1^x by the probability that there is such a path in the vacant set of random interlacements with level slightly higher than u_{**} (which is itself small due to the strong non-percolative character of $\mathcal{V}^u$ when $u > u_{**}$) and a correction term:

$$\widetilde{P}_N\big[x \overset{\mathcal{V}}{\longleftrightarrow} \partial_i A_1^x\big] \le \mathbb{P}\big[x \overset{\mathcal{V}^{u_{**}(1+\varepsilon/8)}}{\longleftrightarrow} \partial_i A_1^x\big] + e^{-c\log^2 N} \le e^{-c'\log^2 N}, \tag{3.11}$$

where $\mathbb{P}$ stands for the law of random interlacements, and $\partial_i A_1^x$ for the inner boundary of the box A_1^x. To prove the above claim, we conduct a local comparison at mesoscopic scale between the trace of the tilted walk, and the occupied set of random interlacements, with a level slightly exceeding u_{**}, via a chain of couplings.

There are two crucial steps in this "chain of couplings", namely Propositions 3.6.2 and 3.6.7. In Proposition 3.6.2, we call upon the estimates on hitting times proved in Section 3.4 and on the results concerning the quasi-stationary measure from Section 3.5. We construct a coupling between the trace in A_1 of excursions of the confined walk up to time T_N, and the trace in A_1 of the excursions of many independent confined walks from A_1 to the boundary of a larger mesoscopic box. This proposition enables us to cut the confined walk into "almost" independent sections, and compare it to the trace of a suitable Poisson point process of excursions. On the other hand, Proposition 3.6.7 uses a result proved in [7], coupling the above mentioned Poisson point process of excursions and the trace of random interlacements. Some of the arguments used in this work are similar to those in [64]. However, in our set-up, special care is needed due to the fact that the stationary measure of the tilted walk is massively non-uniform.

We will now explain how this article is organized. In Section 3.2, we introduce notation and make a brief review of results concerning continuous-time random walk, continuous-time random interlacements, Markov chains, as well as other useful facts and tools. Section 3.3 is devoted to the construction of the tilted random walk and the confined walk, as well as the proof of various properties concerning them. Most important are a lower bound of the spectral gap of the confined walk in Proposition 3.3.12, and an asymptotic upper bound on the relative entropy between the tilted walk and the simple random walk, in Proposition 3.3.14. In Section 3.4, we prove some estimates on the hitting times of some mesoscopic objects, namely Propositions 3.4.5 and 3.4.7 that will be useful in Section 3.6. In Section 3.5, we prove some controls (namely Proposition 3.5.7) on the quasi-stationary measure that will be crucial for the construction of couplings in Section 3.6. In Section 3.6, we develop the chain of couplings and prove that the tilted disconnection probability $\widetilde{P}_N[A_N]$ tends to 1, as N tends to infinity. In the short Section 3.7, we assemble the various pieces and prove the main Theorem 3.1.1.

Finally, we explain the convention we use concerning constants. We denote by c, c', c'', $\overline{c}, \dots$ positive constants with values changing from place to place. Throughout the article, the constants depend on the dimension d. Dependence on additional constants is stated at the beginning of each section.

3.2 Some useful facts

Throughout the article, we assume $d \geq 3$ unless otherwise stated. In this section, we will introduce further notation and recall useful facts concerning continuous-time random walk on $\mathbb{Z}^d$ and its potential theory. We also recall the definition of and some results about continuous-time random interlacements. At the end of the section, we state an inequality on relative entropy and review various results about Markov chains.

We start with some notation. We let $\mathbb{Z}^+ = \{0, 1, \dots\}$ stand for the set of positive integers. We write $|\cdot|$ and $|\cdot|_\infty$ for the Euclidean and l^∞-norms on $\mathbb{R}^d$. We denote by $B(x, r) = \{y \in \mathbb{Z}^d; |x - y| \leq r\}$ the closed Euclidean ball of center x and radius $r \geq 0$ intersected with $\mathbb{Z}^d$ and by $B_\infty(x, r) = \{y \in \mathbb{Z}^d, |x - y|_\infty \leq r\}$ the closed l^∞-ball of center x and radius r intersected with $\mathbb{Z}^d$. When U is a subset of $\mathbb{Z}^d$, we write $|U|$ for the cardinality of U, and $U \subset\subset \mathbb{Z}^d$ means that U is a finite subset of $\mathbb{Z}^d$. We denote by ∂U (resp., $\partial_i U$) the boundary (resp., internal boundary) of U, and by $\overline{U}$ its "closure"

$$
\begin{aligned}
\partial U &= \big\{x \in U^c; \exists y \in U, |x - y| = 1\big\}, \\
\partial_i U &= \big\{y \in U; \exists U^c, |x - y| = 1\big\} \quad \text{and} \quad \overline{U} = U \cup \partial U.
\end{aligned}
\tag{3.12}
$$

When $U \subset \mathbb{R}^d$, and $\delta > 0$, we write $U^\delta = \{z \in \mathbb{R}^d; d(z, U) \leq \delta\}$ for the closed δ-neighbourhood of U, where $d(x, A) = \inf_{y \in A} |x - y|$ is the Euclidean distance of x to A. We define $d_\infty(x, A)$ in a similar fashion, with $|\cdot|_\infty$ in place of $|\cdot|$. To distinguish balls in $\mathbb{R}^d$ from balls in $\mathbb{Z}^d$, we write $B_{\mathbb{R}^d}(x, r) = \{z \in \mathbb{R}^d; |x - z| \leq r\}$ for the closed Euclidean ball of center x and radius r in $\mathbb{R}^d$ and $B^\circ_{\mathbb{R}^d}(x, r) = \{z \in \mathbb{R}^d; |x - z| < r\}$ for the corresponding open Euclidean ball. We also write the N-

discrete blow-up of $U \subseteq \mathbb{R}^d$ as

$$(3.13) \qquad U_N = \{x \in \mathbb{Z}^d; d_\infty(x, NU) \le 1\},$$

where we denote by $NU = \{Nz; z \in U\} \subset \mathbb{R}^d$ the set homothetic to U with ratio N.

We will now collect some notation concerning connectivity properties. We write $x \sim y$ if for $x, y \in \mathbb{Z}^d$, $|x - y| = 1$. We call $\pi : \{1, \dots, n\} \to \mathbb{Z}^d$, with $n \ge 1$, a nearest-neighbour path, when $\pi(i) \sim \pi(i-1)$ for $1 < i \le n$. Given K, L, U subsets of $\mathbb{Z}^d$, we say that K and L are connected by U and write $K \overset{U}{\longleftrightarrow} L$, if there exists a finite nearest-neighbour path π in $\mathbb{Z}^d$ such that $\pi(1)$ belongs to K and $\pi(n)$ belongs to L, and for all $k \in \{1, \dots, n\}$, $\pi(k)$ belongs to U. Otherwise, we say that K and L are not connected by U, and write $K \overset{U}{\nleftrightarrow} L$. Similarly, we say that K is connected to infinity by U, if for K, U subsets of $\mathbb{Z}^d$, $K \overset{U}{\longleftrightarrow} B(0, N)^c$ for all N, and write $K \overset{U}{\longleftrightarrow} \infty$. Otherwise, we say K is not connected to infinity by U, and write $K \overset{U}{\nleftrightarrow} \infty$.

We now turn to the definition of some path spaces and of the continuous-time simple random walk. We consider $\widehat{W}_+$ the spaces of infinite $(\mathbb{Z}^d) \times (0, \infty)$-valued sequences such that the first coordinate of the sequence forms an infinite nearest neighbour path in $\mathbb{Z}^d$, spending finite time in any finite set of $\mathbb{Z}^d$, and the sequence of the second coordinate has an infinite sum. The second coordinate describes the duration at each step corresponding to the first coordinate. We denote by $\widehat{\mathcal{W}}_+$ the respective σ-algebra generated by the coordinate maps, Z_n, ζ_n, $n \ge 0$ [where Z_n is $\mathbb{Z}^d$-valued and ζ_n is $(0, \infty)$-valued]. We denote by P_x the law on $\widehat{W}_+$ under which Z_n, $n \ge 0$, has the law of the simple random walk on $\mathbb{Z}^d$, starting from x, and ζ_n, $n \ge 0$, are i.i.d. exponential variables with parameter 1, independent from Z_n, $n \ge 0$. We denote by E_x the corresponding expectation. Moreover, if α is a measure on $\mathbb{Z}^d$, we denote by P_α and E_α the measure $\sum_{x \in \mathbb{Z}^d} \alpha(x) P_x$ (not necessarily a probability measure) and its corresponding "expectation" (i.e., the integral with respect to the measure P_α).

We attach to $\widehat{w} \in \widehat{W}_+$ a continuous-time process $(X_t)_{t \ge 0}$ and call it the random walk on $\mathbb{Z}^d$ with constant jump rate 1 under P_x, through the following relations:

$$X_t(\widehat{w}) = Z_k(\widehat{w}) \qquad \text{for } t \geq 0, \text{ when } \tau_k \leq t < \tau_{k+1}, \tag{3.14}$$

where for l in $\mathbb{Z}^+$, we set (if $l = 0$, the right sum term is understood as 0),

$$\tau_l = \sum_{i=0}^{l-1} \zeta_i. \tag{3.15}$$

We also introduce the filtration

$$\mathcal{F}_t = \sigma(X_s, s \leq t). \tag{3.16}$$

For I a Borel subset of $\mathbb{R}^+$, we record the set of points visited by $(X_t)_{t\geq 0}$ during the time set I as X_I. Importantly, we denote by $\mathcal{V}$ the vacant set, namely the complement of the entire trace $X_{[0,\infty)}$ of X.

Given a subset U of $\mathbb{Z}^d$, and $\widehat{w} \in \widehat{W}_+$, we write $H_U(\widehat{w}) = \inf\{t \geq 0; X_t(\widehat{w}) \in U\}$ and $T_U = \inf\{t \geq 0; X_t(\widehat{w}) \notin U\}$ for the entrance time in U and exit time from U. Moreover, we write $\widetilde{H}_U = \inf\{s \geq \zeta_1; X_s \in U\}$ for the hitting time of U. If $U = \{x\}$, we then write H_x, T_x and $\widetilde{H}_x$.

Given a subset U of $\mathbb{Z}^d$, we write $\Gamma(U)$ for the space of all right-continuous, piecewise constant functions from $[0,\infty)$ to U, with finitely many jumps on any compact interval. We will also denote by $(X_t)_{t\geq 0}$ the canonical coordinate process on $\Gamma(U)$, and when an ambiguity arises, we will specify on which space we are working. For $\gamma \in \Gamma(U)$, we denote by $\mathrm{Range}(\gamma)$ the trace of γ.

Now, we recall some facts concerning equilibrium measure and capacity, and refer to Section 2, Chapter 2 of [38] for more details. Given $M \subset\subset \mathbb{Z}^d$, we write e_M for the equilibrium measure of M:

$$e_M(x) = P_x[\widetilde{H}_M = \infty] 1_M(x), \qquad x \in \mathbb{Z}^d, \tag{3.17}$$

and cap(M) for the capacity of M, which is the total mass of e_M:

$$\text{cap}(M) = \sum_{x \in M} e_M(x). \tag{3.18}$$

We denote the normalized equilibrium measure of M by

$$\widetilde{e}_M(x) = \frac{e_M(x)}{\text{cap}(M)}. \tag{3.19}$$

There is also an equivalent definition of capacity through the Dirichlet form:

$$\text{cap}(M) = \inf_f \mathcal{E}_{\mathbb{Z}^d}(f, f), \tag{3.20}$$

where $f : \mathbb{Z}^d \to \mathbb{R}$ is finitely supported and $f \geq 1$ on M, and

$$\mathcal{E}_{\mathbb{Z}^d}(f, f) = \frac{1}{2} \sum_{|x-y|=1} \frac{1}{2d} \big(f(y) - f(x)\big)^2 \tag{3.21}$$

is the discrete Dirichlet form for simple random walk.

It is well known that (see, e.g., Section 2.2, pages 52–55 of [38])

$$cN^{d-2} \leq \text{cap}\big(B_\infty(0, N)\big) \leq c' N^{d-2}, \tag{3.22}$$

and that

$$e_{B_\infty(0,N)}(x) \geq c_1 N^{-1} \tag{3.23}$$

for x on the inner boundary of $B_\infty(0, N)$.

Now, we turn to random interlacements. We refer to [13, 23, 56] and [59] for more details. Random interlacements are random subsets of $\mathbb{Z}^d$, governed by a non-negative parameter u (referred to as the "level"), and denoted by $\mathcal{I}^u$. We write $\mathbb{P}$ for the law of $\mathcal{I}^u$. Although the construction of random interlacements is involved, the law of $\mathcal{I}^u$ can be simply characterized by the following relation:

$$\mathbb{P}\big[\mathcal{I}^u \cap K = \varnothing\big] = e^{-u\text{cap}(K)} \qquad \text{for all } K \subset\subset \mathbb{Z}^d. \tag{3.24}$$

We denote by $\mathcal{V}^u = \mathbb{Z}^d \setminus \mathcal{I}^u$ the vacant set of random interlacements at level u.

The connectivity function of the vacant set of random interlacements is known to have a stretched-exponential decay when the level exceeds a certain critical value (see Theorem 4.1 of [58], Theorem 0.1 of [53], or Theorem 3.1 of [46] for recent developments). Namely, there exists a $u_{**} \in (0, \infty)$ which, for our purpose in this article, can be characterized as the smallest positive number such that for all $u > u_{**}$,

$$\mathbb{P}\big[0 \overset{\mathcal{V}^u}{\longleftrightarrow} \partial B_\infty(0, N)\big] \leq c(u) e^{-c'(u) N^{c'(u)}} \qquad \text{for all } N \geq 0, \tag{3.25}$$

(actually, the exponent of N can be chosen as 1, when $d \geq 4$, and as an arbitrary number in $(0,1)$ when $d = 3$, see [46]).

We also wish to recall a classical result about relative entropy, which is helpful in Section 3.3. For $\widetilde{P}$ absolutely continuous with respect to P, the relative entropy of $\widetilde{P}$ with respect to P is defined as

$$H(\widetilde{P}|P) = E^{\widetilde{P}}\left[\log \frac{d\widetilde{P}}{dP}\right] = E^P\left[\frac{d\widetilde{P}}{dP} \log \frac{d\widetilde{P}}{dP}\right] \in [0, \infty]. \tag{3.26}$$

For an event A with positive $\widetilde{P}$-probability, we have the following inequality (see page 76 of [22]):

$$P[A] \geq \widetilde{P}[A] e^{-(H(\widetilde{P}|P) + 1/e)/\widetilde{P}[A]}. \tag{3.27}$$

We end this section with some results regarding continuous-time reversible finite Markov chains.

Let L be the generator for an irreducible, reversible continuous-time Markov chain on a finite set V, with jump rates at each state possibly non-constant. Let π be the stationary measure of this Markov chain. Then $-L$ is self-adjoint in $l^2(\pi)$ and has non-negative eigenvalues $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_{|V|}$. We denote by $\lambda = \lambda_2$ its spectral gap. For any real function f on V, we define its variance with respect to π as $\mathrm{Var}_\pi(f)$. Then the semigroup $H_t = e^{tL}$ satisfies

$$\big\|H_t f - \pi(f)\big\|_2 \le e^{-\lambda t}\sqrt{\mathrm{Var}_\pi(f)}. \tag{3.28}$$

One can further show that, for all x and y in V,

$$\big|P_x(X_t = y) - \pi(y)\big| \le \sqrt{\frac{\pi(y)}{\pi(x)}}\, e^{-\lambda t}, \tag{3.29}$$

see pages 326–328 of [51] for more detail.

We also introduce the so-called "canonical path method" to give a lower bound on the spectral gap λ. We denote by E the edge set

$$\big\{\{x,y\}; x, y \in V, L_{x,y} > 0\big\}, \tag{3.30}$$

where $L_{x,y}$ is the matrix coefficient of L. We investigate the following quantity A

$$A = \max_{e\in E}\left\{\frac{1}{W(e)} \sum_{x,y,\gamma(x,y)\ni e} \mathrm{leng}\big(\gamma(x,y)\big)\pi(x)\pi(y)\right\}, \tag{3.31}$$

where γ is a map, which sends ordered pairs of vertices $(x, y) \in V \times V$ to a finite path $\gamma(x, y)$ between x and y, $\mathrm{leng}(\gamma)$ denotes the length of γ, and

$$W(e) = \pi(x)L_{x,y} = \pi(y)L_{y,x} = (1_x, L1_y)_{l^2(\pi)} = (L1_x, 1_y)_{l^2(\pi)} \tag{3.32}$$

is the edge-weight of $e = \{x, y\} \in E$. Then the proof of Theorem 3.2.1, page 369 of [51] is also valid (note that actually e in [51] is an oriented edge) in the present set-up of possibly non-constant jump rates and shows that

$$\lambda \ge \frac{1}{A}. \tag{3.33}$$

3.3 The tilted random walk

In this section, we construct the main protagonists of this work: a time non-homogenous Markov chain on $\mathbb{Z}^d$, which we will refer to as *the tilted walk*, as well as a continuous-time homogenous Markov chain on a (macroscopic) finite subset of $\mathbb{Z}^d$, which we will refer to as *the confined walk*. The tilted walk coincides with the confined walk up to a certain finite time, which is of order N^d, and then evolves as a simple random walk. We derive a lower bound on the spectral gap of the confined walk in Proposition 3.3.12. In Proposition 3.3.14, we prove that with a suitable limiting procedure, the relative entropy between the tilted random walk and the simple random walk has an asymptotic upper bound given by a quantity involving the Brownian capacity of K that appears in Theorem 3.1.1. In this section, the constants tacitly depend on δ, η, ε and R [see (3.35) and (3.36)].

We recall that K is a compact subset of $\mathbb{R}^d$ as above (3.1). We assume, without loss of generality, that

$$0 \in K. \tag{3.34}$$

Otherwise, as we now explain, we can replace K by $\widetilde{K} = K \cup \{0\}$: on the one hand, by the monotonicity and sub-additivity of Brownian capacity (see, e.g., Proposition 1.12, page 60 of [47]), one has $\mathrm{cap}_{\mathbb{R}^d}(K) = \mathrm{cap}_{\mathbb{R}^d}(\widetilde{K})$; on the other hand, since $K \subseteq \widetilde{K}$, it is more difficult to disconnect $\widetilde{K}_N$ than to disconnect K_N, hence $P_0[K_N \overset{\mathcal{V}}{\nleftrightarrow} \infty] \geq P_0[\widetilde{K}_N \overset{\mathcal{V}}{\nleftrightarrow} \infty]$. This means that the lower bound (3.4) with K replaced by $\widetilde{K}$ implies (3.4), justifying our claim. From now on, for the sake of simplicity, for any $r > 0$ we write $B_{(r)}$ for the open ball $B^{\circ}_{\mathbb{R}^d}(0, r)$ and B_r for the closed ball $B_{\mathbb{R}^d}(0, r)$. We introduce the three parameters

$$0 < \delta, \eta, \varepsilon < 1, \tag{3.35}$$

where δ will be used as a smoothing radius for K, see (3.37), η will be used as a parameter in the construction of $\widetilde{h}$, the smoothed potential function [see (3.39)] and ε will be used as a parameter in the definition of T_N, the time length of "tilting"; see (3.48). We let

$R > 400$ be a large integer (see Remark 3.3.4 for explanations on why we take R to be an integer) such that

$$K \subset B_{R/100}. \tag{3.36}$$

By definition of R, we always have

$$K^{2\delta} \subset B_{R/50}. \tag{3.37}$$

In the next lemma, we show the existence of a function $\widetilde{h}$ that satisfies various properties (among which the most important is an inequality relating its Dirichlet form to the relative Brownian capacity of $K^{2\delta}$), which, as we will later show, make it the right candidate for the main ingredient in the construction of the tilted walk.

We denote by $\mathcal{E}_{\mathbb{R}^d}(f,f) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla f(x)|^2\, dx$ for $f \in H^1(\mathbb{R}^d)$ the usual Dirichlet form on $\mathbb{R}^d$ (see Example 4.2.1, page 167 and (1.2.12), page 11 of [30]). For F and G, respectively, closed and open subsets of $\mathbb{R}^d$ such that $F \subset G$, we define the relative Brownian capacity of F with respect to G by

$$\mathrm{cap}_{\mathbb{R}^d,G}(F) = \inf\big\{\mathcal{E}_{\mathbb{R}^d}(f,f)\big\}, \tag{3.38}$$

where the infimum runs over all $f \in H^1(\mathbb{R}^d)$ which are supported in G and satisfy that $f \geq 1$ on F. We write $C^\infty(B_R)$ for the set of functions having all derivatives of every order continuous in $B_{(R)}$, which all have continuous extensions to B_R (see page 10 of [31] for more details).

