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AN ANALYSIS OF THREE VARIANTS OF FORWARD GUIDANCE CONTRACTS

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日月相推而明生焉
«易經»

献给故土的亲人
流落四方的友人
和我挚爱的祖国

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Abstract

We examine “Forward Guidance Contracts", i.e. contracts that make central bankers’ utility contingent on the precision of their interest-rate forecasts. We integrate these contracts into the New Keynesian Framework and study how they can be used to overcome a liquidity trap.

After an Introduction, in which we explain the motivation, approach, literature and organization of this research, Chapter 2 presents the micro-foundations of our model and in particular a foundation of the central banker’s utility function when the government offers him a wage contract composed of a fixed wage and a variable component that increases with the accuracy of the interest-rate forecast.

Chapter 3 studies the properties of simple renewable Forward Guidance Contracts and characterizes the contracts that the government wants to offer repeatedly. These contracts create favorable tradeoffs between the efficacy of forward guidance at the zero bound and the reduced flexibility in reacting to future events. In addition, we discuss which type of Forward Guidance Contracts can be used when there is uncertainty about natural real interest-rate shocks, a situation which typically calls for moderate incentive intensity.

Long-term contracts are explored in Chapter 4 in an alternative contractual environment. We show that when the size of shock is severe, longer-term contracts could lower social losses further compared to short-term contracts. Severe natural real interest-rate shocks require large incentive intensities with long durations to mitigate the deflation and output collapse in downturns. While such contracts can yield even lower social losses, they also constrain the central bank for a long time and may thus be problematic, as unforeseen events requiring greater flexibility may occur in the interim.

The last chapter of the thesis deals with contracts that are contingent on certain macroeconomic variables, e.g. natural real interest rate or inflation expectation, for instance. The contract, signed in downturns, is in effect as long as certain criteria are fulfilled, e.g. the contract expires one period after the natural real interest rate achieves 2%. With such contracts, the government does not have to re-sign the simple, renewable Forward Guidance Contracts repeatedly, while the same effect can be achieved.

Zusammenfassung

Diese Dissertation befasst sich mit “Forward Guidance Contracts" (FGC), d. h. mit Verträgen, welche die Nutzenfunktion eines Zentralbankers von der Präzision seiner Zinsprognosen abhängig machen. Wir integrieren solche Verträge in einen Neu-Keynesianische Rahmen und analysieren, wie sie eingesetzt werden können, um eine Liquiditätsfalle zu überwinden.

In der Einführung legen wir die Motivation für unsere Arbeit dar, entwerfen das Verfahren, nach dem wir vorgehen wollen, und stellen den Bezug zur Literatur zu diesem Thema her. Kapitel 2 präsentiert die mikroökonomischen Grundlagen des Modells, insbesondere wird die Nutzenfunktion des Zentralbankers erarbeitet, wenn ihm die Regierung einen Vertrag anbietet, in dem sowohl ein festes Gehalt als auch eine variable Gehaltskomponente enthalten sind. Die variable Komponente ist von der Präzision der Zinsprognose abhängig, welche der Zentralbanker gemacht hat.

Kapitel 3 untersucht die Eigenschaften von einfachen, erneuerbaren FGCs und beschreibt diejenigen Verträge, welche eine Regierung wiederholt anbieten wird, weil sie einen vorteilhaften Kompromiss darstellen zwischen der Wirksamkeit von “forward guidance" an der Nullzinsgrenze und der mit dem Vertrag einhergehenden Reduzierung der Flexibilität, auf zukünftige Ereignisse zu reagieren. Wir analysieren auch, welche Art von Verträgen genutzt werden kann, wenn eine Unsicherheit bezüglich der Ausprägung des natürlichen Realzins-Schocks besteht—eine Situation, die in der Regel eine moderate Anreiz-Intensität erfordert.

In Kapitel 4 werden langfristige Verträge untersucht, welche in alternativen Vertragsumgebungen implementiert werden. Wir zeigen, dass langfristige Verträge dann bessere Ergebnisse bringen, wenn der Realzins-Schock beträchtlich ist: Sie können Wohlfahrtsverluste effizienter mindern als kurzfristige Verträge. Grosse natürliche Realzins-Schocks erfordern hohe Anreize und lange Laufzeiten, wenn sie Deflationen und Produktionseinbrüche in Abschwüngen abmildern sollen. Solche Verträge können zwar die Wohlfahrtsverluste mindern, doch sie binden die Zentralbank auf lange Sicht, was insbesondere beim Auftreten von unvorhergesehenen Ereignissen nachteilig ist—weil solche Ereignisse nach höherer Flexibilität verlangen, als es die Vertäge zulassen könnten.

Das letzte Kapitel der Dissertation wendet sich einer besonderen Art von FGC zu, denjenigen, welche an bestimmte makroökonomische Variablen gebunden sind, wie zum Beispiel an das erwartete Ausmass der Inflation oder an den Realzinssatz. Ein Vertrag, welcher in einem Abschwung unterzeichnet worden ist, gilt so lange wie bestimmte Kriterien erfüllt sind. Der Vertrag endet zum Beispiel sobald der natürliche Realzins bei 2% liegt. Die Regierung muss solche Verträge nicht immer wieder unterzeichnen und erreicht doch dieselben Ziele wie mit den regelmässig erneuerten Forward Guidance Contracts.

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1 Introduction

1.1 Forward Guidance in a Liquidity Trap

Since the financial crisis of 2007-08, the global economy has been suffering from deflation and depressed output. Major central banks, e.g. the Federal Reserve (Fed), the Bank of Japan (BoJ), and the European Central Bank (ECB), for instance, have taken various actions to lower the real interest rate to stimulate the economy.

We briefly review and summarize the Fed's responses[1] to the financial crisis as follows.

In 2008, the Federal Open Market Committee (FOMC) lowered its policy rate—federal funds rate—rapidly to the near-zero level. When the policy rate is in the vicinity of the zero lower bound, the substitutability between money and bonds becomes very high. In such circumstances, further monetary easing becomes ineffective since the opportunity cost of holding money is zero and the economic agents start hoarding money—a situation called liquidity trap.

In the presence of a liquidity trap, it is vital to create inflationary expectation to lower the real interest rate. The Fed started to make public statements about its future actions. In particular, it pledged to refrain from increasing the short-term interest rates until certain criteria have been fulfilled.[2]

In December 2008, the Fed started to do forward guidance[3]:

> "*[...] the Committee anticipates that weak economic conditions are likely to warrant exceptionally low levels of the federal funds rate* ***for some time****.*"

Three months later, the Fed made quite a similar statement except that the term "some time" was replaced by "an extended period".

[1] In major advanced economies, other central banks have adopted similar policies in the aftermath of the financial crisis.

[2] Apart from lowering the policy rate rapidly to the near-zero level and signaling its intention to keep it lower for a longer period, the Fed has launched several Large Scale Asset Purchasing (LSAP) programs. Besides, fiscal stimulus was carried out along with this aggressive unconventional monetary policy. Eggertsson and Woodford (2003), Eggertsson and Woodford (2004), Eggertsson (2006), and Correia et al. (2013) analyze the use of fiscal policy to reduce saving and to stimulate aggregate demand in the presence of zero lower bound. However, in this thesis, we solely examine how to conduct forward guidance to undo the liquidity trap.

[3] See Federal Reserve Board (2008).

These two announcements can be categorized as “open-ended forward guidance", as no date to exit from the “low levels of the federal funds rate" was mentioned explicitly. The open-ended forward guidance gives central bank more flexibility to react to unforeseen events. However, the efficacy of creating inflationary expectation is then substantially reduced, since the public expects the central bank to raise the interest rate as soon as the economy recovers.
In August 2011, the Fed started to do forward guidance with a more specific outlook[4]:

> “*The Committee currently anticipates that economic conditions—including low rates of resource utilization and a subdued outlook for inflation over the medium run—are likely to warrant exceptionally low levels for the federal funds rate* ***at least through mid-2013***."

In January 2012, the expected end date was shifted to late 2014. In September 2012, it was further shifted to mid-2015.
These types of forward guidance with explicit-time span are called “calendar-based forward guidance". Later in 2012, the Fed started forward guidance that is contingent on certain macroeconomic variables—known as “state-contingent forward guidance"[5]:

> “*[...] the Committee decided to keep the target range for the federal funds rate at* 0 *to* $1/4$ *percent and currently anticipates that this exceptionally low range for the federal funds rate will be appropriate at least as long as* ***the unemployment rate remains above*** 6.5 ***percent, inflation between one and two years ahead is projected to be no more than a half percentage point above the Committee's*** 2 ***percent longer-run goal, and longer-term inflation expectations continue to be well anchored***."