Lemma 3.3.1. *There exists a continuous function* $\widetilde{h} : \mathbb{R}^d \to \mathbb{R}$*,* *satisfying the following properties:*

$$
(3.39)\quad \begin{cases}
1.\ \widetilde{h} \text{ is a } C^\infty(B_R) \text{ function when restricted} \\
\quad \text{to } B_R, \text{ and harmonic on } B_{(R)} \setminus B_{R/2}; \\
2.\ 0 \le \widetilde{h}(z) \le 1 \text{ for all } z \in \mathbb{R}^d, \widetilde{h} = 1 \text{ on } K^{2\delta}, \\
\quad \text{and } \widetilde{h}(z) = 0 \text{ outside } B_{(R)}; \\
3.\ \mathcal{E}_{\mathbb{R}^d}(\widetilde{h}, \widetilde{h}) \le (1+\eta)^2 \mathrm{cap}_{\mathbb{R}^d, B_{(R)}}(K^{2\delta}); \\
4.\ c w_1 \le \widetilde{h} \le c' w_2 \text{ where } w_1, w_2 \text{ are defined} \\
\quad \text{respectively in (3.43) and (3.44);} \\
5.\ \widetilde{h}(z_1) \ge c \widetilde{h}(z_2) \text{ for all } z_1, z_2 \in \mathbb{R}^d \\
\quad \text{such that } |z_1| \le |z_2| \le R.
\end{cases}
$$

Proof. We now construct $\widetilde{h}$. We define, with δ as in (3.35),

$$
(3.40)\qquad h(z) = W_z[H_{K^{2\delta}} < T_{B_{(R)}}] \qquad \forall z \in \mathbb{R}^d,
$$

the Brownian relative equilibrium potential function, where W_z stands for the Wiener measure starting from $z \in \mathbb{R}^d$, and $H_{K^{2\delta}}$ and $T_{B_{(R)}}$, respectively, stand for the entrance time of the canonical Brownian motion in $K^{2\delta}$ and its exit time from $B_{(R)}$.

We let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth, non-decreasing and concave function such that $0 \le (\psi)'(z) \le 1$ for all $z \in \mathbb{R}$, $\psi(z) = z$ for $z \in (-\infty, \frac{1}{2}]$, and $\psi(z) = 1$ for $z \in [1 + \frac{\eta}{2}, \infty)$. We consider

$$
(3.41)\qquad \widetilde{h} = \psi \circ \big((1+\eta)h\big).
$$

Now we prove the claims.

We first prove claim 1 in (3.39). It is classical that h is C^∞ on $B_{(R)} \setminus K^{2\delta}$. In addition, h is continuous, equal to 1 on $K^{2\delta}$ and to 0 outside $B_{(R)}$ (note that every point in $K^{2\delta}$ is regular for $K^{2\delta}$). In particular, $(1+\eta)h \ge 1 + \eta/2$ on an open neighbourhood of $K^{2\delta}$, which implies that $\widetilde{h}$ is identically equal to 1 on this neighbourhood. It follows that $\widetilde{h}$ is C^∞ on $B_{(R)}$. Now we show that $\widetilde{h}$ is C^∞ on B_R. To prove this, it suffices to prove that h is C^∞ on $B_R \setminus B_{R-c}$

for some $c > 0$, where h coincides with $\widetilde{h}$. We then represent h as $G^{B_{(R)}}\mu$, where we denote by $G^{B_{(R)}}$ and μ, respectively, the killed Green function for $B_{(R)}$ and the (Brownian) equilibrium measure of $K^{2\delta}$ relative to $B_{(R)}$. Since μ is supported on $K^{2\delta}$ and $G^{B_{(R)}}(x,y)$ is C^∞ for all $x \in B_R \setminus B_{(R-c)}$ and $y \in K^{2\delta} \subset B_{R/50}$ [by the explicit formula of the killed Green function for a ball (see, e.g., (41) in Section 2.2, page 40 of [28])], we know that h is C^∞ on $B_R \setminus B_{R-c}$, which implies that $\widetilde{h}$ is C^∞ on B_R. This completes the proof of claim 1.

Claim 2 follows directly from the definition of $\widetilde{h}$: for all $z \in \mathbb{R}^d$, $(1+\eta)h(z) \in [0, 1+\eta]$, hence by the definition of ψ, $\widetilde{h}(z) \in [0,1]$; $\widetilde{h} = 1$ on $K^{2\delta}$ is already shown in claim 1 of (3.39); moreover, by (3.40), outside $B_{(R)}$, $h = 0$, hence $\widetilde{h} = 0$.

We now prove claim 3. By $(\mathcal{E}.4)''$, page 5 of [30], an equivalent characterization of Markovian Dirichlet form, one knows that since ψ is a normal contraction,

$$
\begin{aligned}
\mathcal{E}_{\mathbb{R}^d}(\widetilde{h},\widetilde{h}) &\leq \mathcal{E}_{\mathbb{R}^d}\big((1+\eta)h,(1+\eta)h\big) = (1+\eta)^2\mathcal{E}_{\mathbb{R}^d}(h,h) \\
&= (1+\eta)^2 \mathrm{cap}_{\mathbb{R}^d, B_{(R)}}\big(K^{2\delta}\big),
\end{aligned}
$$

where the last equality follows from [30], pages 152 and 71.

We now turn to claim 4. Because $B_\delta \subset K^{2\delta} \subset B_{R/50}$ by (3.34), we know that

$$
w_1 \leq h \leq w_2 \qquad \text{on } B_R, \tag{3.42}
$$

where

(3.43)

$$
w_1(z) = W_z[H_{B_\delta} < T_{B_{(R)}}] = \begin{cases} 1, & |z| \in [0,\delta), \\ \dfrac{|z|^{2-d} - R^{2-d}}{\delta^{2-d} - R^{2-d}}, & |z| \in [\delta, R), \\ 0, & |z| \in [R, \infty) \end{cases}
$$

and

$$
(3.44)\quad \begin{aligned} w_2(z) &= W_z[H_{B_{R/50}} < T_{B_{(R)}}] \\ &= \begin{cases} 1, & |z| \in [0, R/50), \\ \dfrac{|z|^{2-d} - R^{2-d}}{(R/50)^{2-d} - R^{2-d}}, & |z| \in [R/50, R), \\ 0, & |z| \in [R, \infty) \end{cases} \end{aligned}
$$

are respectively the Brownian relative equilibrium potential functions of B_δ and $B_{R/50}$ (see (4) in Section 1.7, page 29 of [25] for the explicit formula of w_1 and w_2). By the definition of ψ, we also know that, $cr \leq \psi(r) \leq c'r$ for $0 \leq r \leq 1 + \eta$. Hence by the definition of $\widetilde{h}$, we find that

$$
(3.45)\qquad \widetilde{c}w_1 \overset{(3.42)}{\leq} ch \leq \widetilde{h} \leq c'h \overset{(3.42)}{\leq} \widetilde{c'}w_2.
$$

Claim 4 hence follows.

Finally, claim 5 follows by claim 4 and the observation from the explicit formula of w_1 and w_2 that $w_1 \geq cw_2$ uniformly for some positive c on B_R and both w_1 and w_2 are radially symmetric and radially non-increasing:

$$
(3.46)\qquad \begin{aligned} &\widetilde{h}(z_1) \geq cw_1(z_1) \geq c'w_2(z_1) \geq c'w_2(z_2) \geq c''\widetilde{h}(z_2) \\ &\qquad\qquad \text{for } z_1, z_2 \text{ s.t. } |z_1| \leq |z_2| \leq R. \end{aligned}
$$

□

We then introduce the restriction to $\mathbb{Z}^d$ of the blow-up of $\widetilde{h}$ and its $l^2(\mathbb{Z}^d)$-normalization as

$$
(3.47)\qquad h_N(x) = \widetilde{h}\left(\frac{x}{N}\right) \qquad \text{for } x \in \mathbb{Z}^d \quad \text{and} \quad f(x) = \frac{h_N(x)}{\|h_N\|_2},
$$

and also set [see (3.35) for the definition of ε]

$$
(3.48)\qquad T_N = u_{**}(1+\varepsilon)\|h_N\|_2^2,
$$

[recall that u_{**} is the threshold of random interlacements defined above (3.25)]. We define T_N in a way such that the quantity $T_N f^2$ is bigger than u_{**} on K_N^δ, which, roughly speaking, makes the occupational time profile of the tilted random walk (which we will later define) at time T_N on K_N^δ bigger than that of the random interlacement with intensity u_{**}. We also set

$$U^N = B_{(NR)} \cap \mathbb{Z}^d. \tag{3.49}$$

This will be the state space of the confined random walk that we will later define.

In the following lemma, we record some basic properties of f. Intuitively speaking, f is a volcano-shaped function, with maximal value on K_N^δ that vanishes outside U^N. Note that f tacitly depends on δ, η and R.

Lemma 3.3.2. *For large N, one has*

$$\begin{cases} 1.\ f \text{ is supported on } U^N \text{ and } f > 0 \text{ on } U^N; \\ 2.\ f^2 \text{ is a probability measure on } \mathbb{Z}^d \text{ supported on } U^N; \\ 3.\ T_N f^2(\cdot) = u_{**}(1+\varepsilon) \text{ on } K_N^\delta. \end{cases} \tag{3.50}$$

Proof. Claims 1 and 2 follow by the definition of f [see (3.47)] and U^N [see (3.49)], note that by (3.49) $x \in U^N$ implies $\frac{x}{N}$ belongs to the *open* ball $B_{(R)}$. Claim 3 follows from the definition of T_N [see (3.48)] and the fact that $h_N = 1$ on K_N^δ for large N. □

We introduce a subset of U^N (which will be used in Lemma 3.3.11)

$$\begin{aligned} O^N &= \left\{U^N \setminus \left(\partial_i U^N \cup B_{NR/2}\right)\right\} \\ &\quad \cup \left\{x \in \partial_i U^N; |y| = NR \text{ for all } y \sim x, y \notin U^N\right\} \end{aligned} \tag{3.51}$$

(note that both N and R are integers). Intuitively speaking, O^N denotes the set of points in U^N which have distance at least $NR/2$ from 0 such that all their neighbours outside U^N (if there exists any) are on the sphere ∂B_{NR}. In the next lemma we collect some properties of h_N and T_N for later use, in particular in the proofs of

Lemmas 3.3.10, 3.3.11 and Propositions 3.3.13, 3.3.14.

Lemma 3.3.3. *For large N, one has*

$$(3.52)\qquad \begin{cases} 1.\ cN^{-2} \le h_N(x) \le 1 \text{ for all } x \in U^N; \\ 2.\ h_N(x) \le cN^{-1} \text{ for all } x \in \partial_i U^N; \\ 3.\ h_N(x) \ge c'N^{-1} \text{ for all } x \in O^N; \\ 4.\ c'N^d \le \|h_N\|_2^2 \le c''N^d; \\ 5.\ cN^d \le T_N \le c'N^d. \end{cases}$$

Proof. We first prove claim 1. The right-hand side inequality follows by the definition of h_N [see (3.47)] and $\widetilde{h}$ [see (3.41)]. We now turn to the left-hand side inequality of claim 1. For all $x \in U^N$, one has $|x|^2 < (NR)^2$ by the definition of U^N [see (3.49)]. Since x has integer coordinates, this implies $|x| \le \sqrt{(NR)^2 - 1}$, hence for all $x \in U^N$,

$$(3.53)\qquad |x| \le NR - cN^{-1}.$$

Thus, by claim 4 of (3.39) and (3.43) one has

$$(3.54)\qquad \widetilde{h}(z) \ge c'N^{-2} \qquad \text{for all } |z| \le R - \frac{c}{N^2}.$$

This implies that for large N, for all $x \in U^N$,

$$(3.55)\qquad h_N(x) = \widetilde{h}\Big(\frac{x}{N}\Big) \ge c''N^{-2}.$$

Similarly, to prove claims 2, and 3, again by claim 4 of (3.39) and respectively (3.44) and (3.43), it suffices to prove that

$$(3.56)\qquad |x| \ge NR - 1 \qquad \forall x \in \partial_i U^N$$

and that

$$(3.57)\qquad |x| \le NR - c' \qquad \forall x \in O^N.$$

To prove (3.56), we observe that, if $x \in \partial_i U^N$, there exists $y \notin U^N$, such that $x \sim y$. Since $|y| \geq NR$ and $|x-y| = 1$, the claim (3.56) follows by triangle inequality. Now we prove (3.57). We consider $x = (a_1, \ldots, a_d) \in O^N$. By definition of O^N [see (3.51)], $a_1^2 + \cdots + a_d^2 \geq c(NR)^2$, hence without loss of generality, we assume that $|a_1| \geq cNR$. By the definition of O^N, we also know that $(|a_1|+1)^2 + a_2^2 + \cdots + a_d^2 \leq (NR)^2$, which implies that $|x| = \sqrt{a_1^2 + \cdots + a_d^2} \leq NR(1 - c'/N)^{1/2} \leq NR - c''$, and hence (3.57).

Claim 4 follows by the observation that by claim 2 of (3.39), on the one hand $h_N \leq 1$ on $\mathbb{Z}^d$ and h_N is supported on U^N, and on the other hand $h_N = 1$ on $(NK^\delta) \cap \mathbb{Z}^d$.

Claim 5 follows as a consequence of claim 4 and the definition of T_N; see (3.48). □

Remark 3.3.4. With Lemma 3.3.3 we reveal the reason for choosing R to be an integer: because we wish that the lattice points are not too close to the boundary of B_{NR} [see (3.53)]. This enables us to show, for example, that h_N is not too small on U^N, as in claim 1 of (3.52).

Now, we introduce a non-negative martingale that plays an important role in our construction of the tilted random walk. Given a real-valued function g on $\mathbb{Z}^d$, we denote its discrete Laplacian by

$$\Delta_{\text{dis}} g(x) = \frac{1}{2d} \sum_{|e|=1} g(x+e) - g(x). \tag{3.58}$$

For the finitely supported non-negative f defined in (3.47), for all x in U^N, we introduce under the measure P_x the stochastic process

$$M_t = \frac{f(X_{t \wedge T_{U^N}})}{f(x)} e^{\int_0^{t \wedge T_{U^N}} v(X_s)\, ds}, \qquad t \geq 0, P_x\text{-a.s.}, \tag{3.59}$$

where

$$v = -\frac{\Delta_{\text{dis}} f}{f}. \tag{3.60}$$

We define for all $T \geq 0$, a non-negative measure $\widehat{P}_{x,T}$ (on $\widehat{W}_+$) with density M_T with respect to P_x,

$$\widehat{P}_{x,T} = M_T P_x. \tag{3.61}$$

In the next lemma, we show that $\widehat{P}_{x,T}$ is the law of a Markov chain and identify its infinitesimal generator.

Lemma 3.3.5. *For all $x \in U^N$, one has*

$$\begin{aligned}&\widehat{P}_{x,T} \text{ is the probability measure for a}\\ &\text{Markov chain up to time } T \text{ on } U^N.\end{aligned} \tag{3.62}$$

Its semi-group (acting on the finite dimensional space of functions on U^N) admits a generator given by the bounded operator:

$$\widetilde{L}g(x) = \frac{1}{2d} \sum_{y\in U^N, y\sim x} \frac{f(y)}{f(x)}\big(g(y) - g(x)\big). \tag{3.63}$$

Proof. To prove the claims (3.62) and (3.63), we first prove that

$$M_t \text{ is an } (\mathcal{F}_t)\text{-martingale under } P_x. \tag{3.64}$$

For $\zeta \in (0,1)$, we define $f^{(\zeta)} = f + \zeta$ and $v^{(\zeta)} = -\frac{\Delta_{\text{dis}} f^{(\zeta)}}{f^{(\zeta)}} = -\frac{\Delta_{\text{dis}} f}{f^{(\zeta)}}$. We denote by $M_t^{(\zeta)}$, $t \geq 0$, the stochastic process similarly defined as M_t in (3.59) by

$$M_t^{(\zeta)} = \frac{f^{(\zeta)}(X_{t\wedge T_{U^N}})}{f^{(\zeta)}(x)} e^{\int_0^{t\wedge T_{U^N}} v^{(\zeta)}(X_s)\, ds} \qquad t \geq 0, P_x\text{-a.s.} \tag{3.65}$$

By Lemma 3.2 in Chapter 4, page 174 of [27], $M_t^{(\zeta)}$ is an $(\mathcal{F}_t)$-martingale under P_x. Since N is fixed, $f^{(\zeta)}$ is uniformly for $\zeta \in (0,1)$ bounded from above and below on U^N, $v^{(\zeta)}$ is uniformly in ζ bounded on U^N as well. Hence, for all $t \geq 0$, $M_t^{(\zeta)}$ is bounded above uniformly for all $\zeta \in (0,1)$. Therefore, the claim (3.64) follows from the dominated convergence theorem since for all $x \in U^N$, P_x-a.s.,

$\lim_{\zeta \to 0} M_t^{(\zeta)} = M_t$. To prove the claim (3.62), we just note that

$$E_x[M_T] = M_0 = 1. \tag{3.66}$$

Moreover, for all x in U^N by claim 1 of (3.50)

$$f(X_{T_{U^N}}) = 0, \tag{3.67}$$

thus $\widehat{P}_{x,T}$ vanishes on all paths which exit U^N before T. Then the claim (3.63) follows by Theorem 2.5, page 61 of [21]. □

Remark 3.3.6. When we apply the lemma from [27] mentioned in the proof above, we need that $\inf_{x \in \overline{U^N}} f(x) > 0$. However, by claim 1 of (3.50), we know that $f(x) = 0$ for all $x \in \partial U^N$. To cope with this problem, we introduce a perturbation term ζ, and apply the lemma to the perturbated objects instead of the original ones.

We then denote the law of the "tilted random walk" by

$$\widetilde{P}_N = \widehat{P}_{0,T_N}. \tag{3.68}$$

Remark 3.3.7. Intuitively speaking, $\widetilde{P}_N$ is the law of a tilted random walk, which restrains itself up to time T_N from exiting U^N and then, after the deterministic time T_N, continues as the simple random walk. It is absolutely continuous with respect to P_0.

It is convenient for us to define $\{\overline{P}_x\}_{x \in U^N}$, a family of finite-space Markov chains on U^N, with generator $\widetilde{L}$ defined in (3.63). We will call this Markov chain "the confined walk", since it is supported on $\Gamma(U^N)$ [see below (3.16) for the definition]. We will also tacitly regard it as a Markov chain on $\mathbb{Z}^d$, when no ambiguity rises. We denote by $\overline{E}_x$ the expectation with respect to $\overline{P}_x$, for all $x \in U^N$.

Thus, the following corollary is immediate.

Corollary 3.3.8.

(3.69) *Up to time $T_N, \widetilde{P}_N$ coincides with $\overline{P}_0$.*

Proof. It suffices to identify the finite time marginals of the two measures with the help of the Markov property and (3.63). □

Remark 3.3.9. Since the confined walk is time-homogenous, in Sections 3.4, 3.5 and 3.6 we will actually perform the analysis on the confined walk instead of the tilted walk, and transfer the result concerning the time period $[0, T_N)$ back to the tilted walk thanks to the above corollary. See, for instance, (3.251).

We now state and prove some basic estimates about the confined walk.

Lemma 3.3.10. *One has*

(3.70)

$$\begin{cases} 1. \textit{ The measure } \pi(x) = f^2(x), x \in U^N, \textit{ is a reversible} \\ \quad \textit{measure for the (irreducible) confined walk } \{\overline{P}_x\}_{x \in U^N}; \\ 2. \textit{ The Dirichlet form associated with } \{\overline{P}_x\}_{x\in U^N} \textit{ and } \pi \textit{ is} \\ \quad \overline{\mathcal{E}}(g,g) = (-\widetilde{L}g, g)_{l^2(\pi)} = \frac{1}{2}\sum_{x,y\in U^N, x\sim y} \frac{f(x)f(y)}{2d}(g(x)-g(y))^2 \\ \quad \textit{with } g : U^N \to \mathbb{R}^+; \\ 3. \textit{ If } x, y \in U^N, |x| \le |y|, \textit{ then one has } h_N(x) \ge c h_N(y) \\ \quad \textit{and } \pi(x) \ge c'\pi(y); \\ 4. \textit{ For all } x \in U^N, cN^{-d-4} \le \pi(x) = f^2(x) \le c'N^{-d}. \end{cases}$$

Proof. Claim 1 follows from claims 1 and 2 of (3.50) and the observation that by (3.63) $\widetilde{L}$ is self-adjoint in $l^2(\pi)$. Claim 2 follows from claim 1 and (3.63). Claim 3 follows from (3.47) and claim 5 of (3.39). Claim 4 follows from claims 1 and 4 of (3.52) and the definition of f [see (3.47)]. □

In the next lemma, we control the fluctuation of v with a rough lower bound and a more refined upper bound.

Lemma 3.3.11. *One has [recall v is defined in (3.60)], for all x in U^N,*

$$-cN^2 \le v(x) \le c'N^{-2}. \tag{3.71}$$

Proof. We first record an identity for later use:

$$v(x) \overset{(3.60)}{=} -\frac{\Delta_{\mathrm{dis}} f(x)}{f(x)} \overset{(3.47)}{=} -\frac{\Delta_{\mathrm{dis}} h_N(x)}{h_N(x)}. \tag{3.72}$$

The inequality on the left-hand side of (3.71) is very rough and follows from

$$v \underset{h_N \geq 0}{\overset{(3.72)}{\geq}} -\frac{\max_{x\in U^N} h_N(x)}{\min_{x\in U^N} h_N(x)} \overset{(3.52)1.}{\geq} -cN^2. \tag{3.73}$$

Next, we prove the inequality on the right-hand side of (3.71). We split U^N into three parts and call them by I^N, O^N and S^N, respectively. Before we go into detail, we describe roughly the division, and what it entails. The region $I^N = B_{NR/2} \cap \mathbb{Z}^d$ is the "inner part" of U^N; the region O^N that already appears in (3.51) is the "outer part" of U^N that does not feel the push of the "hard" boundary, that is, all neighbours of its points belong to B_{NR}; the region $S^N = \partial_i U^N \setminus O^N$ is a subset of the inner boundary of U^N, where all points have a least one neighbour outside $B_{NR} \cap \mathbb{Z}^d$, and thus "feel the hard push" from outside U^N. As we will later see, in the microscopic region that corresponds to I^N, $\widetilde{h}$ is a smooth function; in the region O^N, h_N is at least of order N^{-1} and $|\Delta_{\mathrm{dis}} h_N|$ is at most of order N^{-3}; in the region S^N, one has $\Delta_{\mathrm{dis}} h_N > 0$.

We first record an estimate. Using a Taylor formula at second order with Lagrange remainder (see Theorem 5.16, pages 110–111 of [50]), since for all $x \in U^N \setminus S^N$, all y adjacent to x belongs to B_{NR}, we know from (3.47) that

$$\Delta_{\mathrm{dis}} h_N(x) \geq \frac{1}{N^2}\left(\frac{1}{2d}\Delta\widetilde{h}\left(\frac{x}{N}\right) - cN^{-1}\right) \qquad \text{for all } x \in U^N \setminus S^N. \tag{3.74}$$

We first treat points in $I^N = B_{NR/2} \cap \mathbb{Z}^d$. On $B_{R/2}$, we know that $\widetilde{h} \geq c$ and $\widetilde{h}$ is C^∞ by claim 1 of (3.39). We thus obtain that for all x in I^N,

$$-\frac{\Delta_{\text{dis}} h_N(x)}{h_N(x)} \underset{(3.74)}{\overset{(3.47)}{\leq}} -\frac{\Delta\widetilde{h}(\frac{x}{N}) - cN^{-1}}{\widetilde{h}(\frac{x}{N})N^2} \leq cN^{-2}. \tag{3.75}$$

We then recall that $O^N = \{U^N \setminus (\partial_i U^N \cup B_{NR/2})\} \cup \{x \in \partial_i U^N; |y| = NR \text{ for all } y \sim x, y \notin U^N\}$, as defined in (3.51). By claim 1 of (3.39), we know that for all $x \in O^N$, $\Delta\widetilde{h}(\frac{x}{N}) = 0$. Hence, we find that

$$v(x) \overset{(3.74)}{\leq} -\frac{\Delta\widetilde{h}(x/N) - cN^{-1}}{h_N(x)N^2} \overset{(3.52)3.}{\leq} \frac{cN^{-1}}{c'N^{-1}\cdot N^2} = c''N^{-2} \quad \text{for all } x \in O^N. \tag{3.76}$$

We finally treat points in $S^N = \partial_i U^N \setminus O^N$. By Lemma 6.37, page 136 of [31], $\widetilde{h}$ can be extended to a C^3 function w on $B_{(R+1)}$ such that $w = \widetilde{h}$ in B_R and all the derivatives of w up to order three are uniformly bounded in $B_{(R+1)}$. Hence, we have for all $x \in S^N$,

$$\begin{aligned} -\Delta_{\text{dis}} h_N(x) &= \left(w\left(\frac{x}{N}\right) - \frac{1}{2d}\sum_{y\sim x} w\left(\frac{y}{N}\right)\right) \\ &\quad + \frac{1}{2d}\sum_{y\sim x, y\notin U^N}\left(w\left(\frac{y}{N}\right) - \widetilde{h}\left(\frac{y}{N}\right)\right) = \mathrm{I} + \mathrm{II}. \end{aligned} \tag{3.77}$$

On the one hand, by a second-order Taylor expansion with Lagrange remainder, and since $\Delta w = 0$ in $B_R \setminus B_{R/2}$, we have

$$\mathrm{I} \leq \frac{1}{N^2}\left(\frac{1}{2d}\Delta w\left(\frac{x}{N}\right) + \frac{c}{N}\right) = \frac{c'}{N^3} \qquad \text{for } x \in S^N. \tag{3.78}$$

On the other hand, we know that by claim 2 of (3.39)

$$\widetilde{h}\left(\frac{y}{N}\right) = 0 \qquad \text{for all } y \notin U^N. \tag{3.79}$$

Moreover, by definition of S^N, there exists a point y in $\mathbb{Z}^d$, adjacent to x, such that $NR < |y| \le NR + 1$. This implies that

$$(3.80)\qquad \begin{aligned} (NR+1)^2 &\ge |y|^2 \ge (NR)^2 + 1 \quad \text{and hence} \\ R + \frac{1}{N} &\ge \frac{|y|}{N} \ge R + c' N^{-2}. \end{aligned}$$

By claim 4 of (3.39), since $\widetilde{h}$ is bounded from above and below by two functions having (constant) negative outer normal derivatives on ∂B_R, we find that

$$(3.81)\qquad \frac{\partial \widetilde{h}}{\partial n}(z) < -c \qquad \text{uniformly for all } z \in \partial B_R,$$

where $\frac{\partial \widetilde{h}}{\partial n}$ denotes the outer normal derivative of $\widetilde{h}$. Thus, we find that for large N,

$$(3.82)\qquad w\left(\frac{y}{N}\right) \le -\overline{c} N^{-2}.$$

This implies that

$$(3.83)\qquad \mathrm{II} \overset{(3.79)}{=} \frac{1}{2d} \sum_{y \sim x, y \notin U^N} w\left(\frac{y}{N}\right) \le -c'' N^{-2}.$$

Combining (3.78) and (3.83), it follows that for large N and all $x \in S^N$,

$$(3.84)\qquad v(x) \overset{(3.72)}{=} -\frac{\Delta_{\mathrm{dis}} h_N(x)}{h_N(x)} \underset{(3.83)}{\overset{(3.77),(3.78)}{\le}} \frac{cN^{-3} - c'' N^{-2}}{h_N(x)} < 0.$$

Since I^N, O^N and S^N form a partition of U^N, the inequality in the right-hand side of (3.71) follows by collecting (3.75), (3.76) and (3.84). □

We will now derive a lower bound for the spectral gap of the confined walk, which we denote by $\overline{\lambda}$. We use the method introduced at the end of Section 3.2 and derive an upper bound for the quantity A [recall that A is defined in (3.31)]. However, we first need to specify our choice of paths γ. For $x = (x_1, \dots, x_d), y = (y_1, \dots, y_d) \in U^N$, we assume, without loss of generality, that for some $l \in \{0, \dots, d\}$ we have

$$\begin{cases} |x_i| \geq |y_i|, & \text{for i=1,...,l} \\ |x_i| < |y_i|, & \text{for i=l+1, ...,d,} \end{cases} \tag{3.85}$$

($l = 0$ means that $|x_i| < |y_i|$ for all $i = 1, \dots, d$, and $l = d$ means that $|x_i| \geq |y_i|$ for all $i = 1, \dots, d$). For $p, q \in \mathbb{Z}^d$, which differ only in one coordinate, we denote by $\beta(p, q)$ the straight (and shortest) path between them. Then $\gamma(x, y)$ is defined as follows:

$$\begin{aligned} \gamma(x,y) \quad = \quad & \text{the concatenation of the paths} \\ & \beta\big((y_1, \dots, y_{i-1}, x_i, \dots, x_d), (y_1, \dots, y_i, x_{i+1}, \dots, x_d)\big) \\ & \text{as } i \text{ goes from 1 to } d. \end{aligned} \tag{3.86}$$

Loosely speaking, $\gamma(x, y)$ successively "adjusts" each coordinate of x with the corresponding coordinate of y by first "decreasing" the coordinates where $|x_i|$ is bigger or equal to $|y_i|$ and then "increasing" the coordinates where $|y_i|$ is bigger than $|x_i|$. It is easy to check that this path lies entirely in U^N, since by (3.85), for all $\{p, q\} \in \gamma(x, y)$, one has

$$\max\big(|p|, |q|\big) \leq \max\big(|x|, |y|\big). \tag{3.87}$$

Proposition 3.3.12. *One has*

$$\overline{\lambda} \geq cN^{-2}. \tag{3.88}$$

Proof. Recall that the quantity

$$A = \max_{e \in E} \left\{ \frac{1}{W(e)} \sum_{x,y,\gamma(x,y) \ni e} \text{leng}\big(\gamma(x,y)\big)\pi(x)\pi(y) \right\}$$

is defined in (3.31). By (3.33), to prove (3.88), it suffices to prove that

$$A \leq c' N^2. \tag{3.89}$$

On the one hand, by (3.87) and claim 3 of (3.70) one obtains that for all $\{p,q\} \in \gamma(x,y)$,

$$\min\big(\pi(p), \pi(q)\big) \geq c \min\big(\pi(x), \pi(y)\big). \tag{3.90}$$

This implies that for all $\{p,q\} \in \gamma(x,y)$

$$\begin{aligned} W\big(\{p,q\}\big) &\overset{(3.63)}{\underset{(3.32)}{=}} \pi(p) \frac{f(q)}{2df(p)} \overset{(3.70)1.}{=} \frac{1}{2d} f(p) f(q) \\ &\overset{(3.90)}{=} \frac{1}{2d} \sqrt{\pi(p)\pi(q)} \geq c' \min\big(\pi(x), \pi(y)\big). \end{aligned} \tag{3.91}$$

On the other hand, for any $x, y \in U^N$, one has

$$\text{leng}\big(\gamma(x,y)\big) \leq cN. \tag{3.92}$$

Now we estimate the maximal possible number of paths that could pass through a certain edge. We claim that, for any edge $e \in E^N$, where we denote by E^N the edge set of U^N consisting of unordered pairs of neighbouring vertices in U^N:

$$E^N = \big\{\{x,y\}; x, y \in U^N, |x-y| = 1\big\}, \tag{3.93}$$

there are at most cN^{d+1} paths passing through e. We now prove the claim. To fix a pair of $\{x,y\}$ such that $e = \{(a_1, \dots, a_k, \dots, a_d), (a_1, \dots, a_k + 1, \dots, a_d)\}$ belongs to $\gamma(x,y)$, where $k \in \{1, \dots, d\}$, there are $2d$ coordinates to be chosen. Actually, for $i = 1, \dots, k-1, k+1, \dots, d$ the ith coordinate of either x or y must be a_i. This leaves us at most 2^{d-1} ways of choosing $(d-1)$ coordinates of x and y to be fixed by $a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_d$. For the other $(d+1)$ coordinates, we have no more than cN choices for each of them, since both x and y must lie in U^N. This implies that there are no more that $c' N^{d+1}$ pairs of $\{x,y\} \subset U^N$, such that $e \in \gamma(x,y)$ is possible.

Combining the argument in the paragraph above with (3.91) and

(3.92), one has

$$
\begin{aligned}
A & \overset{(3.31)}{=} \max_{e\in E^N} \frac{1}{W(e)} \sum_{x,y,\gamma(x,y)\ni e} \mathrm{leng}\big(\gamma(x,y)\big)\pi(x)\pi(y) \\
& \underset{(3.92)}{\overset{(3.91)}{\leq}} \max_{e\in E^N} \sum_{x,y,\gamma(x,y)\ni e} c'N\cdot\max\big(\pi(x),\pi(y)\big) \\
& \overset{(3.70)4.}{\leq} c''N^{d+1}\cdot N\cdot N^{-d} = c''N^2.
\end{aligned} \tag{3.94}
$$

This proves (3.89), and hence (3.88). □

We then define for $\{\overline{P}_x\}_{x\in U^N}$ the regeneration time

$$\overline{t_*} = N^2\log^2 N. \tag{3.95}$$

In view of above proposition, $\overline{t_*}$ is much larger than the relaxation time $1/\overline{\lambda}$, which is of order $O(N^2)$. Hence, for all x in U^N, $\overline{P}_x[X_t = \cdot]$ becomes very close to the stationary distribution π, when $t\geq\overline{t_*}$. More precisely, by (3.29) and (3.88)

$$
\begin{aligned}
\sup_{x,y\in U^N}\big|\overline{P}_x[X_t=y]-\pi(y)\big| & \leq \sup_{x,y\in U^N}\sqrt{\frac{\pi(y)}{\pi(x)}}e^{-\overline{\lambda}t} \\
& \underset{(3.88),(3.95)}{\overset{(3.70)4.}{\leq}} e^{-c\log^2 N} \quad \forall t\geq\overline{t_*}.
\end{aligned} \tag{3.96}
$$

We now relate the relative entropy between $\widetilde{P}_N$ (which tacitly depends on R, η, δ and ε) and P_0 to the Dirichlet form of h_N and derive an asymptotic upper bound for it by successively letting $N\to\infty$, $\eta\to 0$, $R\to\infty$, $\delta\to 0$ and $\varepsilon\to 0$ in the following Propositions 3.3.13 and 3.3.14. The Brownian capacity of K will appear as the limit in the above sense of the properly scaled Dirichlet form of h_N.

Proposition 3.3.13. *One has*

$$H(\widetilde{P}_N|P_0)\leq u_{**}(1+\varepsilon)\mathcal{E}_{\mathbb{Z}^d}(h_N,h_N)+o\big(N^{d-2}\big). \tag{3.97}$$

Proof. By definition of the relative entropy [see (3.26)], we have

$$\begin{aligned}
H(\widetilde{P}_N|P_0) &\overset{(3.26)}{=} E^{\widetilde{P}_N}\left[\log \frac{d\widetilde{P}_N}{dP_0}\right] \underset{(3.68)}{\overset{(3.61)}{=}} E^{\widetilde{P}_N}[\log M_{T_N}] \\
&\overset{(3.69)}{=} \overline{E}_0[\log M_{T_N}] \\
&\overset{(3.59)}{=} \overline{E}_0\left[\int_0^{T_N} v(X_s)\,ds + \log f(X_{T_N}) - \log f(X_0)\right] \\
&= \overline{E}_0\left[\int_0^{\overline{t_*}} v(X_s)\,ds\right] + \overline{E}_0\left[\int_{\overline{t_*}}^{T_N} v(X_s)\,ds\right] \\
&\quad + \overline{E}_0\big[\log f(X_{T_N}) - \log f(X_0)\big] \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

For an upper bound of I, by (3.71) and the definition of $\overline{t_*}$ [see (3.95)], we have

$$\mathrm{I} \le \overline{t_*} \max_{x \in U^N} v(x) \le c \log^2 N. \tag{3.98}$$

For an upper bound of II, we notice that applying (3.96) for $t \in (\overline{t_*}, T_N)$,

(3.99)

$$\begin{aligned}
&\left|\overline{E}_0\left[\int_{\overline{t_*}}^{T_N} v(X_t)\,dt\right] - (T_N - \overline{t_*}) \int v\,d\pi\right| \\
&\quad \le (T_N - \overline{t_*}) \sup_{t \in [\overline{t_*}, T_N]} \sup_{y \in U^N} \left|\overline{P}_0[X_t = y] - \int v\,d\pi\right| \cdot \max_{y \in U^N} \big|v(y)\big| \\
&\quad \overset{(3.96)}{\le} e^{-c \log^2 N} (T_N - \overline{t_*}) \max_{y \in U^N} \big|v(y)\big| \underset{(3.52)5.}{\overset{(3.71)}{\le}} e^{-c' \log^2 N}.
\end{aligned}$$

Since f is supported on U^N by claim 1 of (3.50), we may enlarge the range for summation in the second equality in the following calculation without changing the sum and see that

$$\begin{aligned}\int v\,d\pi &\overset{(3.60)}{\underset{(3.70)1.}{=}} \sum_{x\in U^N} \frac{-\Delta_{\mathrm{dis}} f(x)}{f(x)} f^2(x) \\ &\overset{(3.48)}{=} \frac{u_{**}(1+\varepsilon)}{T_N} \sum_{x\in\mathbb{Z}^d} -f(x)\Delta_{\mathrm{dis}} f(x) \|h_N\|_2^2 \\ &\overset{(3.47)}{=} \frac{u_{**}(1+\varepsilon)}{T_N} \sum_{x\in\mathbb{Z}^d} -h_N(x)\Delta_{\mathrm{dis}} h_N(x).\end{aligned} \tag{3.100}$$

By the discrete Green–Gauss theorem and the definition of Dirichlet form, we have

$$\begin{aligned}\sum_{x\in\mathbb{Z}^d} -h_N(x)\Delta_{\mathrm{dis}} h_N(x) &= \frac{1}{2} \sum_{\substack{x,x'\in\mathbb{Z}^d \\ x\sim x'}} \frac{1}{2d}\big(h_N(x') - h_N(x)\big)^2 \\ &= \mathcal{E}_{\mathbb{Z}^d}(h_N, h_N).\end{aligned} \tag{3.101}$$

Hence by (3.100) and (3.101), we know that

$$(T_N - \overline{t_*}) \int v\,d\pi \le u_{**}(1+\varepsilon)\mathcal{E}_{\mathbb{Z}^d}(h_N, h_N). \tag{3.102}$$

Thus, we obtain from (3.102) and (3.99) that

$$\mathrm{II} \le u_{**}(1+\varepsilon)\mathcal{E}_{\mathbb{Z}^d}(h_N, h_N) + e^{-c' \log^2 N}. \tag{3.103}$$

For the calculation of III, we know that

$$\begin{aligned}\overline{E}_0\big[\log f(X_{T_N}) - \log f(X_0)\big] &\le \log \max_{x\in U^N} f(x) - \log \min_{x\in U^N} f(x) \\ &\overset{(3.70)4.}{\le} c\log N.\end{aligned} \tag{3.104}$$

Combining (3.98), (3.103) and (3.104), we obtain that

$$H(\widetilde{P}_N | P_0) \le u_{**}(1+\varepsilon)\mathcal{E}_{\mathbb{Z}^d}(h_N, h_N) + o\big(N^{d-2}\big), \tag{3.105}$$

which is (3.97). □

Proposition 3.3.14. *One has*

$$\limsup_{\varepsilon\to 0}\limsup_{\delta\to 0}\limsup_{R\to\infty}\limsup_{\eta\to 0}\limsup_{N\to\infty}\frac{1}{N^{d-2}}H(\widetilde{P}_N|P_0) \leq \frac{u_{**}}{d}\mathrm{cap}_{\mathbb{R}^d}(K). \tag{3.106}$$

Proof. By (3.97), we have

$$\limsup_{N\to\infty}\frac{1}{N^{d-2}}H(\widetilde{P}_N|P_0) \leq u_{**}(1+\varepsilon)\limsup_{N\to\infty}\frac{1}{N^{d-2}}\mathcal{E}_{\mathbb{Z}^d}(h_N,h_N). \tag{3.107}$$

By the definition of h_N, we have

$$\begin{aligned}\frac{1}{N^{d-2}}&\mathcal{E}_{\mathbb{Z}^d}(h_N,h_N)\\ &= \frac{1}{4dN^{d-2}}\sum_{x\sim y\in\mathbb{Z}^d}\big(h_N(y)-h_N(x)\big)^2\\ &\overset{(3.47)}{=} \frac{1}{4dN^{d-2}}\sum_{x\sim y\in\mathbb{Z}^d}\left(\widetilde{h}\left(\frac{y}{N}\right)-\widetilde{h}\left(\frac{x}{N}\right)\right)^2.\end{aligned} \tag{3.108}$$