One year later, a similar announcement was made to reemphasize that the Fed's effective zero-rate policy would be kept up until certain criteria are fulfilled.
After the financial crisis, announcing the future stance of monetary policy has become a common component in the toolbox of central banks. A broad academic and political debate has emerged on the potential and limits of such forward guidance. A challenging phenomenon has attracted particular attention: If an economy is hit by adverse shocks—e.g. by a negative shock to financial intermediation—and the central bank's reaction is constrained by the zero bound on nominal interest rates, such a downturn will cause excessively high costs. Figuring out how the central bank can reduce the economic costs of this downturn and can provide appropriate stimulus for the economy is a major challenge for monetary policy—and the subject of ongoing debate.

[4] See Federal Reserve Board (2011).
[5] See Federal Reserve Board (2012).

Several solutions to this problem have been proposed in the academic literature. Krugman (1998), Eggertsson and Woodford (2003), and recently Werning (2011) have investigated how the commitment to keeping the nominal interest rate at zero for several periods—even beyond the duration in the discretionary solution—can generate favorable tradeoffs between current downturns and a future boom, and can lower the intertemporal costs of adverse shocks.

1.2 Approach and Results

As pointed out in Thornton (2012), central banks' verbal guidance of "lower for longer" tends to have little impact on the efficacy of forward guidance at the zero lower bound due to a lack of credibility since central banks do not incur losses when they deviate from their announcements[6]. Chehal and Trehan (2009) shows that central banks' commitment to keep future policy rates at low levels has a limited impact on the market's expectation and that there is no significant difference between the calendar-based and state-contingent forward guidances. The reason behind this: In the period when the economy recovers, the central banker would bring its accommodative monetary policy stance to an end regardless of his past announcement since the announcement has no binding effect and the loss of reputation does not enter his objective function. Then private parties realize this time-inconsistency in downturns and thus do not expect inflation regardless of the central bank's announcements. In this thesis, we consider a general approach to make forward guidance effective at the zero lower bound by introducing Forward Guidance Contracts (henceforth FGCs) as an alternative and flexible commitment device. In particular, we combine the standard New Keynesian Framework with FGCs to examine the zero-bound problem[7]. These contracts work as follows: Central bankers announce their policy rate for a particular time frame. The central bankers' intertemporal utility is made dependent on the accuracy of this forecast. For example, their pay, pension or the length of their term could monotonically decrease with the size of the deviation of the actual interest-rate choice from the forecast. Utility losses could also occur when scrupulous bankers are appointed who are intrinsically reluctant to deviate from their own forecasts and thus suffer utility losses when they do indeed deviate.

The gist of our model is that FGCs create partial commitment. Central bankers will try to stick to the forecast but still deviate to some extent if future developments make such

[6] One example is the Bank of Japan. After lowering its policy rate to 0.15%, the Bank of Japan announced in April 1999 that the rate would be maintained at this level "until deflationary concerns are dispelled". However, the rate was soon raised against the backdrop of a weak economy (inflation at around -0.5% and the output gap still in the negative territory). See Shirai (2013).

[7] See Eggertsson and Woodford (2003) and Eggertsson (2003).

a commitment too costly. We show that repeated short-term FGCs can yield favorable tradeoffs between the efficacy of forward guidance in helping to jump-start the economy and a reduction of flexibility in responding to future developments.

At a more specific level, our results are as follows: First, we integrate simple renewable FGCs offered by the government into the New Keynesian Framework and provide a microfoundation of these contracts. Second, we characterize and analyze optimal FGCs when the government commits to using such contracts in downturns. Under these contracts, the central banker sets interest rates in a downturn at zero and sets interest rates immediately after the downturn at levels lower than the ones he would set under discretion. The induced higher levels of inflation and output at the beginning of the future boom feed back into higher current output and inflation. Third, we characterize the contracts that the government chooses when it decides in each period whether to offer FGCs or not. Two insights are central. On the one hand, short-term renewable FGCs can achieve a large fraction of the possible welfare gains and long-term contracts are typically undesirable. If, however, the natural real interest-rate shock is extremely severe, renewable longer-term contracts can further improve welfare. On the other hand, the government may not be able to commit to repeatedly using FGCs. However, in the numerical specification the inability of governments to commit to using short-term contracts has no welfare costs.