By claim 2 of (3.39), the summation in the right member of the second equality in (3.108) can be reduced to $x,y\in U^N\cup\partial U^N$. Then we split the sum into two parts:

$$\sum_{x\sim y\in\mathbb{Z}^d}\left(\widetilde{h}\left(\frac{y}{N}\right)-\widetilde{h}\left(\frac{x}{N}\right)\right)^2 = \Sigma_1+\Sigma_2, \tag{3.109}$$

where

$$\Sigma_1 = \sum_{x,y\in U^N, x\sim y}\left(\widetilde{h}\left(\frac{y}{N}\right)-\widetilde{h}\left(\frac{x}{N}\right)\right)^2 \tag{3.110}$$

contains all summands with $x,y\in U^N$, and

$$\Sigma_2 = 2\sum_{x\in U^N, y\notin U^N, x\sim y}\big(h_N(y)-h_N(x)\big)^2 \tag{3.111}$$

contains all summands with x in U^N and y in ∂U^N. By claim 2 of (3.39), we find that

$$\lim_{N\to\infty} \frac{1}{4dN^{d-2}}\Sigma_1 = \frac{1}{2d}\int_{\mathbb{R}^d} s\big|\nabla\widetilde{h}(y)\big|^2\,dy \tag{3.112}$$

by a Riemann sum argument. While by claim 2 of (3.52), we obtain that

$$\Sigma_2 \le c \sum_{x\in\partial_i U^N} h_N(x)^2 \le c' N^{d-1}\left(\frac{c}{N}\right)^2 = c'' N^{d-3}. \tag{3.113}$$

This implies that

$$\lim_{N\to\infty} \frac{1}{N^{d-2}}\Sigma_2 = 0. \tag{3.114}$$

Therefore, we have

$$\begin{aligned} &\limsup_{N\to\infty} \frac{1}{N^{d-2}}\mathcal{E}_{\mathbb{Z}^d}(h_N,h_N) \\ \le\ &\lim_{N\to\infty} \frac{1}{4dN^{d-2}}(\Sigma_1+\Sigma_2) \\ =\ &\frac{1}{2d}\int_{\mathbb{R}^d}\big|\nabla\widetilde{h}(y)\big|^2\,dy = \frac{1}{d}\mathcal{E}_{\mathbb{R}^d}(\widetilde{h},\widetilde{h}). \end{aligned} \tag{3.115}$$

Therefore, by claim 3 of (3.39) we see that

$$\begin{aligned} &\limsup_{\eta\to 0}\limsup_{N\to\infty} \frac{1}{N^{d-2}} H(\widetilde{P}_N|P_0) \\ &\le \limsup_{\eta\to 0} \frac{u_{**}(1+\varepsilon)}{d}\mathcal{E}_{\mathbb{R}^d}(\widetilde{h},\widetilde{h}) \\ &\le \frac{u_{**}(1+\varepsilon)}{d}\mathrm{cap}_{\mathbb{R}^d,B_{(R)}}\big(K^{2\delta}\big). \end{aligned} \tag{3.116}$$

As $R\to\infty$, the relative capacity converges to the usual Brownian capacity (this follows for instance from the variational characterization of the capacity in Theorem 2.1.5 on pages 71 and 72 of [30]):

$$\text{cap}_{\mathbb{R}^d, B_{(R)}}\big(K^{2\delta}\big) \to \text{cap}_{\mathbb{R}^d}\big(K^{2\delta}\big) \qquad \text{as } R \to \infty. \tag{3.117}$$

Then, letting $\delta \to 0$, by Proposition 1.13, page 60 of [47], we have

$$\text{cap}_{\mathbb{R}^d}\big(K^{2\delta}\big) \to \text{cap}_{\mathbb{R}^d}(K) \qquad \text{as } \delta \to 0. \tag{3.118}$$

Finally, by letting $\varepsilon \to 0$ the claim then follows. □

Remark 3.3.15. In this section, guided by the heuristic strategy described below (3.7), we have constructed the tilted random walk. In essence, this continuous-time walk spends up to T_N, chosen in (3.48), at each point $x \in K_N^\delta$ an expected time equal to $u_{**}(1+\varepsilon)h_N^2(x) = u_{**}(1+\varepsilon)$, when started with the stationary measure π of the confined walk. The low entropic cost of the tilted walk with respect to the simple random walk is quantified by the above Proposition 3.3.14. We will now see in the subsequent sections that in the vicinity of points of K_N^δ, the geometric trace left by the tilted walk by time T_N stochastically dominates random interlacements at a level "close to $u_{**}(1+\varepsilon)$".

3.4 Hitting time estimates

In this section, we relate the entrance time (of the confined walk) into mesoscopic boxes inside K_N^δ to the capacity of these boxes and T_N [see (3.48)] and establish a pair of asymptotically matching bounds in the Propositions 3.4.5 and 3.4.7. It is a key ingredient for the construction of couplings in Section 3.6. The arguments in this section are similar to those in Section 3.4 and the Appendix of [64]. However, in our set-up, special care is needed due to the fact that the stationary measure is massively non-uniform. In this section, the constants tacitly depend on δ, η, ε and R [see (3.35) and (3.36)], r_1, r_2, r_3, r_4 and r_5 [see (3.119)].

We start with the precise definition of objects of interest for the current and the subsequent sections. We denote by $\Gamma^N = \partial K_N^{\delta/2}$ the boundary in $\mathbb{Z}^d$ of the discrete blow-up of $K^{\delta/2}$ (we recall (3.13) and

(3.13) for the definition of the boundary and of the discrete blow-up). The above Γ^N will serve as a set "surrounding" K_N. We choose real numbers

$$0 < r_1 < r_2 < r_3 < r_4 < r_5 < 1. \tag{3.119}$$

We define for x_0 in Γ^N six boxes centered at x_0 (when there is ambiguity we add a superscript for their center x_0):

$$A_i = B_\infty\big(x_0, \lfloor N^{r_i} \rfloor\big),\ \ 1 \le i \le 5 \quad \text{and} \ \ A_6 = B_\infty\left(x_0, \left\lfloor \frac{\delta}{100} N \right\rfloor\right), \tag{3.120}$$

and we tacitly assume that N is sufficiently large so that for all $x_0 \in \Gamma^N$, the following inclusions hold:

$$A_1 \subset A_2 \subset A_3 \subset A_4 \subset A_5 \subset A_6 \subset B_N^\delta \subset\subset \mathbb{Z}^d. \tag{3.121}$$

In view of (3.121) and claim 3 of (3.50) we find that, by (3.63), for large N and all x in U^N

$$\text{the stopped processes } X_{\cdot \wedge T_{A_6}} \text{ under } P_x \text{ and } \overline{P}_x \text{ have the same law.} \tag{3.122}$$

Remark 3.4.1. Recall that the regeneration time $\overline{t_*}$ is defined in (3.95) as $\overline{t_*} = N^2 \log^2 N$, and for all $k = 1, \dots, 5$, A_k are mesoscopic objects of size $O(N^r)$ where $r \in (0,1)$. Informally, Propositions 3.4.5 and 3.4.7 will imply that for all x "far away" from A_k, with a high $\overline{P}_x$-probability,

$$T_N \gg H_{A_k} \gg \overline{t_*}. \tag{3.123}$$

Given any x_0 in Γ^N, we write

$$D = U^N \setminus A_2, \tag{3.124}$$

and let

$$g(x) = \overline{P}_x[H_{A_1} \le T_{A_2}] \overset{(3.122)}{=} P_x[H_{A_1} \le T_{A_2}], \qquad x \in U^N, \tag{3.125}$$

be the (tilted) potential function of A_1 relative to A_2. We also let

$$f_{A_1}(x) = 1 - \frac{\overline{E}_x[H_{A_1}]}{\overline{E}_\pi[H_{A_1}]} \tag{3.126}$$

be the centered fluctuation of the scaled expected entrance time of A_1 (relative to the stationary measure).

The following lemma shows that the inverse of $\overline{E}_\pi[H_{A_1}]$ is closely related to $\overline{\mathcal{E}}(g,g)$. (Actually we are going to prove that they are approximately equal later in this section; see Propositions 3.4.7 and 3.4.5, as well as Remark 3.4.8.)

Lemma 3.4.2. *One has,*

$$\overline{\mathcal{E}}(g,g)\Big(1 - 2\sup_{x\in D}\big|f_{A_1}(x)\big|\Big) \le \frac{1}{\overline{E}_\pi[H_{A_1}]} \le \overline{\mathcal{E}}(g,g)\frac{1}{\pi(D)^2}. \tag{3.127}$$

The proof is omitted due to its similarity to the proof of Lemma 3.2 in [14] (which further calls Proposition 3.41 in [3], which is originally intended for Markov chains with constant jump rate).

In the next lemma, we collect some properties of entrance probabilities for later use, namely Propositions 3.4.5, 3.4.7, 3.5.7 and 3.6.1.

Lemma 3.4.3. *For large N, one has*

$$\overline{P}_x[H_{A_1} < \overline{t_*}] \le N^{-c} \qquad \text{for all } x \in D, \tag{3.128}$$

and similarly

$$\overline{P}_x[H_{A_2} < \overline{t_*}] \le N^{-c'} \qquad \text{for all } x \in U^N \setminus A_3. \tag{3.129}$$

Uniformly for all $x \in \partial_i A_1$, one has

$$\begin{aligned} e_{A_1}(x) &\le P_x[T_{A_6} < \widetilde{H}_{A_1}] \le P_x[T_{A_3} < \widetilde{H}_{A_1}] \\ &\le P_x[T_{A_2} < \widetilde{H}_{A_1}] \le e_{A_1}(x)\big(1 + N^{-c''}\big). \end{aligned} \tag{3.130}$$

Proof. We start with (3.128). First, we explain that to prove (3.128), it suffices to show that

$$(3.131)\quad \sup_{x\in\partial A_2} \overline{P}_x[H_{A_1} < \widetilde{t}] \le \sup_{x\in\partial A_2} \overline{P}_x[H_{A_1} < \widetilde{t} - t_\#] + N^{-c'} \quad \text{for all } 0 \le \widetilde{t} \le \overline{t_*},$$

where we write $t_\#$ for $N^2/\log N$. Indeed, with (3.131), the claim (3.128) follows by an induction argument:

$$(3.132)\quad \begin{aligned} &\sup_{x\in D} \overline{P}_x[H_{A_1} < \overline{t_*}] \\ &\le \sup_{x\in\partial A_2} \overline{P}_x[H_{A_1} < \overline{t_*}] \le \sup_{x\in\partial A_2} \overline{P}_x[H_{A_1} < \overline{t_*} - t_\#] + N^{-c'} \\ &\le \cdots \le \sup_{x\in\partial A_2} \overline{P}_x\left[H_{A_1} < \overline{t_*} - \left\lceil \frac{\overline{t_*}}{t_\#}\right\rceil t_\#\right] + \left\lceil \frac{\overline{t_*}}{t_\#}\right\rceil N^{-c'} \\ &\overset{(3.95)}{\le} 0 + c\log^3 N \cdot N^{-c'} \le N^{-c''}. \end{aligned}$$

Now we prove (3.131). We pick $\widetilde{t}$ in $[0, \overline{t_*}]$. One has

$$(3.133)\quad \begin{aligned} &\sup_{x\in\partial A_2} \overline{P}_x[H_{A_1} < \widetilde{t}] \\ &\le \sup_{x\in\partial A_2} \overline{P}_x[H_{A_1} < T_{A_6}] + \sup_{x\in\partial A_2} \overline{P}_x[T_{A_6} < H_{A_1} < \widetilde{t}]. \end{aligned}$$

On the one hand, by Proposition 1.5.10, page 36 of [38], one has

$$(3.134)\quad \sup_{x\in\partial A_2} \overline{P}_x[H_{A_1} < T_{A_6}] \overset{(3.122)}{=} \sup_{x\in\partial A_2} P_x[H_{A_1} < T_{A_6}] \le N^{-c}.$$

Now we seek an upper bound for the second term in the right member of (3.133). We write

$$\begin{aligned} &\sup_{x\in\partial A_2} \overline{P}_x[T_{A_6} < H_{A_1} < \widetilde{t}] \\ \le\ &\sup_{x\in\partial A_2} \overline{P}_x[t_\# < T_{A_6} < H_{A_1} < \widetilde{t}] + \sup_{x\in\partial A_2} \overline{P}_x[T_{A_6} \le t_\#] \\ =\ &\mathrm{I} + \mathrm{II}. \end{aligned}$$

To bound I, we can assume that $t_\# < \widetilde{t}$ (otherwise $\mathrm{I} = 0$). Applying

Markov property successively (first at time $t_\#$, then at time T_{A_6}, and finally at time H_{A_2}), we find

$$\begin{aligned} \text{I} &\le \sup_{y \in U^N} \overline{P}_y[T_{A_6} < H_{A_1} < \widetilde{t} - t_\#] \\ &\le \sup_{x \in \partial A_6} \overline{P}_x[H_{A_1} < \widetilde{t} - t_\#] \qquad (3.135) \\ &\le \sup_{x \in \partial A_2} \overline{P}_x[H_{A_1} < \widetilde{t} - t_\#]. \end{aligned}$$

Hence to prove (3.131), it suffices to prove that

$$\text{II} \le N^{-c}. \qquad (3.136)$$

Recalling that $d_\infty(\partial A_2, \partial A_6) \ge cN$, we find that

$$\text{II} \overset{(3.122)}{=} \sup_{x \in \partial A_2} P_x[T_{A_6} < t_\#] \le dP[T_{[-mN,mN]} \le t_0], \qquad (3.137)$$

where P is the probability law of a one-dimensional random walk started from 0 (and we denote by E the corresponding expectation), $t_0 = t_\#/d$, and $m = \delta/200$. We know that

$$P[T_{[-mN,mN]} \le t_0] = P\Big[\max_{0 \le t \le t_0} |X_t| \ge mN\Big]. \qquad (3.138)$$

By Doob's inequality, we have for $\lambda > 0$, using symmetry

$$\begin{aligned} &P\Big[\max_{0 \le t \le t_0} |X_t| \ge mN\Big] \\ =\;& 2P\Big[\max_{0 \le t \le t_0} \exp(\lambda X_t) \ge \exp(\lambda mN)\Big] \qquad (3.139) \\ \le\;& \frac{2E[\exp(\lambda X_{t_0})]}{\exp(\lambda mN)}. \end{aligned}$$

Note that $\exp\{\lambda X_t - t(\cosh\lambda - 1)\}$, $t \ge 0$, is a martingale under P, so

$$E\big[\exp(\lambda X_{t_0})\big] = \exp\big\{t_0(\cosh\lambda - 1)\big\}. \qquad (3.140)$$

Hence by taking $\lambda = \frac{mN}{2t_0} = cN^{-1}\log^{-1} N$, we obtain that the right-hand term of (3.139) is bounded from above by

(3.141)
$$2\exp\left\{t_0(\cosh\lambda - 1) - \frac{m^2N^2}{2t_0}\right\} \leq 2\exp\left(-c\frac{m^2N^2}{2t_0}\right) \leq N^{-c'}.$$

This implies that

(3.142)
$$P[T_{[-mN,mN]} \leq t_0] \leq N^{-c}.$$

Thus, one obtains (3.136) by collecting (3.137) and (3.142). This completes the proof of (3.131), and hence of (3.128).

The claim (3.129) follows by a similar argument.

Now we turn to (3.130). All, except the rightmost inequality of (3.130), are immediate. For the rightmost inequality, we first notice that by an estimate similar to the discussion below (3.25) of [64] we have,

(3.143)
$$P_x[T_{A_2} < \widetilde{H}_{A_1} < \infty] \leq N^{-c} e_{A_1}(x) \qquad \text{for all } x \in \partial_i A_1.$$

And hence we get that for all $x \in \partial_i A_1$,

(3.144)
$$\begin{aligned} & P_x[T_{A_2} < \widetilde{H}_{A_1}] \\ = \; & P_x[\widetilde{H}_{A_1} = \infty] + P_y[T_{A_2} < \widetilde{H}_{A_1} < \infty] \\ \leq \; & \big(1 + N^{-c}\big) e_{A_1}(x). \end{aligned}$$

This completes the proof of (3.130), and hence of Lemma 3.4.3. □

Now we make a further calculation of the tilted Dirichlet form of g defined in (3.125).

Proposition 3.4.4. *For large N, one has*

(3.145)
$$\frac{\mathrm{cap}(A_1)}{T_N} u_{**}(1+\varepsilon) \leq \overline{\mathcal{E}}(g,g) \leq \big(1+N^{-c}\big)\frac{\mathrm{cap}(A_1)}{T_N} u_{**}(1+\varepsilon).$$

Proof. Combining the fact that $\pi = f^2$ [from claim 1 of (3.70)], and the observation that g is discrete harmonic in $A_2 \setminus A_1$, $g = 1$ on A_1

and $g = 0$ outside A_2, one has [recall that Z_1 is the first step of the discrete chain attached to X_t, $t \geq 0$, see (3.14)]

$$(3.146)\quad \begin{aligned} &\overline{\mathcal{E}}(g,g) = (g, -\widetilde{L}g)_{l^2(\pi)} \\ = \;& \frac{u_{**}(1+\varepsilon)}{T_N} \sum_{y \in \partial_i A_1} g(y)\Big(g(y) - \sum_{x \sim y} \frac{1}{2d} g(x)\Big) \\ \overset{(3.125)}{=} \;& \frac{u_{**}(1+\varepsilon)}{T_N} \sum_{y \in \partial_i A_1} \Big(1 - \sum_{x \sim y} P_y[Z_1 = x] P_x[H_{A_1} < T_{A_2}]\Big) \\ \overset{\text{Markov}}{=} \;& \frac{u_{**}(1+\varepsilon)}{T_N} \sum_{y \in \partial_i A_1} P_y[T_{A_2} < \widetilde{H}_{A_1}]. \end{aligned}$$

On the one hand, by the rightmost inequality in (3.130), one has

$$(3.147)\quad \begin{aligned} &\sum_{y \in \partial_i A_1} P_y[T_{A_2} < \widetilde{H}_{A_1}] \\ &\leq \big(1 + N^{-c}\big) \sum_{y \in \partial_i A_1} e_{A_1}(y) = \big(1 + N^{-c}\big) \mathrm{cap}(A_1). \end{aligned}$$

On the other hand, one also knows that

$$(3.148)\quad \mathrm{cap}(A_1) = \sum_{y \in \partial_i A_1} e_{A_1}(y) \overset{(3.130)}{\leq} \sum_{y \in \partial_i A_1} P_y[T_{A_2} < \widetilde{H}_{A_1}].$$

Thus, the claim (3.145) follows by collecting (3.146), (3.147) and (3.148). □

Next, we prove the first half of the main estimate of this section, namely the upper bound on $1/\overline{E}_\pi[H_{A_1}]$. Let us mention that this upper bound will actually be needed in the proof of Lemma 3.4.6.

Proposition 3.4.5. *For large N, one has*

$$(3.149)\quad \frac{1}{\overline{E}_\pi[H_{A_1}]} \leq \big(1 + N^{-c}\big) \frac{\mathrm{cap}(A_1)}{T_N} u_{**}(1+\varepsilon).$$

As a consequence, one has

$$\overline{E}_\pi[H_{A_1}] \geq cN^{2+c'}. \tag{3.150}$$

Proof. We first prove (3.149). We apply the right-hand inequality in (3.145) to the right-hand estimate in (3.127). Note that

$$\pi(D) = 1 - \pi(A_2) \geq 1 - cN^{(r_2-1)d} \qquad \text{for large } N, \tag{3.151}$$

for large N, with the help of (3.145) we thus find that

$$\begin{aligned} \frac{1}{\overline{E}_\pi[H_{A_1}]} &\overset{(3.127)}{\leq} \frac{\overline{\mathcal{E}}(g,g)}{\pi(D)^2} \\ &\overset{(3.151)}{\leq} \big(1 - cN^{(r_2-1)d}\big)^{-2}\overline{\mathcal{E}}(g,g) \\ &\overset{(3.145)}{\leq} \big(1 + N^{-c}\big)\frac{\mathrm{cap}(A_1)}{T_N}u_{**}(1+\varepsilon). \end{aligned} \tag{3.152}$$

This yields (3.149). Then the claim (3.150) follows by observing (3.22) and claim 5 of (3.52). □

In the following Lemma 3.4.6 and Proposition 3.4.7, we build a corresponding lower bound by controlling the fluctuation function f_{A_1} defined in (3.126).