Fourth, we characterize FGCs that yield welfare gains for an entire range of negative natural real interest-rate shocks when the contract parameters have to be chosen under a veil of uncertainty about such shocks. Typically, the optimal intensity of central bankers' incentives to stick to their forecasts is moderate in such circumstances. Fifth, we consider an alternative contractual environment in which FGCs are signed at the beginning of a given period t, become effective immediately, last two periods, and do not constrain the interest-rate forecast in the contract. Such contracts can achieve welfare gains similar to one-period contracts. In addition, they can easily be extended to longer-term contracts, which can further improve welfare if the natural real interest-rate shock is extremely severe. Last but not least, state-contingent FGCs can achieve welfare gains that are similar to the simple renewable FGCs, while state-contingent FGCs exempt the government from re-signing the contract repeatedly in downturns.

1.3 Literature

FGCs are a new type of contract for central bankers. They are related to earlier and recent literature. Walsh (1995)[8] proposes incentive contracts for central bankers and

[8] The theory of incentive contracts was further developed in the influential papers by Persson and Tabellini (1993), Beetsma and Jensen (1998), Beetsma and Jensen (1999), Jensen (1997), Lockwood (1997), and Svensson (1997).

shows that such contracts can eliminate the inflation bias and can induce socially desirable shock stabilization when central bankers face a classic time-inconsistency problem. Gersbach and Hahn (2014) show that making deviations from inflation forecast costly for central bankers can improve welfare in a standard New Keynesian Framework with a time-inconsistency problem due to the so-called stabilization bias.

In this thesis we examine FGCs in which the central bankers' utility is contingent on the accuracy of their own forecast regarding their future policy choices. Our contribution to the literature on contracts for central bankers is twofold. First, we examine FGCs in the New Keynesian Framework with the zero lower bound. The credibility problem is unrelated to the inflation bias or the stabilization bias but a consequence of the possibility of a liquidity trap (see Eggertsson (2006)). Second, we provide a microfoundation of such contracts and analytical solutions for optimal renewable FGCs. We illustrate that such type of contracts can harvest a large fraction of the possible welfare gains. For moderate shocks optimal renewable, short-term FGCs are preferable over long-term contracts. Repeated short-term FGCs can create favorable tradeoffs between the commitment to zero interest rates when the economy is hit by a negative natural real interest-rate shock and the desired flexibility in increasing interest rates when the economy returns to normal levels. However, when the negative natural real interest-rate shock is extreme, longer-term contracts generate large welfare gains.

This study belongs to a recent strand of the literature on the benefits and costs of forward guidance and the optimal way of implementing it. Woodford (2012), Campbell (2008) and Gersbach and Hahn (2011) stress the social value of publishing central bank interest rate projections, and Campbell et al. (2012) and Gurkaynak et al. (2005) find that policy inclinations about the forward path of interest rates reveal information and can affect market expectations.[9] Bodenstein et al. (2012) show that, with imperfect credibility captured by a probability of discarding promises, central banks could achieve considerable welfare gains. However, the credibility of the U.S. Federal Reserve and the Swedish Riksbank has been low in the aftermath of 2008 crisis. Lim and Goodhart (2011) are critical of forward guidance, arguing that it may have little impact on expectations. We add to this literature by proposing to implement forward guidance in the form of FGCs because these contracts make forward guidance credible and therefore effective in influencing expectations. In addition, we discuss which type of FGCs may help in jump-starting an economy.

While we focus on FGCs with a microfoundation on how variation of payments, pension or the length of the term of central banks can be used to motivate central bankers to commit partially, other proposals have been made that work through the central bank balance

[9] Mirkov and Natvik (2013) find that central banks may be unwilling to deviate from previous interest-rate projections.

sheet. Krippner and Thornton (2012) suggest that large-scale purchases of interest-rate-derivative contracts could significantly raise the credibility (and ultimately the efficacy) of central bank's guidance, as central bank would incur great capital loss from breaking its early promise. Levin et al. (2010) have suggested that large-scale asset purchase programs may increase the commitment power of forward guidance. As long as those measures affect the utility of central bankers, our framework could be applied for such proposals.

1.4 Organization of the Thesis

This thesis is organized as follows: In the next chapter we present the micro-foundations of the model used in this thesis. Simple renewable FGCs are presented in Chapter 3. In Chapter 4 we investigate longer-term FGCs. State-contingent FGCs are studied in Chapter 5.

Chapter 2: Microfoundation Chapter 2 presents the micro-foundations of the model and in particular a foundation of the central banker's utility function when the government offers him a wage contract composed of a fixed wage and a variable component increasing with the accuracy of the interest-rate forecast.

Chapter 3: Simple Forward Guidance Contracts[10] Chapter 3 studies the properties of simple renewable Forward Guidance Contracts and characterizes the contracts that the government wants to offer repeatedly. These contracts create favorable tradeoffs between the efficacy of forward guidance at the zero bound and the reduced flexibility in reacting to future events. In addition, we discuss which type of Forward Guidance Contracts can be used when there is uncertainty about natural real interest-rate shocks, a situation which typically calls for moderate incentive intensity.

Chapter 4: Longer-term Forward Guidance Contracts Long-term contracts are explored in Chapter 4 in an alternative contractual environments. It shows that when the size of the shock is severe, longer-term contracts could lower social losses further compared to short-term contracts. Severe natural real interest-rate shocks require large incentive intensities with long contract durations to mitigate deflation and output collapse in downturns. While those contracts can yield even lower social losses in such circumstances, they also constrain the central bank for a long time and may thus be problematic, as unforeseen events requiring greater flexibility may occur in the interim.

[10] This chapter is based on joint research with Hans Gersbach and Volker Hahn and was published as a CESifo working paper (see Gersbach et al. (2015)).

Chapter 5: State-contingent Forward Guidance Contracts The last part of the thesis deals with FGCs that are contingent on macroeconomic variables. In such contracts, the central bankers' remuneration loss, which occurs if they deviate from the forecasts, would itself depend on macroeconomic variables such as the natural real interest rate and inflation expectation, for instance. We show that stage-contingent FGCs can achieve the same welfare gains as calendar-based FGCs while the government does not have to re-sign the same contracts repeatedly in downturns.

2 Microfoundation

In this chapter, we provide the microfoundation of our model as the reference framework for the remainder of the thesis and for the IS (Investment/Saving) Equation, Phillips Curve and the intertemporal social losses in particular. This approach has been extensively studied in the past decades, e.g. in Woodford (2003) and Eggertsson (2005). Nevertheless, we provide a detailed account of all necessary microfoundations. In particular, we derive the objective function of the central bank in the presence of an incentive contract.

2.1 Set-up

There is a continuum of identical infinitely-lived households, which allows to focus on a representative household. The representative household can buy one-period riskless government bonds and it owns firms. There is a continuum of firms indexed by $i \in [0, 1]$, which produce differentiated goods. The index identifies each firm and the variety of good it produces. In period t ($t = 0, 1, 2, \cdots$), firm i employs labor $N_t(i)$, produces $Y_t(i)$ of variety i, sets price $P_t(i)$ and pays wage $W_t(i)$. As labor is homogeneous, all firms pay the same wage. The prevailing wage in the labor market in period t is denoted by W_t. Total labor supply is denoted by N_t.

The utility function of the representative household in a particular period depends on four arguments: aggregate consumption C_t, real money holdings $\frac{M_t}{P_t}$, labor supply N_t and government consumption G_t. The Dixit-Stiglitz index for aggregate consumption is given by $C_t \equiv (\int_0^1 C_t(i)^{\frac{\theta-1}{\theta}} di)^{\frac{\theta}{\theta-1}}$, with $C_t(i)$ denoting the consumption of differentiated good i in period t and $\theta > 1$ denoting the elasticity of substitution between differentiated goods. The aggregate price index is defined by $P_t \equiv (\int_0^1 P_t(i)^{1-\theta} di)^{\frac{1}{1-\theta}}$.

In period t, the expected intertemporal utility is given by

$$\mathbb{E}_t \sum_{j=0}^{\infty} \beta^j U_{t+j} = \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \left(u(C_{t+j}, \frac{M_{t+j}}{P_{t+j}}, \xi_{t+j}) - \int_0^1 v(N_{t+j}(i), \xi_{t+j}) di + g(G_{t+j}, \xi_{t+j}) \right). \tag{2.1}$$

$u(\cdot)$ is increasing and concave in consumption C_t and is increasing and concave in real money holding $\frac{M_t}{P_t}$ up to a satiation level. $v(\cdot)$ is increasing and convex in labor supply N_t. $g(\cdot)$ is increasing and concave in government consumption G_t.