Lemma 3.4.6. *For large N, one has*

$$f_{A_1}(x) \geq -N^{-c} \qquad \textit{for all } x \in U^N. \tag{3.153}$$

and in the notation of (3.124)

$$\begin{aligned} \overline{E}_x[H_{A_1}] \;\;&\geq\;\; \overline{E}_\pi[H_{A_1}] - e^{-c'\log^2 N} \\ &-\;\; \overline{P}_x[H_{A_1} \leq \overline{t_*}]\big(\overline{t_*} + \overline{E}_\pi[H_{A_1}]\big) \;\forall x \in D. \end{aligned} \tag{3.154}$$

Proof. As we now explain, to prove (3.153), it suffices to show that

$$\big|\overline{E}_x\big[\overline{E}_{X_{\overline{t_*}}}[H_{A_1}]\big] - \overline{E}_\pi[H_{A_1}]\big| \leq e^{-c'\log^2 N} \qquad \text{for all } x \in U^N. \tag{3.155}$$

Indeed, since $H_{A_1} \leq \overline{t_*} + H_{A_1} \circ \theta_{\overline{t_*}}$, the simple Markov property applied at time $\overline{t_*}$ and (3.155) imply that

$$\sup_{x \in U^N} \overline{E}_x[H_{A_1}] \leq \overline{t_*} + e^{-c\log^2 N} + \overline{E}_\pi[H_{A_1}]. \tag{3.156}$$

It then follows that

$$\begin{aligned} &\frac{\sup_{x\in U^N} \overline{E}_x[H_{A_1}]}{\overline{E}_\pi[H_{A_1}]} - 1 \\ &\underset{(3.149)}{\overset{(3.156)}{\leq}} \big(\overline{t_*} + e^{-c\log^2 N}\big) c' \frac{\mathrm{cap}(A_1)}{T_N} \\ &\underset{(3.48)}{\overset{(3.22)}{\leq}} \big(\overline{t_*} + e^{-c\log^2 N}\big) c'' N^{(d-2)r_1 - d} \underset{(3.119)}{\overset{(3.95)}{\leq}} N^{-\widetilde{c}}. \end{aligned} \tag{3.157}$$

This proves (3.153). We now prove (3.155). Let us consider the expectation of H_{A_1} when started from $X_{\overline{t_*}}$. We first note that for all $x \in U^N$,

$$\begin{aligned} &\big|\overline{E}_x\big[\overline{E}_{X_{\overline{t_*}}}[H_{A_1}]\big] - \overline{E}_\pi[H_{A_1}]\big| \\ &\quad \leq \sum_{y\in U^N} \big|\overline{P}_x[X_{\overline{t_*}} = y] - \pi(y)\big| \sup_{y\in U^N} \overline{E}_y[H_{A_1}]. \end{aligned} \tag{3.158}$$

By the relaxation to equilibrium estimate (3.96), one has

$$\sum_{y\in U^N} \big|\overline{P}_x[X_{\overline{t_*}} = y] - \pi(y)\big| \leq e^{-c\log^2 N} \qquad \text{for all } x \in U^N. \tag{3.159}$$

Thus, to prove (3.155) it suffices to obtain a very crude upper bound for the supremum of the expected entrance time in A_1 as the starting point varies in U^N:

$$\overline{E}_y[H_{A_1}] \leq cN^{5+d} \qquad \text{for all } y \in U^N. \tag{3.160}$$

This follows, for example, by a corollary of the commute time identity (see Corollary 4.28, page 59 of [5]):

$$\overline{E}_y[H_{A_1}] \leq r_{\mathrm{eff}}(y, A_1)\pi\big(U^N\big) \qquad \text{for all } y \in U^N, \tag{3.161}$$

where $r_{\text{eff}}(y, A_1)$ stands for the effective resistance between y and A_1. On the one hand, by the third equality of (3.91) and claim 4 of (3.70), for all $x, y \in U^N$ such that $x \sim y$, we know that

$$W(x,y) = \frac{1}{2d}\sqrt{\pi(x)\pi(y)} \in \big(cN^{-(4+d)}, 1\big], \tag{3.162}$$

hence the resistance on $\{p, q\}$ does not exceed cN^{4+d}. We know that for any y in U^N, for some $x \in \partial_i A_1$, the effective resistance between y and x [which we denote by $r_{\text{eff}}(y,x)$] is less or equal to the effective resistance between y and x on the path $\gamma(y,x)$ [which we denote by $r^{\gamma}_{\text{eff}}(y,x)$] defined above Proposition 3.3.12 [note that $\gamma(y,x)$ is a subgraph of U^N]. Since by (3.92) $\gamma(y,x)$ is of length no more than cN, $r^{\gamma}_{\text{eff}}(y,x)$ does not exceed $c'N^{5+d}$ by (3.162). Hence, we obtain that

$$r_{\text{eff}}(y, A_1) \leq r_{\text{eff}}(y,x) \leq r^{\gamma}_{\text{eff}}(y,x) \leq cN^{5+d}. \tag{3.163}$$

On the other hand, one has $\pi(U^N) = 1$ [by claim 1 of (3.70)]. Thus, (3.161) and (3.163) yield that

$$\sup_{y \in U^N} \overline{E}_y[H_{A_1}] \leq cN^{5+d}. \tag{3.164}$$

This completes the proof of (3.160), and hence of (3.153).

We now turn to (3.154). We consider any $x \in D$. By the simple Markov property applied at time $\overline{t_*}$, we find that

$$\begin{aligned}
& \overline{E}_x[H_{A_1}] \\
\geq\; & \overline{E}_x\big[\mathbf{1}_{\{H_{A_1} > \overline{t_*}\}} \overline{E}_{X_{\overline{t_*}}}[H_{A_1}]\big] \\
=\; & \overline{E}_x\big[\overline{E}_{X_{\overline{t_*}}}[H_{A_1}]\big] - E_x\big[\mathbf{1}_{\{H_{A_1} \leq \overline{t_*}\}} \overline{E}_{X_{\overline{t_*}}}[H_{A_1}]\big] \\
\overset{(3.155)}{\geq}\; & \overline{E}_\pi[H_{A_1}] - e^{-c\log^2 N} \\
& -\overline{P}_x[H_{A_1} \leq \overline{t_*}] \sup_{y \in U^N} \overline{E}_y[H_{A_1}] \\
\overset{(3.156)}{\geq}\; & \overline{E}_\pi[H_{A_1}] - e^{-c'\log^2 N} \\
& -\overline{P}_x[H_{A_1} \leq \overline{t_*}]\big(\overline{t_*} + \overline{E}_\pi[H_{A_1}]\big).
\end{aligned} \tag{3.165}$$

This proves (3.154) and finishes Lemma 3.4.6. □

We now prove the second main estimate.

Proposition 3.4.7. *For large N, one has that*

(3.166) $$\frac{1}{\overline{E}_\pi[H_{A_1}]} \geq \big(1 - N^{-c}\big)\frac{\mathrm{cap}(A_1)}{T_N} u_{**}(1+\varepsilon).$$

Proof. By applying (3.145) and the left-hand inequality of (3.127), for large N, one has,

(3.167) $$\frac{1}{\overline{E}_\pi[H_{A_1}]} \geq \Big(1 - 2\sup_{x\in D}\big|f_{A_1}(x)\big|\Big)\frac{\mathrm{cap}(A_1)}{T_N} u_{**}(1+\varepsilon).$$

Thus, with (3.153) in mind, to prove (3.166), it suffices to show that for large N,

(3.168) $$\sup_{x\in D} f_{A_1}(x) \leq N^{-c}.$$

Dividing by $\overline{E}_\pi[H_{A_1}]$ on both sides of (3.154) and taking the infimum over all $x \in D$, one obtains

(3.169) $$\begin{aligned} &\inf_{x\in D}\frac{\overline{E}_x[H_{A_1}]}{\overline{E}_\pi[H_{A_1}]} \\ &\overset{(3.95)}{\geq} 1 - \frac{e^{-c'\log^2 N}}{\overline{E}_\pi[H_{A_1}]} - \sup_{x\in D}\overline{P}_x[H_{A_1} \leq \overline{t_*}]\left(\frac{N^2\log^2 N}{\overline{E}_\pi[H_{A_1}]} + 1\right) \\ &\underset{(3.128)}{\overset{(3.150)}{\geq}} 1 - e^{-\overline{c}'\log^2 N} - N^{-\widetilde{c}'}\big(c''(\log N)^2 N^{-\widetilde{c}} + 1\big) \\ &\geq\ 1 - N^{-c}. \end{aligned}$$

Together with (3.167), this proves (3.168) as well as (3.166). □

Remark 3.4.8. The combination of Propositions 3.4.5 and 3.4.7

forms a pair of asymptotically tight bounds on $\overline{E}_\pi[H_{A_1}]$, namely

$$
\begin{aligned}
&\big(1 - N^{-c}\big)\frac{\mathrm{cap}(A_1)}{T_N}u_{**}(1+\varepsilon) \\
\le\ & 1/\overline{E}_\pi[H_{A_1}] \\
\le\ & \big(1 + N^{-c}\big)\frac{\mathrm{cap}(A_1)}{T_N}u_{**}(1+\varepsilon).
\end{aligned}
\tag{3.170}
$$

3.5 Quasi-stationary measure

In this section, we introduce the quasi-stationary distribution (abbreviated below as q.s.d.) induced on D [recall that D is defined in (3.124)] and collect some of its properties. This will help us show in the next section that carefully chopped sections of the confined random walk are approximately independent, allowing us to bring into play excursions of random walk and furthermore random interlacements. In Proposition 3.5.5, we prove that the q.s.d. on D is an appropriate approximation of the stationary distribution of the random walk conditioned to stay in D. In Proposition 3.5.7, we show that the hitting distribution of A_1 of the confined walk starting from the q.s.d. on D is very close to the normalized equilibrium measure of A_1. In this section, the constants tacitly depend on δ, η, ε and R [see (3.35) and (3.36)], r_1, r_2, r_3, r_4 and r_5 [see (3.119)].

We fix the choice of A_1 and A_2 as in the last section [see (3.120)]. The arguments in Lemma 3.5.2, Propositions 3.5.3, 3.5.5 and 3.5.7 below are similar to those of Section 3.2 and the Appendix of [64]. However, in our set-up, special care is needed due to the fact that the stationary measure is massively non-uniform in the present context.

We now define the q.s.d. on $D(= U^N \setminus A_2)$. We denote by $\{H_t^D\}_{t\ge 0}$ the semi-group of $\{\overline{P}_x\}_{x\in U^N}$ killed outside D, so that for all $f \in U^N \to \mathbb{R}$

$$
H_t^D f(x) = \overline{E}_x\big[f(X_t), H_{A_2} > t\big]. \tag{3.171}
$$

We denote by L^D the generator of $\{H_t^D\}_{t\ge 0}$. It is classical fact that for $f : D \to \mathbb{R}$,

$$L^D f(x) = \widetilde{L}\widetilde{f}(x) \qquad \forall x \in D, \tag{3.172}$$

where $\widetilde{f}$ is the extension of f to U^N vanishing outside D and $\widetilde{L}$ [defined in (3.63)] is the generator for the tilted walk. We denote by π^D the restriction of the measure π onto D. So, $\{H_t^D\}_{t\geq 0}$ and L^D are self-adjoint in $l^2(\pi^D)$ and

$$H_t^D = e^{tL^D}. \tag{3.173}$$

We then denote by $\lambda_i^D, i = 1, \dots, |D|$, with

$$0 \leq \lambda_i^D \leq \lambda_{i+1}^D, \qquad i = 1, \dots, |D| - 1, \tag{3.174}$$

the eigenvalues of $-L^D$ and by f_i, $i = 1, \dots, |D|$, an $l^2(\pi^D)$-orthonormal basis of eigenfunctions associated to λ_i. Because D is connected, by the Perron–Frobenius theorem, all entries of f_1 are positive. The quasi-stationary distribution on D is the probability measure on D with density with respect to π^d proportional to f_1, that is,

$$\sigma(y) = \frac{(f_1, \delta_y)_{l^2(\pi^D)}}{(f_1, \mathbf{1})_{l^2(\pi^D)}}, \qquad x \in D, \tag{3.175}$$

where, for $y \in D$, $\delta_y : D \to \mathbb{R}$ is the point mass function at y. It is known that the q.s.d. on D is the limit distribution of the walk conditioned on never entering A_2, that is, for all $x, y \in D$, one has (see (6.6.3), page 91 of [37]),

$$\sigma(y) = \lim_{t\to\infty} \overline{P}_x[X_t = y | H_{A_2} > t]. \tag{3.176}$$

We now prove a lemma which is useful in the proof of Proposition 3.5.3 below.

Lemma 3.5.1. *For all $x, y \in D$, one has that*

$$\sigma(y) \geq N^{-c}\overline{P}_y[H_x < H_{A_2}]\sigma(x). \tag{3.177}$$

Proof. By the $l^2(\pi^D)$-self-adjointness of the killed semi-group $(H_t^D)_{t\geq 0}$,

we have that for all $x, y \in D$, $t > 0$,
(3.178)
$$\overline{P}_x[X_t = y | H_{A_2} > t] = \overline{P}_y[X_t = x | H_{A_2} > t] \frac{\pi(y)}{\pi(x)} \frac{\overline{P}_y[H_{A_2} > t]}{\overline{P}_x[H_{A_2} > t]}.$$

On the one hand, by the strong Markov property applied at time H_x, we know that for all $x, y \in D$,

$$\overline{P}_y[H_{A_2} > t] \geq \overline{P}_y[H_x < H_{A_2}] \overline{P}_x[H_{A_2} > t], \tag{3.179}$$

On the other hand, by claim 4 of (3.70), we know that for all $x, y \in D$, $t > 0$,

$$\frac{\pi(y)}{\pi(x)} \geq cN^{-4}. \tag{3.180}$$

Thus, the claim (3.177) follows by taking limits in t on both sides of (3.178) and incorporating (3.180) and (3.179). □

The next lemma is also a preparation for Proposition 3.5.3.

Lemma 3.5.2. *For all $x \in D \setminus A_4$, one has*

$$\max_{y \in \partial A_3} \overline{P}_y[H_x < H_{A_2}] \geq N^{-c}. \tag{3.181}$$

Proof. We fix an $x \in D \setminus A_4$ in the proof. Applying the Markov property at time T_{A_3} under $\overline{P}_{y'}$ for $y' \in \partial A_2$, we see that

$$\begin{aligned} \max_{y' \in \partial A_2} \overline{P}_{y'}[H_x < H_{A_2}] &= \max_{y' \in \partial A_2} \overline{P}_{y'}[T_{A_3} < H_x < H_{A_2}] \\ &\leq \max_{y \in \partial A_3} \overline{P}_y[H_x < H_{A_2}]. \end{aligned} \tag{3.182}$$

We now develop a lower bound on the left-hand side of (3.182) via effective resistance estimates. We denote by U^{col} the graph obtained by collapsing A_2 into a single vertex a in U^N. With some abuse of notation, we use the same symbol for the vertices in U^{col} as in U^N except for a. We denote by $W^{\text{col}} : U^{\text{col}} \times U^{\text{col}} \to \mathbb{R}^+$ the induced edge-weight. Let

$$w_a = \sum_{y \in \partial A_2} W^{\mathrm{col}}(a, y) = \sum_{z \in A_2, y \in \partial A_2, z \sim y} W(z, y) \tag{3.183}$$

be the sum of the weights of edges that touch a in U^{col}. We denote by $\{P_z^{\mathrm{col}}\}_{z \in U^{\mathrm{col}}}$ the discrete-time reversible Markov chain with edge-weight W^{col}. The reversible measure of this Markov chain π^{col} is given through

$$\pi^{\mathrm{col}}(z) = \begin{cases} w_a, & z = a, \\ \sum_{y \sim z} W(z, y), & \text{otherwise.} \end{cases} \tag{3.184}$$

Then we have
(3.185)

$$\max_{y' \in \partial A_2} \overline{P}_{y'}[H_x < H_{A_2}] = \max_{y' \in \partial A_2} P_{y'}^{\mathrm{col}}[H_x < H_a] \geq P_a^{\mathrm{col}}[H_x < \widetilde{H}_a].$$

By a classical result on electrical networks (see Proposition 3.10, page 69 of [3]), the escape probability in the right-hand side of (3.185) equals

$$P_a^{\mathrm{col}}[H_x < \widetilde{H}_a] = \big(w_a r^{\mathrm{col}}(a, x)\big)^{-1}, \tag{3.186}$$

where $r^{\mathrm{col}}(a, x)$ is the effective resistance between a and x on U^{col}. We know that $r^{\mathrm{col}}(a, x)$ is smaller or equal to the effective resistance between a and x along a path between a and x of length no more than cN and along this path the edge-weight is no less than N^{-c} by (3.162). Hence, we obtain that

$$r^{\mathrm{col}}(a, x) \leq N^c. \tag{3.187}$$

Moreover, we know that

$$w_a = \sum_{z \in A_2, y \in \partial A_2, z \sim y} W(z, y) \leq N^c \max_{z \in D} W(z, y) \overset{(3.162)}{\leq} N^c. \tag{3.188}$$

Therefore, we conclude from (3.186), (3.187) and (3.188) that

$$P_a^{\mathrm{col}}[H_x < \widetilde{H}_a] \geq N^{-c'}. \tag{3.189}$$

The claim (3.181) follows by collecting (3.182), (3.185) and (3.189). □

The next proposition is a crucial estimate for us, showing that σ is not too small at any point in D. This fact will be used in Proposition 3.5.5. In the proof, we mainly rely on the reversibility of the confined walk, hitting probability estimates of simple random walk, and the Harnack principle.

Proposition 3.5.3. *For large N, one has the following lower bound:*

$$\inf_{x \in D} \sigma(x) \geq N^{-c}, \tag{3.190}$$

and for all $x \in D$,

$$N^{c'} \geq f_1(x) \geq N^{-c''}. \tag{3.191}$$

Proof. We first prove (3.190). The claim (3.191) will then follow. Because σ is a probability measure, and

$$|D| \leq cN^d, \tag{3.192}$$

there must exist some x' in D such that

$$\sigma\big(x'\big) \geq cN^{-d}. \tag{3.193}$$

By (3.177), to prove (3.190) it suffices to prove that for all $x \in D$,

$$\overline{P}_x[H_{x'} < H_{A_2}] \geq N^{-c'}. \tag{3.194}$$

We now prove (3.194) by treating two cases according to the location of x'.

Case 1: $x' \in A_4 \setminus A_2$ [recall the definition of A_4 in (3.120)]. By (3.122) and a standard hitting estimate (see Proposition 1.5.10, page 36 of [38]) for simple random walk, for all x in D, we have that [recall the definition of A_5 in (3.120)],

$$\overline{P}_x[H_{\partial A_5} < H_{A_2}] \geq N^{-c}, \tag{3.195}$$

(note that the left-hand side equals 1 if $x \notin A_5$). We write

$$l(x) = \overline{P}_x[H_{x'} < H_{A_2}]. \tag{3.196}$$

By the strong Markov property applied at $H_{\partial A_5}$,
(3.197)

$$l(x) \overset{\text{Markov}}{\geq} \overline{P}_x[H_{\partial A_5} < H_{A_2}] \min_{y \in \partial A_5} l(y) \overset{(3.195)}{\geq} N^{-c} \min_{y \in \partial A_5} l(y).$$

We now develop a lower bound on the right-hand side of (3.197). Let $S_1 = B_\infty(x_0, 3N^{r_5}) \setminus B_\infty(x_{0,} \frac{1}{3}N^{r_5})$ and $S_2 = B_\infty(x_0, 2N^{r_5}) \setminus B_\infty(x_{0,} \frac{1}{2}N^{r_5})$, (we tacitly assume that N is sufficiently large that $S_1 \subset A_6$, and $S_2 \subset D$). It is straight-forward to see that $l(x)$ is $\widetilde{L}$-harmonic in $D \backslash \{x'\}$ and that $\widetilde{L}$ coincides with Δ_{dis} in S_1 [see (3.122)]. By the Harnack inequality (see Theorem 6.3.9, page 131 of [39]), we know that (note that $\partial A_5 \subset S_2$)

$$\min_{y \in \partial A_5} l(y) \geq c' \max_{y \in \partial A_5} l(y). \tag{3.198}$$

This implies by (3.197) that

$$\min_{x \in D} l(x) \geq c' N^{-c} \max_{y \in \partial A_5} l(y). \tag{3.199}$$

We now take any point $y' \in \partial A_5$ of least distance (in the sense of l^∞-norm) to x' on ∂A_5 and sharing $(d-1)$ common coordinates with x' and fix y'. We set $B = B_\infty(y', |y'-x'|_\infty - 1)$. Our way of choosing y' ensures that $x' \in \partial B$. Then by (3.122), we have
(3.200)

$$l\big(y'\big) = \overline{P}_{y'}[H_{x'} < H_{A_2}] \geq \overline{P}_{y'}\big[X_{T_B} = x'\big] \overset{(3.122)}{=} P_{y'}\big[X_{T_B} = x'\big].$$

By a classical estimate (see Lemma 6.3.7, pages 158–159 of [39]), we have

(3.201) $$P_{y'}\big[X_{T_B} = x'\big] \geq cN^{(1-d)r_5}.$$

Thus, the claim (3.194) follows by collecting (3.199), (3.200) and (3.201).