The budget constraint amounts to[1]

$$\begin{aligned}&\int_0^1 P_t(i)C_t(i)di + B_t + M_t \\ &\leq (1+i_{t-1})B_{t-1} + (1+i_{t-1}^M)M_{t-1} + \int_0^1 N_t(i)W_t di + \int_0^1 Z_t(i)di - P_tT_t,\end{aligned} \tag{2.2}$$

where $\beta \in (0,1)$ is the discount factor and $\xi_t = (\xi_t^C, \xi_t^M, \xi_t^N, \xi_t^G, \xi_t^O)$ is a vector of shocks, where ξ_t^C, ξ_t^M, ξ_t^N, ξ_t^G and ξ_t^O are exogenous shocks on consumption, money demand, labor supply, government consumption and other factors, respectively. $Z_t(i)$ denotes the profit of firm i. T_t are real lump-sum taxes. i_t is the nominal interest rate on the one-period riskless government bond whose stock is given by B_t. i_t^M is the nominal interest rate on the monetary base M_t. We follow the literature in setting i_t^M to zero (see e.g. Walsh (2003)).

2.2 The Representative Household's Problem

We first derive the optimal consumption bundle of the representative household[2]. We maximize aggregate consumption for a given expenditure level, denoted by Ω_t.

$$\max_{\{C_t(i)\}_{i=0}^1} C_t = \left(\int_0^1 C_t(i)^{\frac{\theta-1}{\theta}} di\right)^{\frac{\theta}{\theta-1}}$$

s.t.

$$\int_0^1 P_t(i)C_t(i)di \leq \Omega_t.$$

We write the corresponding Lagrangian as

$$\mathcal{L} = \left(\int_0^1 C_t(i)^{\frac{\theta-1}{\theta}} di\right)^{\frac{\theta}{\theta-1}} - \eta_t \left(\int_0^1 P_t(i)C_t(i)di - \Omega_t\right), \tag{2.3}$$

where η_t is the Lagrange multiplier.
The first-order condition with respect to $C_t(i)$ is given by

$$C_t(i)^{-1/\theta} C_t^{1/\theta} = \eta_t P_t(i).$$

Rearranging terms yields

$$C_t(i) = C_t(j) \left(\frac{P_t(i)}{P_t(j)}\right)^{-\theta} \tag{2.4}$$

[1] The budget constraint is the same as the one on page 66 in Woodford (2003) and the one on page 234 in Walsh (2003). The induced intertemporal budget constraint is also equivalent to the one in Eggertsson (2005).
[2] The derivation is similar to the one in Galí (2008).

for two differentiated goods i and j.

Note that the elasticity of substitution is constant and equal to

$$\theta = -\frac{d \ln \frac{C_t(i)}{C_t(j)}}{d \ln \frac{P_t(i)}{P_t(j)}}.$$

With the help of $\Omega_t = \int_0^1 P_t(i)C_t(i)di$, we obtain

$$C_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\theta} \frac{\Omega_t}{P_t} \tag{2.5}$$

for each differentiated good $i \in [0, 1]$.

Inserting Equation (2.5) into the definition of Dixit-Stiglitz aggregate consumption yields

$$C_t = \left(\int_0^1 \left(\left(\frac{P_t(i)}{P_t}\right)^{-\theta} \frac{\Omega_t}{P_t}\right)^{\frac{\theta-1}{\theta}} di\right)^{\frac{\theta}{\theta-1}} = \Omega_t P_t^{\theta-1} \left(\int_0^1 P_t(i)^{1-\theta} di\right)^{\frac{\theta}{\theta-1}} = \Omega_t P_t^{-1}. \tag{2.6}$$

Substituting Equation (2.6) in Equation (2.5), finally yields

$$C_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\theta} C_t. \tag{2.7}$$

Equation (2.6) implies

$$P_t C_t = \int_0^1 P_t(i)C_t(i)di.$$

Thus, the budget constraint of the representative household can be rewritten as

$$P_t C_t + B_t + M_t \leq (1 + i_{t-1})B_{t-1} + M_{t-1} + \int_0^1 N_t(i)W_t di + \int_0^1 Z_t(i)di - P_t T_t. \tag{2.8}$$

The Lagrangian of the representative household's problem is

$$\begin{aligned} \mathcal{L} =& \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \{[u(C_{t+j}, \frac{M_{t+j}}{P_{t+j}}, \xi_{t+j}) - \int_0^1 v(N_{t+j}(i), \xi_{t+j})di + g(G_{t+j}, \xi_{t+j})] \\ & - \lambda_{t+j}[P_{t+j}C_{t+j} + B_{t+j} + M_{t+j} - (1 + i_{t+j-1})B_{t+j-1} - M_{t+j-1} \\ & - \int_0^1 N_{t+j}(i)W_{t+j}di - \int_0^1 Z_{t+j}(i)di + P_{t+j}T_{t+j}]\}, \end{aligned} \tag{2.9}$$

where λ_{t+j} is the Lagrange multiplier associated with the budget constraint in period $t+j$.