Case 2: $x' \in D \setminus A_4$. Since $\partial A_3 \subset A_4 \setminus A_2$, if we can prove that for some $y \in \partial A_3$,

(3.202) $$\sigma(y) \geq N^{-c},$$

then we are brought back to case 1 by taking the y in (3.202) as the x' in (3.193). Now we show that we can indeed find such y that (3.202) holds. By (3.177) and our assumption that $\sigma(x') \geq N^{-c}$, we have

(3.203)
$$\sigma(y) \overset{(3.177)}{\geq} N^{-c}\overline{P}_y[H_{x'} < H_{A_2}]\sigma\big(x'\big) \overset{(3.193)}{\geq} N^{-c'}\overline{P}_y[H_{x'} < H_{A_2}].$$

Hence, we know that by (3.181), if we pick the y that maximizes the probability in the left-hand side of (3.181), the claim (3.202) is indeed true.

With these two cases, we complete the proof of (3.190).

Now we prove (3.191). By the fact that f_1 is a unit vector in $l^2(\pi^D)$ we know that

(3.204) $$(f_1, f_1)_{l^2(\pi^D)} = 1.$$

To prove the first inequality of (3.191), we observe that, thanks to (3.204):

(3.205)
$$1 = (f_1, f_1)_{l^2(\pi^D)} \geq \max_{x\in D} f_1^2(x) \min_{x\in D} \pi^D(x) \overset{(3.70)4.}{\geq} N^{-c} \max_{x\in D} f_1^2(x).$$

To prove the second inequality of (3.191), we observe that by (3.204)

(3.206) $$\max_{x\in D} \pi^D(x) f_1^2(x) \geq \frac{1}{|D|},$$

which implies that

$$\max_{x\in D} f_1(x) \geq \sqrt{\frac{1}{|D|\max_{x\in D}\pi^D(x)}} \underset{(3.192)}{\overset{(3.70)4.}{\geq}} N^{-c}. \tag{3.207}$$

This implies that for all $x \in D$,

$$\begin{aligned} f_1(x) &\overset{(3.175)}{=} \frac{1}{\pi^D(x)}\sigma(x)(f_1,\mathbf{1})_{l^2(\pi^D)} \\ &\underset{(3.190)}{\overset{(3.70)4.}{\geq}} N^{-c}\max_{x\in D} f_1(x)\min_{x\in D}\pi^D(x) \underset{(3.207)}{\overset{(3.70)4.}{\geq}} N^{-c'}. \end{aligned} \tag{3.208}$$

This completes the proof of (3.191), and concludes the proof of Proposition 3.5.3. □

In the following proposition, we show that the spectral gap of L^D is at least of order N^{-2}.

Lemma 3.5.4. *One has that for large N*

$$\lambda_2^D - \lambda_1^D \geq cN^{-2}. \tag{3.209}$$

Proof. Recall that $\overline{\lambda}_2$ stand for the second smallest eigenvalue of $-\widetilde{L}$. By the eigenvalue interlacing inequality (see Theorem 2.1 of [34]), we have

$$\lambda_2^D \geq \overline{\lambda}_2. \tag{3.210}$$

While by the paragraph below equation (12) of [2], we have

$$\lambda_1^D = \frac{1}{\overline{E}_\sigma[H_{A_2}]}. \tag{3.211}$$

By Lemma 10(a) of [2], we have

$$\overline{E}_\sigma[H_{A_2}] \geq \overline{E}_\pi[H_{A_2}] \text{ or equivalently } \frac{1}{\overline{E}_\sigma[H_{A_2}]} \leq \frac{1}{\overline{E}_\pi[H_{A_2}]}. \tag{3.212}$$

By an argument similar to the proof of Proposition 3.4.5 (by replacing A_1, A_2 by A_2, A_3), we find that

$$\frac{1}{\overline{E}_\pi[H_{A_2}]} \leq cN^{-d+(d-2)r_2}. \tag{3.213}$$

This implies by (3.211) and (3.212) that

$$\lambda_1^D \leq cN^{-d+(d-2)r_2}. \tag{3.214}$$

Hence, we obtain that for large N

$$\lambda_2^D - \lambda_1^D \underset{(3.210)}{\overset{(3.214)}{\geq}} \overline{\lambda}_2 - cN^{-d+(d-2)r_2} \overset{(3.88)}{\geq} c'N^{-2}. \tag{3.215}$$

This finishes the proof of (3.209). □

The next proposition shows, with the help of the spectral gap estimate obtained in Lemma 3.5.4, that the q.s.d. on D is very close to the distribution of the confined walk at time $\overline{t_*}$ conditioned on not hitting A_2 [see (3.95) for the definition of $\overline{t_*}$].

Proposition 3.5.5. *One has that for large N,*

$$\sup_{x,y\in D} \left|\overline{P}_x[X_{\overline{t_*}} = y \middle| H_{A_2} > \overline{t_*}] - \sigma(y)\right| \leq e^{-c\log^2 N}. \tag{3.216}$$

Proof. The conditional probability in (3.216) is expressed through $H^D_{\overline{t_*}}$ as

$$P_x[X_{\overline{t_*}} = y | H_{A_2} > \overline{t_*}] = H^D_{\overline{t_*}}\delta_y(x) \Big/ (H^D_{\overline{t_*}}\mathbf{1})(x). \tag{3.217}$$

Now we calculate the numerator in the right-hand side of (3.217). We decompose δ_y in the $l^2(\pi^D)$ base $\{f_i\}_{i=1,\dots,|D|}$:

$$\begin{aligned} &\delta_y = \sum_{i=1}^{|D|} a_i f_i, \text{ where} \\ &a_i = (\delta_y, f_i)_{l^2(\pi^D)} = f_i(y)\pi^D(y), \text{ for } 1 \leq i \leq |D|. \end{aligned} \tag{3.218}$$

Hence, one can decompose $H^D_{\overline{t_*}}\delta_y(x)$ into a linear combination of $a_i f_i(x)$:

$$H^D_{\overline{t_*}}\delta_y(x) = e^{-\lambda_1^D \overline{t_*}}\left(a_1 f_1(x) + \sum_{i=2}^{|D|} e^{(\lambda_1^D - \lambda_i^D)\overline{t_*}} a_i f_i(x)\right). \tag{3.219}$$

Now we show that the first term inside the brackets on the right-hand side of (3.219) is significantly larger than the other terms. By Proposition 3.5.3, one has

$$a_1 f_1(x) \overset{(3.218)}{=} \pi^D(y) f_1(y) f_1(x) \underset{(3.70)4.}{\overset{(3.191)}{\geq}} N^{-c}. \tag{3.220}$$

For large N, thanks to Lemma 3.5.4, the reminder term inside the brackets of (3.219) is bounded by

$$\begin{aligned}
\left|\sum_{i=2}^{|D|} e^{(\lambda_1^D - \lambda_i^D)\overline{t_*}} a_i f_i(x)\right| &\underset{(3.218)}{\overset{(3.174)}{\leq}} \sum_{i=2}^{|D|} e^{(\lambda_1^D - \lambda_2^D)\overline{t_*}} \left|\pi^D(y) f_i(y) f_i(x)\right| \\
&\underset{(3.95)}{\overset{(3.209)}{\leq}} |D| e^{-c\log^2 N}\left|\pi^D(y) f_i(y) f_i(x)\right| \\
&\underset{(3.70)4.}{\overset{(3.192)}{\leq}} e^{-c''\log^2 N}.
\end{aligned} \tag{3.221}$$

This implies that

$$\left|\frac{H^D_{\overline{t_*}}\delta_y(x)}{e^{-\lambda_1^D \overline{t_*}} a_1 f_1(x)} - 1\right| \overset{(3.219)\text{–}(3.221)}{\leq} e^{-c\log^2 N}. \tag{3.222}$$

We now turn to the denominator of the right-hand side of (3.217). By an argument which is very similar to that leading to (3.222), one can show that

$$\left|\frac{(H^D_{\overline{t_*}}\mathbf{1})(x)}{e^{-\lambda_1^D \overline{t_*}} f_1(x)(f_1, \mathbf{1})_{l^2(\pi^D)}} - 1\right| \leq e^{-c\log^2 N}. \tag{3.223}$$

Combining (3.222) and (3.223), one has that for large N and uniformly for all $x, y \in D$ [remind the definition of $\sigma(\cdot)$ in (3.175)]

$$\begin{aligned} \big|P_x[X_{\overline{t_*}} = y | H_{A_2} > \overline{t_*}] - \sigma(y)\big| &\overset{(3.217)}{=} \left| \frac{H^D_{\overline{t_*}}\delta_y(x)}{(H^D_{\overline{t_*}}\mathbf{1})(x)} - \sigma(y) \right| \\ &\underset{(3.223)}{\overset{(3.222)}{\leq}} e^{-c\log^2 N}\sigma(y) \leq e^{-c\log^2 N}, \end{aligned} \tag{3.224}$$

which is exactly the claim (3.216). □

We define the stopping time V as the first time when the confined random walk has stayed outside A_2 for a consecutive duration of $\overline{t_*}$:

$$V = \inf\{t \geq \overline{t_*} : X_{[t-\overline{t_*}, t]} \cap A_2 = \varnothing\}. \tag{3.225}$$

The next lemma is a preparatory result for Proposition 3.5.7 below. This lemma shows that the probability $\overline{P}_x[V < \widetilde{H}_{A_1}]$, when normalized by the sum of such probabilities as x varies in the inner boundary of A_1, is approximately equal to $\widetilde{e}_{A_1}(x)$, the normalized equilibrium measure of A_1.

Lemma 3.5.6. *For large N, one has that for all $x \in D$,*

$$\left| \frac{\overline{P}_x[V < \widetilde{H}_{A_1}]}{\sum_{y \in \partial_i A_1} \overline{P}_y[V < \widetilde{H}_{A_1}]\widetilde{e}_{A_1}(x)} - 1 \right| \leq N^{-c}. \tag{3.226}$$

Proof. For any $y \in \partial_i A_1$, by (3.129) and the strong Markov property applied at time T_{A_3}, we obtain that

$$\begin{aligned} &\overline{P}_y[V < \widetilde{H}_{A_1}] \\ &\underset{(3.225)}{\overset{\text{Markov}}{\geq}} \overline{P}_y[T_{A_3} < \widetilde{H}_{A_1}] \inf_{x \in U^N \setminus A_3} \overline{P}_x[H_{A_2} > \overline{t_*}] \\ &\underset{(3.122)}{\overset{(3.129)}{\geq}} P_y[T_{A_3} < \widetilde{H}_{A_1}]\big(1 - N^{-c}\big) \overset{(3.130)}{\geq} e_{A_1}(y)\big(1 - N^{-c}\big). \end{aligned} \tag{3.227}$$

On the other hand, $\overline{P}_y[V < \widetilde{H}_{A_1}]$ is bounded from above by [recall that $V > T_{A_2}$ by definition of V, see (3.225)]

$$\begin{aligned} \overline{P}_y[V < \widetilde{H}_{A_1}] &\leq \overline{P}_y[T_{A_2} < \widetilde{H}_{A_1}] \\ \overset{(3.122)}{=} P_y[T_{A_2} < \widetilde{H}_{A_1}] &\overset{(3.130)}{\leq} e_{A_1}(y)\big(1 + N^{-c}\big). \end{aligned} \tag{3.228}$$

Together with (3.227), we find that for any $y \in \partial_i A$

$$\big(1 - N^{-c}\big)e_{A_1}(y) \leq \overline{P}_y[V < \widetilde{H}_{A_1}] \leq \big(1 + N^{-c'}\big)e_{A_1}(y). \tag{3.229}$$

Summing over $y \in \partial_i A_1$ we obtain that

$$\begin{aligned} &\big(1 - N^{-c}\big) \sum_{y \in \partial_i A_1} \overline{P}_y[V < \widetilde{H}_{A_1}] \\ &\leq \operatorname{cap}(A_1) \leq \big(1 + N^{-c'}\big) \sum_{y \in \partial_i A_1} \overline{P}_y[V < \widetilde{H}_{A_1}]. \end{aligned} \tag{3.230}$$

The claim (3.226) follows by combining (3.229) and (3.230), recalling that by the definition of normalized equilibrium measure, $\widetilde{e}_{A_1}(x) = e_{A_1(x)}/\operatorname{cap}(A_1)$. □

The following proposition shows that the hitting distribution of the confined walk on A_1 started from the q.s.d. on D is very close to the normalized equilibrium measure of A_1. The proof of the next proposition is close to the proof of Lemma 3.10 of [64]. and can be found in the Appendix at the end of this Chapter.

Proposition 3.5.7. *For large N and any $x_0 \in \Gamma^N$ (recall that A_1 tacitly depends on x_0), one has*

$$\sup_{x \in \partial_i A_1} \left| l \frac{\overline{P}_\sigma[X_{H_{A_1}} = x]}{\widetilde{e}_{A_1}(x)} - 1 \right| \leq N^{-c}. \tag{3.231}$$

3.6 Chain coupling of excursions

In this section, we prove in Theorem 3.6.9 that the tilted random walk disconnects K_N from infinity with a probability, which tends to 1 as N tends to infinity. For this purpose, we show that the confined random walk visits the mesoscopic boxes A_1 centered at Γ^N[defined in (3.120)] sufficiently often so that at time T_N the trace of the walk "locally" dominates (via a chain of couplings) random interlacements with intensity higher than u_{**}. Hence, it disconnects in each such box the center from its boundary with very high probability. Some arguments in this section are based on Section 4 of [64], with necessary adaptations. In this section, the constants tacitly depend on δ, η, ε and R [see (3.35) and (3.36)], r_1, r_2, r_3, r_4 and r_5 [see (3.119)].

Throughout this section, we fix $x_0 \in \Gamma^N$, the center of the boxes A_1 through A_6, except in Proposition 3.6.8 and Theorem 3.6.9.

We recall the definition of V in (3.225). For a path in $\Gamma(U^N)$, we denote by R_k and V_k the successive entrance times H_{A_1} and stopping times V:

$$\begin{aligned} R_1 &= H_{A_1}; \qquad V_1 = R_1 + V \circ \theta_{R_1}; \quad \text{and for } i \geq 2, \\ R_i &= V_{i-1} + H_{A_1} \circ \theta_{V_{i-1}}; \qquad V_i = R_i + V \circ \theta_{R_i}. \end{aligned}$$

Colloquially, we call such sections $X_{[R_i,V_i)}$ "long excursions" in contrast to the "short excursions" we will later define [see above (3.247)]. We set

$$J = \big\lfloor (1+\varepsilon/2) u_{**} \mathrm{cap}(A_1) \big\rfloor. \tag{3.232}$$

The next proposition shows that, with high probability, the confined random walk has already made at least J "long excursions" before time T_N.

Proposition 3.6.1. *For large N, one has*

$$\overline{P}_0[R_J \geq T_N] \leq e^{-N^c}. \tag{3.233}$$

The proof is deferred to the Appendix at the end of this Chapter because it is rather technical and similar to the proof of Lemma 4.3 of [64].

Next, we construct a chain of couplings. Simply speaking, it is a sequence of couplings involving multiple random sets, in which the preceding set stochastically dominate the following set with probability close (or sometimes equal) to 1.

We start with the first coupling. The following proposition shows that one can construct a probability space where $(J-1)$ "long excursions" (counted from the second excursion) coincide with high probability with $(J-1)$ independent "long excursions" started from the q.s.d. We write $\overline{P_2^J} = \bigotimes_{i=2}^J \overline{P}_\sigma^i$ for the product of $(J-1)$ independent copies of $\overline{P}_\sigma$. We denote by $\mathcal{A}_i$ the random set $X_{[R_i,V_i)} \cap A_1$ and set $\mathcal{A} = \bigcup_{i=2}^J \mathcal{A}_i$.

Proposition 3.6.2. *For large N, there exists a probability space $(\Omega_0, \mathcal{B}_0, Q_0)$, endowed with a random set $\mathcal{A}$ with the same law as $\mathcal{A}$ under $\overline{P}_0$ and random sets $\check{\mathcal{A}}_i$, $i = 2, \ldots, J$, distributed as $\check{X}^i_{[0,V_1)} \cap A_1$ where for $i \geq 2$, $\check{X}^i$'s are i.i.d. distributed as X under $\overline{P}_\sigma$, such that*

$$Q_0[\mathcal{A} \neq \check{\mathcal{A}}] \leq e^{-c'' \log^2 N}, \tag{3.234}$$

where $\check{\mathcal{A}} = \bigcup_{i=2}^J \check{\mathcal{A}}_i$.

Proof. For each $x \in D$, we use Proposition 4.7, page 50 in [40] and Proposition 3.5.5 to construct a coupling q_x of random variables Ξ with the law of $X_{\overline{t_*}}$ under $\overline{P}_x[\cdot | H_{A_2} > \overline{t_*}]$ and Σ with the law of σ such that

$$\max_{x \in D} q_x[\Xi \neq \Sigma] \leq |D| e^{-c \log^2 N} \leq e^{-c' \log^2 N}. \tag{3.235}$$

We introduce L, the index of last "step" of the path in A_2 before time V [see (3.15) and the paragraph above (3.14) for the definition of τ_l and Z_l, resp.]:

$$L = \sup\{l \geq 0 : \tau_l \leq V, Z_l \in A_2\}. \tag{3.236}$$

We then introduce $L_i = L \circ \theta_{R_i} + l_i$, where l_i satisfies $\tau_{l_i} = R_i$ for $i \geq 1$ as the last step at which the i-th excursion is in A_2.

We now construct Q_0 with the help of (3.235) in a similar fashion to the proof of Lemma 4.2 in [64]. The procedure goes inductively. We start by choosing $x_1^+ \in \partial A_2$ according to $\overline{P}_0[Z_{L_1+1} = \cdot]$. For $i \geq 1$, if x_i^+ is chosen, we choose x_{i+1} and $\check{x}_{i+1}$ points in $D = U^N \setminus A_2$ according to $q_{x_i^+}[\Xi = \cdot, \Sigma = \cdot]$. If x_{i+1} and $\check{x}_{i+1}$ coincide (which is the typical case, that is, if the coupling is successful at step $i+1$), we choose $\mathcal{A}_{i+1} = \check{\mathcal{A}}_{i+1}$ subsets of A_1 and $x_{i+1}^+ = \check{x}_{i+1}^+$ points in ∂A_2 according to $\overline{P}_{x_{i+1}}[\mathcal{A}_1 = \cdot, Z_{L_1+1} = \cdot]$. Otherwise, if x_{i+1} differs from $\check{x}_{i+1}$ (which means that the coupling fails at step $i+1$), then we choose independently $\mathcal{A}_{i+1}, x_{i+1}^+$ according to $\overline{P}_{x_{i+1}}[\mathcal{A}_1 = \cdot, Z_{L_1+1} = \cdot]$ and $\check{\mathcal{A}}_{i+1}, \check{x}_{i+1}^+$ according to $\overline{P}_{\check{x}_{i+1}}[\mathcal{A}_1 = \cdot, Z_{L_1+1} = \cdot]$. In both cases, we repeat the above procedure until step J. Then we write $\mathcal{A} = \bigcup_{i=2}^J \mathcal{A}_i$ and $\check{\mathcal{A}} = \bigcup_{i=2}^J \check{\mathcal{A}}_i$.

By a procedure as in the proof of Lemma 4.2 in [64], (we replace A by A_1, B by A_2, t_* by $\overline{t_*}$, $\mathbb{T}$ by U^N, X_i by Z_i, Y_t by X_t, k by J, U_1 by V_1, $\overline{x}_i$ and $\overline{x}_i^+$ by $\check{x}_i$ and $\check{x}_i^+$), we can check that Q_0 is a coupling of $\mathcal{A}$ and $\check{\mathcal{A}}$, and the probability that the coupling fails has an upper bound

$$
\begin{aligned}
Q_0[\mathcal{A} \neq \check{\mathcal{A}}] \quad &\leq \quad (J-1) \max_{x \in D} q_x[\Xi \neq \Sigma] \\
&\underset{(3.235)}{\overset{(3.232)}{\leq}} \quad c' N^{d-2} e^{-c \log^2 N} \leq e^{-c'' \log^2 N},
\end{aligned}
\tag{3.237}
$$

which is exactly what we want. □

Now, on an auxiliary probability space $(\mathcal{O}_1, \mathcal{F}_1, \mathcal{P}^{\mathcal{I}_1})$, we denote by η_1 the Poisson point process on $\Gamma(U^N)$ with intensity $(1+\varepsilon/3) \cdot u_{**} \mathrm{cap}(A_1) \kappa_1$, where κ_1 is defined as the law of the stopped process $X_{(H_{A_1}+\cdot) \wedge V_1}$ under $\overline{P}_\sigma$. In other words, κ_1 is the law of "long excursions" started from σ and recorded from the first time it enters A_1. We denote by

$$
\mathcal{I}_1 = \bigcup_{\gamma \in \mathrm{supp}(\eta_1)} \mathrm{Range}(\gamma) \cap A_1
\tag{3.238}
$$

the trace of η_1 on A_1. In the next proposition, we construct a second coupling such that $\check{\mathcal{A}}$ dominates $\mathcal{I}_1$ with high probability.

Proposition 3.6.3. *There exists a probability space $(\Omega_1, \mathcal{B}_1, Q_1)$, endowed with random sets $\mathcal{I}_1$ with the same law as $\mathcal{I}_1$ under $\mathcal{P}^{\mathcal{I}_1}$ and $\check{\mathcal{A}}$ with the same law as $\check{\mathcal{A}}$ under $\overline{P_2^J}$, such that*

$$
Q_1[\check{\mathcal{A}} \supseteq \mathcal{I}_1] \geq 1 - e^{-N^c}.
\tag{3.239}
$$

Proof. We pick a Poisson random variable ξ with parameter $(1 + \varepsilon/3)u_{**} \times \mathrm{cap}(A_1)$. Then we generate (independently from ξ) an infinite sequence $\{\check{X}^i\}_{i\geq 1}$ of i.i.d. confined walks under $\overline{P}_\sigma$. We then let $\mathcal{I}_1 \sim \bigcup_{i=2}^{\xi+1} \check{X}^i_{[0,V_1)} \cap A_1$ and $\check{\mathcal{A}} = \bigcup_{i=2}^{J} \check{X}^i_{[0,V_1)} \cap A_1$, both having the respective required laws. Moreover $\{\check{\mathcal{A}} \supseteq \mathcal{I}_1\} = \{J \geq \xi + 1\}$, by the definition of J [see (3.232)] and a standard estimate on the deviation of Poisson random variables, we have

$$Q_1[\check{\mathcal{A}} \supseteq \mathcal{I}_1] = Q_1[J \geq \xi + 1] \geq 1 - e^{-N^c}, \tag{3.240}$$

which is exactly (3.239). □

Now on another auxiliary probability space $(\mathcal{O}_2, \mathcal{F}_2, \mathcal{P}^{\mathcal{I}_2})$, we denote by η_2 the Poisson point process on $\Gamma(U^N)$ with intensity $(1+\varepsilon/4)u_{**}\mathrm{cap}(A_1)\kappa_2$, where κ_2 is defined as the law of the stopped process $X_{\cdot\wedge V_1}$ under $\overline{P}_{\widetilde{e}_{A_1}}$. In other words, it is the law of "long excursions" started from the normalized equilibrium measure of A_1 (note that, since in this case the excursions start from inside A_1, we start recording directly from time 0). We denote by

$$\mathcal{I}_2 = \bigcup_{\gamma \in \mathrm{supp}(\eta_2)} \mathrm{Range}(\gamma) \cap A_1 \tag{3.241}$$

the trace of η_2 on A_1. The next proposition and corollary construct the third coupling so that $\mathcal{I}_1$ dominates $\mathcal{I}_2$ almost surely. This is shown by proving that the intensity measure of $\mathcal{I}_1$ is bigger than that of $\mathcal{I}_2$ with the help of Proposition 3.5.7.

Proposition 3.6.4. *For large N, one has*

$$\left(1 + \frac{\varepsilon}{3}\right)\kappa_1 \geq \left(1 + \frac{\varepsilon}{4}\right)\kappa_2. \tag{3.242}$$

Proof. By the definition of κ_1 and κ_2, and the strong Markov property applied at time H_{A_1}, we can represent the Radon–Nikodym derivative of κ_1 and κ_2 through a function of the starting point of the trajectory

$$\frac{d\kappa_1}{d\kappa_2} = \phi(X_0) \tag{3.243}$$

where $\phi(x) = \overline{P}_\sigma[X_{H_{A_1}} = x]/\widetilde{e}_{A_1}(x)$ for all $x \in \partial_i A_1$ and 0 otherwise. Hence we obtain, via (3.231), that for large N,

$$\frac{d(\kappa_1 - \kappa_2)}{d\kappa_2} = \phi(X_0) - 1 \geq -N^{-c} \geq \frac{-\varepsilon/12}{(1+\varepsilon/3)}, \quad \kappa_2\text{-a.s.} \tag{3.244}$$

This implies (3.242) after rearrangement. □

As a consequence, we have the following corollary.

Corollary 3.6.5. *For large N, there exists a probability space $(\Sigma_2, \mathcal{B}_2, Q_2)$ endowed with random sets $\mathcal{I}_1$ with the same law as $\mathcal{I}_1$ under $P^{\mathcal{I}_1}$ and $\mathcal{I}_2$ with the same law as $\mathcal{I}_2$ under $P^{\mathcal{I}_2}$, such that*

$$\mathcal{I}_1 \supseteq \mathcal{I}_2, \qquad Q_2\text{-a.s.} \tag{3.245}$$

Proof. This follows immediately from the domination of measures. Indeed, we first construct $\mathcal{I}_2$ on some probability space. Then we consider the positive measure on $\Gamma(U^N)$

$$\alpha = (1+\varepsilon/3)\kappa_1 - (1+\varepsilon/4)\kappa_2, \tag{3.246}$$

and construct (independently from $\mathcal{I}_2$) a Poisson point process $\widehat{\eta}$ on $\Gamma(U^N)$ with intensity measure α. Then $\mathcal{I}_1 = (\bigcup_{\gamma \in \mathrm{supp}(\widehat{\eta})} \mathrm{Range}(\gamma) \cap A_1) \cup \mathcal{I}_2$ has the required law. □

On another auxiliary probability space $(\mathcal{O}'_2, \mathcal{F}'_2, \mathcal{P}^{\mathcal{I}'_2})$, we denote by η'_2 the law of the Poisson point process on $\Gamma(U^N)$ with intensity $(1+\varepsilon/4)u_{**}\mathrm{cap}(A_1)$
κ'_2, where κ'_2 is defined as the stopped process $X_{\cdot \wedge T_{A_2}}$ under $P_{\widetilde{e}_{A_1}}$, or equivalently $\overline{P}_{\widetilde{e}_{A_1}}$. Contrary to the definition of a "long excursion", we would like to call $X_{[H_{A_1}, T_{A_2})}$ a "short excursion", since we stop the excursion earlier than a "long excursion" (this is because $T_{A_2} < V_1$).

In other words, κ_2' is the measure of "short excursions" started from the normalized equilibrium measure of A_1. We denote by

$$(3.247) \qquad \mathcal{I}_2' = \bigcup_{\gamma \in \mathrm{supp}(\eta_2')} \mathrm{Range}(\gamma) \cap A_1$$

the trace of η_2' in A_1. Hence, we can naturally construct the fourth coupling such that $\mathcal{I}_2$ dominates $\mathcal{I}_2'$ almost surely, which is stated in the corollary below.

Corollary 3.6.6. *When N is large, there exists a probability space $(\Sigma_2', \mathcal{B}_2', Q_2')$, endowed with random sets $\mathcal{I}_2'$ with the same law as $\mathcal{I}_2'$ under $\mathcal{P}^{\mathcal{I}_2'}$, and $\mathcal{I}_2$ with the same law as $\mathcal{I}_2$ under $\mathcal{P}^{\mathcal{I}_2}$ such that*

$$(3.248) \qquad \mathcal{I}_2 \supseteq \mathcal{I}_2', \qquad Q_2'\text{-a.s.}$$

The fifth coupling establishes the stochastic domination of $\mathcal{I}_2'$ on the trace of $\mathcal{I}^{(1+\varepsilon/8)u_{**}}$ in A_1. It is reproduced from [7].

Proposition 3.6.7. *When N is large, there exists a probability space $(\Sigma_3, \mathcal{B}_3, Q_3)$ endowed with random sets $\mathcal{I}$ with the same law as $\mathcal{I}^{u_{**}(1+\varepsilon/8)} \cap A_1$ under $\mathbb{P}$ and $\mathcal{I}_2'$ with the same law as $\mathcal{I}_2'$ under $P^{\mathcal{I}_2'}$, such that*

$$(3.249) \qquad Q_3\big[\mathcal{I}_2' \supseteq \mathcal{I}\big] \geq 1 - e^{-N^c}.$$

We refer the readers to Proposition 5.4 of [7] and to Section 9 of [7] for its proof.

The next proposition links together the above couplings from Propositions 3.6.2, 3.6.3, Corollaries 3.6.5, 3.6.6, and Proposition 3.6.7. We prove that for any x_0 in the "strip" Γ^N, the probability that it is connected in $\mathcal{V}$ [i.e., the vacant set of the random walk, see below (3.16)] to the (inner) boundary of $A_1^{x_0}$ is small.

Proposition 3.6.8. *For large N and all $x_0 \in \Gamma^N$, one has*

$$(3.250) \qquad \widetilde{P}_N\big[x_0 \stackrel{\mathcal{V}}{\longleftrightarrow} \partial_i A_1^{x_0}\big] \leq e^{-c \log^2 N}.$$

Proof. First, by Corollary 3.3.8, Proposition 3.6.1 and the first two couplings [namely Proposition 3.6.2 (see ibid. for notation) and Corollary 3.6.3], one knows that for large N,

$$\begin{aligned}
\widetilde{P}_N\big[x_0 \overset{\mathcal{V}}{\longleftrightarrow} \partial_i A_1^{x_0}\big] &\overset{(3.69)}{\leq} \overline{P}_0\big[x_0 \overset{(X_{[R_2,T_N)})^c}{\longleftrightarrow} \partial_i A_1^{x_0}\big] \\
&\underset{(3.234)}{\overset{(3.233)}{\leq}} \overline{P_2^J}\big[x_0 \overset{\check{\mathcal{A}}^c}{\longleftrightarrow} \partial_i A_1^{x_0}\big] + e^{-c\log^2 N} \\
&\overset{(3.239)}{\leq} P^{\mathcal{I}_1}\big[x_0 \overset{\mathcal{I}_1^c}{\longleftrightarrow} \partial_i A_1^{x_0}\big] + e^{-c'\log^2 N}.
\end{aligned} \tag{3.251}$$

Then, by the third, fourth and fifth couplings, namely Corollaries 3.6.5, 3.6.6 and Proposition 3.6.7, and the strong super-criticality of random interlacements [see (3.25)], for large N, one obtains the following inequalities:

$$\begin{aligned}
\mathcal{P}^{\mathcal{I}_1}\big[x_0 \overset{\mathcal{I}_1^c}{\longleftrightarrow} \partial_i A_1^{x_0}\big] &\overset{(3.245)}{\leq} \mathcal{P}^{\mathcal{I}_2}\big[x_0 \overset{\mathcal{I}_2^c}{\longleftrightarrow} \partial_i A_1^{x_0}\big] \\
&\overset{(3.248)}{\leq} \mathcal{P}^{\mathcal{I}_2'}\big[x_0 \overset{\mathcal{I}_2'^c}{\longleftrightarrow} \partial_i A_1^{x_0}\big] \\
&\overset{(3.249)}{\leq} Q_3\big[x_0 \overset{\mathcal{I}^c}{\longleftrightarrow} \partial_i A_1^{x_0}\big] + e^{-N^c} \overset{(3.25)}{\leq} e^{-N^{c'}},
\end{aligned} \tag{3.252}$$

which show that the first term to the right of the last inequality in (3.251) has a stretched exponential decay in N. The claim (3.250) hence follows by inserting (3.252) into (3.251). □

We are ready now to state and prove the main result of this section, namely that the tilted disconnection probability tends to 1 as N tends to infinity.

Theorem 3.6.9.

$$\lim_{N\to\infty} \widetilde{P}_N[K_N \overset{\mathcal{V}}{\nleftrightarrow} \infty] = 1. \tag{3.253}$$

Proof. Note that for large N, if a nearest-neighbour path connects K_N and infinity, it must go through the set Γ^N at some point x_0 [see above (3.119) for the definition of Γ^N]. Hence, it connects x_0 to the inner boundary of $A_1^{x_0}$, so that

$$\{K_N \overset{\mathcal{V}}{\nleftrightarrow} \infty\}^c \subset \bigcup_{x_0 \in \Gamma^N} \{x_0 \overset{\mathcal{V}}{\longleftrightarrow} \partial_i A_1^{x_0}\}. \tag{3.254}$$

Thus, we see that for large N,

$$\widetilde{P}_N\big[\{K_N \overset{\mathcal{V}}{\nleftrightarrow} \infty\}^c\big] \le \sum_{x_0 \in \Gamma^N} \widetilde{P}_N\big[x_0 \overset{\mathcal{V}}{\longleftrightarrow} \partial_i A_1^{x_0}\big]. \tag{3.255}$$

By Proposition 3.6.8, we find that for large N, uniformly for each $x_0 \in \Gamma^N$, we can bound each term on right-hand side of (3.255), and find

$$\widetilde{P}_N\big[\{K_N \overset{\mathcal{V}}{\nleftrightarrow} \infty\}^c\big] \le \big|\Gamma^N\big| e^{-c \log^2 N} \underset{N\to\infty}{\longrightarrow} 0. \tag{3.256}$$

This completes the proof of Theorem 3.6.9. □

3.7 Denouement and epilogue

In this section, we combine the main ingredients, namely Theorem 3.6.9 and Proposition 3.3.14 and prove Theorem 3.1.1.

Proof. Proof of Theorem 3.1.1 We recall the entropy inequality [see (3.27)], and apply it to P_0 and $\widetilde{P}_N$ (which is defined in Section 3.3). By Theorem 3.6.9, one has

$$\lim_{N\to\infty} \widetilde{P}_N[K_N \overset{\mathcal{V}}{\nleftrightarrow} \infty] = 1, \tag{3.257}$$

thus the relative entropy inequality (3.27) yields that

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}} \log\big(P_0[K_N \overset{\mathcal{V}}{\nleftrightarrow} \infty]\big) \ge -\limsup_{N\to\infty} \frac{1}{N^{d-2}} H(\widetilde{P}_N | P_0). \tag{3.258}$$

Then, as in the proof of Proposition 3.3.14, taking consecutively the lim sup as $\eta \to 0$, $R \to \infty$, $\delta \to 0$ and $\varepsilon \to 0$, one has

$$\limsup_{\varepsilon\to 0}\limsup_{\delta\to 0}\limsup_{R\to\infty}\limsup_{\eta\to 0}\limsup_{N\to\infty}\frac{1}{N^{d-2}}H(\widetilde{P}_N|P_0) \leq \frac{u_{**}}{d}\mathrm{cap}_{\mathbb{R}^d}(K), \tag{3.259}$$

proving Theorem 3.1.1. □

Remark 3.7.1. Assume for simplicity that the compact K is regular. Notice that unlike what happens for $d \geq 5$, when $d = 3, 4$, the function h defined in (3.8) is not in $L^2(\mathbb{R}^d)$, and $h_N(x) = h(\frac{x}{N})$ is not in $l^2(\mathbb{Z}^d)$. This fact affects T_N defined in (3.48) (T_N/N^d diverges if $R \to \infty$ when $d = 3, 4$, but not when $d \geq 5$).

One can wonder whether this feature reflects different qualitative behaviours of the random walk path under the conditional measure $P_0[\cdot|K_N \overset{\mathcal{V}}{\nleftrightarrow} \infty]$ when N becomes large.

Appendix

In the appendix, we include the proof of Propositions 3.5.7 and 3.6.1.

Proof of Proposition 3.5.7. We first prove that for $x \in \partial_i A_1$

$$\left|\overline{P}_x[V < \widetilde{H}_{A_1}] - \overline{P}_\sigma[X_{H_{A_1}} = x]\sum_{y\in\partial_i A_1}\overline{P}_y[V < \widetilde{H}_{A_1}]\right| \leq e^{-c\log^2 N}, \tag{3.260}$$

and, as we will see, the claim (3.231) will then follow. We consider in the left-hand side of (3.261) the probability that the random walk started from $x \in \partial_i A_1$ stays in D for a time interval of length $\overline{t_*}$ before returning to A_1, and then returns to A_1 through some vertex other than x. By reversibility of the confined walk, and the fact that by claim 3 of (3.50) and claim 1 of (3.70), $\pi(y) = \pi(x)$ for all

$y \in \partial_i A_1$, this probability can be written as

$$(3.261)\qquad \begin{aligned} &\sum_{y \in \partial_i A_1 \setminus \{x\}} \overline{P}_x[V < \widetilde{H}_{A_1}, X_{H_{A_1}} = y] \\ &= \sum_{y \in \partial_i A_1 \setminus \{x\}} \overline{P}_y[V < \widetilde{H}_{A_1}, X_{H_{A_1}} = x]. \end{aligned}$$

As in (3.236), we consider L defined by

$$(3.262)\qquad L = \sup\{l : \tau_l \leq V, Z_l \in A_2\}.$$

We consider the summands from (3.261): for all $x, y \in \partial_i A_1$, we sum over all possible values of L and $X_{\tau_L} = Z_L$ [recall the definition of τ_l in (3.15) and the relation between X_{τ_l} and Z_l in (3.14)], and apply Markov property at the times τ_{l+1} and $\tau_{l+1} + \overline{t_*}$:

$$(3.263)\qquad \begin{aligned} &\overline{P}_x[V < \widetilde{H}_{A_1}, X_{\widetilde{H}_{A_1}} = y] \\ &= \sum_{l \geq 0, x' \in \partial_i A_2} \overline{P}_x\big[L = l, Z_l = x', V < \widetilde{H}_{A_1}, X_{\widetilde{H}_{A_1}} = y\big] \\ &= \sum_{l \geq 0, x' \in \partial_i A_2} \overline{P}_x\big[Z_l = x', \tau_l < \widetilde{H}_{A_1} \wedge V, H_{A_2} \circ \theta_{\tau_{l+1}} > \overline{t_*}, X_{\widetilde{H}_{A_1}} = y\big] \\ &= \sum_{\substack{l \geq 0, x'' \in D \\ x' \in \partial_i A_2}} \overline{P}_{x''}[X_{H_{A_1}} = y] \overline{E}_x\big[Z_l = x', \tau_l < \widetilde{H}_{A_1} \wedge V, \\ &\quad \overline{P}_{Z_{l+1}}[H_{A_2} > \overline{t_*}] \overline{P}_{Z_{l+1}}\big[X_{\overline{t_*}} = x'' | H_{A_2} > \overline{t_*}\big]\big], \end{aligned}$$

[we will soon use the fact that the conditioned probability in the last expression is close to $\sigma(x'')$ by Proposition 3.5.5]. Similarly, we have

$$(3.264)\qquad \begin{aligned} &\overline{P}_x[V < \widetilde{H}_{A_1}] \\ &= \sum_{l \geq 0, x' \in \partial_i A_2} \overline{E}_x\big[Z_l = x', \tau_l < \widetilde{H}_{A_1} \wedge V, \overline{P}_{Z_{l+1}}[H_{A_2} > \overline{t_*}]\big]. \end{aligned}$$

This implies that

$$\begin{aligned}&\overline{P}_x[V < \widetilde{H}_{A_1}]\overline{P}_\sigma[X_{H_{A_1}} = y] \\ (3.265)\quad &= \sum_{\substack{l\geq 0, x''\in D \\ x'\in\partial_i A_2}} \overline{E}_x\big[Z_l = x', \tau_l < \widetilde{H}_{A_1}\wedge V, \overline{P}_{Z_{l+1}}[H_{A_2} > \overline{t_*}]\big] \\ &\quad\times \sigma\big(x''\big)\overline{P}_{x''}[X_{H_{A_1}} = y].\end{aligned}$$

Hence, by combining (3.263) and (3.265) we have

$$(3.266)\quad \begin{aligned}&\big|\overline{P}_x[V < \widetilde{H}_{A_1}, X_{\widetilde{H}_{A_1}} = y] - \overline{P}_x[V < \widetilde{H}_{A_1}]\overline{P}_\sigma[X_{H_{A_1}} = y]\big| \\ &\overset{(3.216)}{\leq} e^{-c\log^2 N}.\end{aligned}$$

Applying this estimate in both sides in (3.261), we obtain that

(3.267)

$$\begin{aligned}&\Bigg|\overline{P}_x[V < \widetilde{H}_{A_1}]\overline{P}_\sigma[X_{H_{A_1}} \neq x] - \sum_{y\in\partial_i A_1\setminus\{x\}} \overline{P_y}[V < \widetilde{H}_{A_1}]\overline{P}_\sigma[X_{H_{A_1}} = x]\Bigg| \\ &\quad\leq e^{-c\log^2 N}.\end{aligned}$$

Finally, by adding and subtracting $\overline{P}_x[V < \widetilde{H}_{A_1}]\overline{P}_\sigma[X_{H_{A_1}} = x]$, we obtain (3.260) as desired.

Now we prove (3.231). By (3.23) and (3.230) one has that

$$(3.268)\quad \sum_{y\in\partial_i A_1} \overline{P}_y[V < \widetilde{H}_{A_1}]\widetilde{e}_{A_1}(x) \geq N^{-c'}.$$

Hence dividing (3.260) by the left-hand term of (3.268), one obtains

(3.269)

$$\left|\frac{\overline{P}_x[V < \widetilde{H}_{A_1}]}{\sum_{y\in\partial_i A_1}\overline{P}_y[V < \widetilde{H}_{A_1}]\widetilde{e}_{A_1}(x)} - \frac{\overline{P}_\sigma[X_{H_{A_1}} = x]}{\widetilde{e}_{A_1}(x)}\right| \leq e^{-c'\log^2 N},$$

and together with (3.226) the proof of (3.231) is complete. □

Proof of Proposition 3.6.1. In this proof, we always assume that N

is sufficiently large. We recall the definition of T_N in (3.48) and the choice of ε in (3.35). In order to prove (3.233), we observe that, $\overline{P}_0$-a.s.,

(3.270)
$$\begin{aligned}&\{R_J \geq T_N\}\\ &\subset \Big\{H_{A_1} + H_{A_1}\circ\theta_{V_1} + \cdots + H_{A_1}\circ\theta_{V_{J-1}} \geq \Big(1-\frac{\varepsilon}{100}\Big)T_N\Big\}\\ &\cup \Big\{V\circ\theta_{R_1} + \cdots + V\circ\theta_{R_{J-1}} \geq \frac{\varepsilon}{100}T_N\Big\},\end{aligned}$$

that is, the (unlikely) event $\{R_J \geq T_N\}$ happens only when either the sum of H_{A_1}'s exceeds a quantity close to T_N or the sum of shifted V's exceeds a small quantity (but still of order T_N). Now we give an upper bound to their respective probabilities. We define

$$t_N = \sup_{y\in U^N} \overline{E}_y[H_{A_1}], \tag{3.271}$$

which is the maximum of the expected entrance time in A_1 starting from an arbitrary point in U^N (it is not much bigger than $\overline{E}_\pi[H_{A_1}]$ by (3.157)). By the exponential Chebychev inequality and the strong Markov property applied inductively at $V_1,\dots,V_{J-1}$ and $R_1,\dots,R_{J-1}$, we deduce from (3.270) that, for any $\theta > 0$,

$$\begin{aligned}&\overline{P}_0[R_J \geq T_N]\\ &\leq \exp\Big(-\theta\Big(1-\frac{\varepsilon}{100}\Big)\frac{T_N}{t_N}\Big)\Big(\sup_{x\in U^N}\overline{E}_x\Big[\exp\Big(\theta\frac{H_{A_1}}{t_N}\Big)\Big]\Big)^J\\ &\quad + \exp\Big(-\frac{\varepsilon}{100}\frac{T_N}{t_N}\Big)\Big(\sup_{x\in A_1}\overline{E}_x\big[e^{V/t_N}\big]\Big)^J.\end{aligned} \tag{3.272}$$

We now treat the first term on the right-hand side of (3.272). Khaśminskii's lemma (see (4) and (6) in [35]) states that for all B subset of U^N and $n \geq 1$,

$$\sup_{x\in U^N}\overline{E}_x\big[H_B^n\big] \leq n!\sup_{y\in U^N}\overline{E}_y[H_B]^n. \tag{3.273}$$

Hence, we have, for $\theta \in \left(0, \frac{1}{2}\right)$,

$$\sup_{x\in U^N} \overline{E}_x\left[\exp\left(\theta\frac{H_{A_1}}{t_N}\right)\right] \leq \sum_{j=0}^{\infty} \frac{\theta^j}{j!t_N^j} \sup_{x\in U^N} \overline{E}_x\left[H_{A_1}^j\right] \stackrel{(3.273)}{\underset{(3.271)}{\leq}} \sum_{j=0}^{\infty} \theta^j = \frac{1}{1-\theta}. \tag{3.274}$$

Now, we derive an upper bound for $\sup_{x\in A_2} \overline{E}_x[\exp(\frac{V}{t_N})]$ and treat the second term on the right-hand side of (3.272). We first note that, $\overline{P}_x$-a.s. for any $x \in A_2$,

$$\begin{aligned} V &\leq (T_{A_3} + \overline{t_*})1_{\{H_{A_2}\circ\theta_{T_{A_3}} > \overline{t_*}\}} \\ &\quad + (T_{A_3} + \overline{t_*} + V\circ\theta_{H_{A_2}}\circ\theta_{T_{A_3}})1_{\{H_{A_2}\circ\theta_{T_{A_3}} \leq \overline{t_*}\}} \\ &= T_{A_3} + \overline{t_*} + V\circ\theta_{H_{A_2}}\circ\theta_{T_{A_3}}1_{\{H_{A_2}\circ\theta_{T_{A_3}} \leq \overline{t_*}\}}. \end{aligned} \tag{3.275}$$

By the strong Markov property applied at $H_{A_2}\circ\theta_{T_{A_3}} + T_{A_3}$ and T_{A_3}, we have

$$\begin{aligned} &\sup_{x\in A_2} \overline{E}_x\left[e^{V/t_N}\right] \\ &\stackrel{(3.275)}{\leq} \sup_{x\in A_2} \overline{E}_x\left[e^{(T_{A_3}+\overline{t_*})/t_N}\right]\Big(1+ \\ &\qquad \sup_{y\in U^N\setminus A_3} \overline{P}_y[H_{A_2} \leq \overline{t_*}] \sup_{x\in A_2} \overline{E}_x\left[e^{V/t_N}\right]\Big) \\ &\stackrel{(3.129)}{\leq} \sup_{x\in A_2} \overline{E}_x\left[e^{(T_{A_3}+\overline{t_*})/t_N}\right]\Big(1 + N^{-c} \sup_{x\in A_2} \overline{E}_x\left[e^{V/t_N}\right]\Big). \end{aligned} \tag{3.276}$$

By Proposition 3.4.5, we have

$$\begin{aligned} \frac{1}{t_N} &\stackrel{(3.271)}{\leq} \frac{1}{\overline{E}_\pi[H_{A_1}]} \stackrel{(3.149)}{\leq} \big(1+N^{-c}\big)\frac{\mathrm{cap}(A_1)}{T_N}u_{**}(1+\varepsilon) \\ &\stackrel{(3.52)5.}{\underset{(3.22)}{\leq}} cN^{-d+r_1(d-2)}. \end{aligned} \tag{3.277}$$

By an elementary estimate on simple random walk and the observation that the diameter of A_3 is smaller than cN^{r_3}, we have

$$(3.278) \qquad \overline{E}_x[T_{A_3}] \overset{(3.122)}{=} E_x[T_{A_3}] \le cN^{2r_3} \qquad \text{for all } x \in A_3,$$

therefore we obtain that

$$(3.279) \qquad \frac{\sup_{x\in A_3} \overline{E}_x[T_{A_3}]}{t_N} \le cN^{-d+2r_3+(d-2)r_1} \le N^{-c'}.$$

By an argument like (3.274), again with the help of Khaśminskii's lemma [see (3.273)], we obtain that

$$(3.280) \quad \sup_{x\in A_2} \overline{E}_x\left[\exp\left(\frac{T_{A_3}}{t_N}\right)\right] \le \frac{1}{1-N^{-c}} \le e^{N^{-c'}} \qquad \text{for large } N.$$

Moreover, we obtain from (3.277) that

$$(3.281) \qquad \frac{\overline{t_*}}{t_N} \overset{(3.95)}{\le} cN^{-c'}.$$

We apply (3.280) and (3.281) to the right-hand side of (3.276), and conclude after rearrangement [and with an implicit truncation argument where V in (3.275) and (3.276) is replaced by $V \wedge M$] that

$$(3.282) \qquad \sup_{x\in A_2} \overline{E}_x\big[e^{V/t_N}\big] \le e^{N^{-c}}.$$

We now return to (3.272). Substituting (3.274) and (3.282) into (3.272) and using the fact that for $0 \le \theta \le \frac{1}{2}$,

$$(3.283) \qquad (1-\theta)^{-1} \le 1+\theta+2\theta^2 \le e^{\theta+2\theta^2},$$

we deduce that

(3.284)

$$\begin{aligned}
&\overline{P}_0[R_J \geq T_N] \\
\leq\ & \exp\Big(-\theta\Big(1-\frac{\varepsilon}{100}\Big)\frac{T_N}{t_N} + \big(\theta+2\theta^2\big)J\Big) \\
& + \exp\Big(-\frac{\varepsilon}{100}\frac{T_N}{t_N} + N^{-c}J\Big) \\
\underset{(3.283)}{\overset{(3.232)}{\leq}}\ & \exp\Big(-\theta\Big(1-\frac{\varepsilon}{100}\Big)\frac{T_N}{t_N} + \big(\theta+2\theta^2\big)\big\lfloor(1+\varepsilon/2)u_{**}\mathrm{cap}(A_1)\big\rfloor\Big) \\
& + \exp\Big(-\frac{\varepsilon}{100}\frac{T_N}{t_N} + N^{-c}\big\lfloor(1+\varepsilon/2)u_{**}\mathrm{cap}(A_1)\big\rfloor\Big).
\end{aligned}$$

Recall the definition of f_{A_1} in (3.126). Using Lemma 3.4.6, we know that for all $x \in U^N$

$$\frac{\overline{E}_x[H_{A_1}]}{\overline{E}_\pi[H_{A_1}]} = 1 - f_{A_1}(x) \overset{(3.153)}{\leq} 1 + N^{-c} \leq \Big(1-\frac{\varepsilon}{100}\Big)^{-1}. \tag{3.285}$$

Hence, by Proposition 3.4.7 we obtain that

(3.286)

$$\frac{T_N}{t_N} \geq \Big(1-\frac{\varepsilon}{100}\Big)\frac{T_N}{\overline{E}_\pi[H_{A_1}]} \overset{(3.166)}{\geq} \Big(1-\frac{\varepsilon}{50}\Big)(1+\varepsilon)u_{**}\mathrm{cap}(A_1).$$

By choosing an appropriately small θ and applying (3.286) we know that for large N,

$$-\theta\Big(1-\frac{\varepsilon}{100}\Big)\frac{T_N}{t_N} + \big(\theta+2\theta^2\big)\big\lfloor(1+\varepsilon/2)u_{**}\mathrm{cap}(A_1)\big\rfloor \leq -N^c, \tag{3.287}$$

moreover, we also know that for large N

$$-\frac{\varepsilon}{100}\frac{T_N}{t_N} + N^{-c}\big\lfloor(1+\varepsilon/2)u_{**}\mathrm{cap}(A_1)\big\rfloor \leq -N^{c'}. \tag{3.288}$$

Inserting (3.287) and (3.288) into (3.284), we obtain (3.233) as desired. □

Bibliography

[1] R.A. Adams. *Sobolev Spaces.* Academic Press, New York, 1975.

[2] D. J. Aldous and M. Brown. Inequalities for rare events in time-reversible Markov chains I. *Stochastic inequalities.* IMS Lecture Notes-Monograph Series 22. Inst. Math. Statist., Hayward, 1992.

[3] D. J. Aldous and J. A. Fill. *Reversible Markov chains and random walks on graphs.* Unfinished monograph. Available at: http://www.stat.berkeley.edu/users/aldous/RWG/book.pdf

[4] O. Angel, B. Ráth, Q. Zhu. Branching interlacement (*work in progress*).

[5] M. T. Barlow. *Diffusions on Fractals.* Lecture notes in Math. 1690. Springer, Berlin, 1998.

[6] D. Belius. Cover levels and random interlacements. *Ann. Appl. Probab.*, 22(2):522–540, 2012.

[7] D. Belius. Gumbel fluctuations for cover times in the discrete torus. *Probab. Theory Relat. Fields*, 157(3-4):635–689, 2013.

[8] I. Benjamini and A.S. Sznitman. Giant component and vacant set for random walk on a discrete torus. *J. Eur. Math. Soc. (JEMS)*, 10(1):133–172, 2008.

[9] M. van den Berg, E. Bolthausen and F. den Hollander. Moderate deviations for the volume of the Wiener sausage. *Ann. Math.*, 153:355–406, 2001.

[10] P. Billingsley. *Convergence of probability measures.* Wiley, New York, 1968.

[11] E. Bolthausen and J.D. Deuschel. Critical large deviations for Gaussian fields in the phase transition regime, I. *Ann. Probab.*, 21(4):1876–1920, 1993.

[12] R. Cerf. *Large deviations for three dimensional supercritical percolation.* Astérisque 267, Société Mathématique de France, 2000.

[13] J. Černý, A. Teixeira, *From random walk trajectories to random interlacements.* Ensaios Matemáticos 23, 2012.

[14] J. Černý, A. Teixeira, and D. Windisch. Giant vacant component left by a random walk in a random d-regular graph. *Ann. Inst. Henri Poincaré Probab. Stat.*, 47(4):929–968, 2011.

[15] Z.-Q. Chen. Gaugeability and conditional gaugeability. Transactions of the *AMS*, 354(11):4639–4679, 2002.

[16] Z.-Q. Chen and R. Song. General Gauge and conditional Gauge theorems. *Ann. Probab.*, 30(3):1313–1339, 2002.

[17] K.L. Chung and Z. Zhao. *From Brownian motion to Schrödinger's equation.* Springer, New York, 1995.

[18] F. Comets, S. Popov and M. Vachkovskaia. Two-dimensional random interlacements and late points for random walks. To appear in *Comm. Math. Phys.* Also available at arXiv:1502.03470 (44 pages).

[19] G. Dal Maso. *An introduction to* Γ*-convergence.* Birkhäuser, Basel, 1993.

[20] A. Dembo and O. Zeitouni. *Large deviations techniques and applications.* Springer, Berlin, 2nd edition, 1998.

[21] M. Demuth and J.A. van Casteren. *Stochastic spectral theory for selfadjoint Feller operators.* Birkhäuser, Basel, 2000.

[22] J.D. Deuschel and D.W. Stroock. *Large deviations.* Academic Press, Boston, 1989.

[23] A. Drewitz, B. Ráth, A. Sapozhnikov. *An Introduction to Random Interlacements.* SpringerBriefs in Mathematics, Berlin, 2014.

[24] A. Drewitz, B. Ráth, A. Sapozhnikov. Local percolative properties of the vacant set of random interlacements with small intensity. *Ann. Inst. Henri Poincaré Probab. Stat.*, 50(4):1165–1197, 2014.

[25] R. Durrett. *Brownian motion and martingales in analysis.* Wadsworth, Belmont, 1984.

[26] A.W.F. Edwards. Pascal's Problem: The 'Gambler's Ruin'. *International Statistical Review.* 51(1):73–79, 1983.

[27] S.M. Ethier and T.G. Kurtz. *Markov processes.* John Wiley & Sons, New York, 1986.

[28] L. C. Evans. *Partial Differential Equations.* 2nd ed., American Mathematical Society, Providence, 2010.

[29] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet forms and symmetric Markov processes*. Walter de Gruyter, Berlin, 1994.

[30] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet forms and symmetric Markov processes*, volume 19. 2nd revised and extended ed., Walter de Gruyter, Berlin, 2011.

[31] D. Gilbarg, N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. 2nd ed., revised 3rd printing, Springer, Berlin, 1998.

[32] J. Glimm and A. Jaffe. *Quantum Physics*. Springer, Berlin, 1981.

[33] G. Grimmett. *Percolation*. Second edition, Springer, Berlin, 1999.

[34] W. H. Haemers. Interlacing eigenvalues and graphs. *Linear Algebra Appl.* 226/228:593-616, 1995.

[35] R. Z. Khaśminskii. On positive solutions of the equation $AU+Vu=0$. *Theor. Probability Appl.* 4:309-318, 1959.

[36] U. Krengel. *Ergodic theorems*. Walter de Gruyter, Berlin, 1985.

[37] J. Keilson. *Markov chain models-rarity and exponentiality*. Applied Mathematical Sciences 28. Springer, New York, 1979.

[38] G.F. Lawler. *Intersections of random walks*. Birkhäuser, Basel, 1991.

[39] G.F. Lawler and V. Limic. *Random walk: A modern introduction*. Cambridge University Press, 2010.

[40] D. A. Levin, Y. Peres, and E. L. Wilmer. *Markov chains and mixing times*. American Mathematical Society, Providence, 2009.

[41] X. Li. A lower bound for disconnection by simple random walk. To appear in *Ann. Probab.*, also available at arXiv:1412:3959 (39 pages).

[42] X. Li and A.-S. Sznitman. Large deviations for occupation time profiles of random interlacements. *Probab. Theory Relat. Fields*, 161(1):309–350, 2015.

[43] X. Li and A.-S. Sznitman. A lower bound for disconnection by random interlacements. *Electron. J. Probab.*, 19(17):1–26, 2014.

[44] K.R. Parthasarathy. *Probability measures on metric spaces*. Academic Press, New York, 1967.

[45] R.G. Pinsky. *Positive harmonic functions and diffusion*. Cambridge University Press, Cambridge, 1995.

[46] S. Popov and A. Teixeira. Soft local times and decoupling of random interlacements. *J. Eur. Math. Soc.,* 17(10):2545–2593, 2015.

[47] S. Port and C. Stone. *Brownian motion and classical Potential Theory*. Academic Press, New York, 1978.

[48] S.I. Resnick. *Extreme Values, regular variation, and point processes.* Springer, New York, 1987.

[49] W. Rudin. *Functional analysis.* Tata Mc Graw-Hill, New Delhi, 1974.

[50] W. Rudin. *Principles of Mathematical Analysis.* 3rd ed., McGraw-Hill, New York, 1976.

[51] L. Saloff-Coste. *Lecture notes on finite Markov chains.* Lecture Notes in Math. 1665:301-413, Springer, Berlin, 1997.

[52] V. Sidoravicius and A.S. Sznitman. Percolation for the vacant set of random interlacements. *Comm. Pure Appl. Math.*, 62(6):831–858, 2009.

[53] V. Sidoravicius and A.S. Sznitman. Connectivity bounds for the vacant set of random interlacements. *Ann. Inst. Henri Poincaré, Probabilités et Statistiques*, 46(4):976–990, 2010.

[54] A.-S. Sznitman. *Brownian motion, obstacles and random media.* Springer, Berlin, 1998.

[55] A.-S. Sznitman. On the domination of random walk on a discrete cylinder by random interlacements. *Electron. J. Probab.*, 14:1670–1704, 2009.

[56] A.-S. Sznitman. Vacant set of random interlacements and percolation. *Ann. Math.*, 171:2039–2087, 2010.

[57] A.-S. Sznitman. Random interlacements and the Gaussian free field. *Ann. Probab.*, 40(6):2400-2438, 2012.

[58] A.-S. Sznitman. Decoupling inequalities and interlacement percolation on $G \times Z$. *Invent. Math.*, 187(3):645–706, 2012.

[59] A.-S. Sznitman. An isomorphism theorem for random interlacements. *Electron. Commun. Probab.*, 17(9):1–9, 2012.

[60] A.-S. Sznitman. On scaling limits and Brownian interlacements. *Bull. Braz. Math. Soc., New Series*, 44(4):555–592, 2013. Special issue *IMPA 60 years*.

[61] A.-S. Sznitman. Disconnection and level-set percolation for the Gaussian free field. *J. Math. Soc. Japan,* 67(4):1801–1843, 2015.

[62] A.-S. Sznitman. Disconnection, random walks, and random interlacements. To appear in *Probab. Theor. Relat. Fields*, also available at arXiv:1412.3960 (39 pages).

[63] A. Teixeira. On the uniqueness of the infinite cluster of the vacant set of random interlacements. *Ann. Appl. Probab.* 19(1):454–466, 2009.

[64] A. Teixeira and D. Windisch. On the fragmentation of a torus by

random walk. *Commun. Pure Appl. Math.*, 64(12):1599–1646, 2011.

[65] D. Windisch Random walk on a discrete torus and random interlacements. *Electron. C. Probab.*, 13:140–150, 2008.

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