Differentiating the Lagrangian with respect to C_t, C_{t+1}, $\frac{M_t}{P_t}$, $N_t(i)$ and B_t yields

$$u_C(C_t, \frac{M_t}{P_t}, \xi_t) = \lambda_t P_t, \tag{2.10}$$

$$\mathbb{E}_t u_C(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, \xi_{t+1}) = \lambda_{t+1} P_{t+1}, \tag{2.11}$$

$$u_{M/P}(C_t, \frac{M_t}{P_t}, \xi_t) = \lambda_t P_t - \lambda_{t+1} \beta P_t, \tag{2.12}$$

$$v_{N(i)}(N_t(i), \xi_t) = \lambda_t W_t, \tag{2.13}$$

and

$$\lambda_t = \lambda_{t+1} \beta (1 + i_t). \tag{2.14}$$

As explained in the following, the first-order conditions can be combined into three relationships. Combining Equations (2.10), (2.11) and (2.14) leads to the IS Curve

$$\frac{1}{1+i_t} = \mathbb{E}_t[\frac{\beta u_C(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, \xi_{t+1})}{u_C(C_t, \frac{M_t}{P_t}, \xi_t)} \frac{P_t}{P_{t+1}}]. \tag{2.15}$$

Combining Equations (2.10), (2.12) and (2.14) yields the LM (Liquidity/Money) Equation

$$\frac{i_t}{1+i_t} = \frac{u_{M/P}(C_t, \frac{M_t}{P_t}, \xi_t)}{u_C(C_t, \frac{M_t}{P_t}, \xi_t)}. \tag{2.16}$$

Combining Equations (2.10) and (2.13) results in an equation that implicitly describes the labor supply,

$$W_t = \frac{P_t v_{N(i)}(N_t(i), \xi_t)}{u_C(C_t, \frac{M_t}{P_t}, \xi_t)}. \tag{2.17}$$

We further note that Equation (2.16) implies

$$i_t \geq 0, \tag{2.18}$$

where equality applies at the satiation point of the money holding utility, i.e. when $u_{M/P}(C_t, \frac{M_t}{P_t}, \xi_t) = 0$.

We summarize the optimal behavior of the representative household as follows[3]. Its choice of the consumption path evolution is described by the IS Equation in (2.15). For a given amount of aggregate consumption C_t, the household chooses the consumption bundle according to equation (2.7). The LM Equation in (2.16) describes real money

[3] See pages 146-147 in Woodford (2003).

holdings in each period. Finally, for a given wage W_t, the aggregate labor supply for variety i, $N_t(i)$, is given by Equation (2.17).

2.3 Firms and Goods Market Clearing

Firms production function is

$$Y_t(i) = A_t N_t(i), \tag{2.19}$$

where A_t is the technology level that evolves exogenously.
Here we neglect physical capital and consider labor $N_t(i)$ as the only factor of production for the differentiated good i.
Variety i produced by the monopolistic firm i according to the Dixit-Stiglitz demand function[4] (2.7) is given by

$$Y_t(i) = (\frac{P_t(i)}{P_t})^{-\theta} Y_t. \tag{2.20}$$

As in Eggertsson (2005), firms have a price-changing cost denoted by $\gamma(\frac{P_t}{P_{t-1}})$. We assume a closed economy and thus the goods market clearing condition is $Y_t = C_t + F_t + \gamma(\frac{P_t}{P_{t-1}})$, where Y_t is the aggregate output and F_t denotes total government spending.

2.4 Government

We assume a tax-collecting cost $s(T_t)$, where $s'(\cdot) > 0$ and $s''(\cdot) \geq 0$. Thus, the government's total consumption is $F_t = G_t + s(T_t)$.
The government finances its total spending F_t by levying lump-sum taxes T_t and issuing B_t one-period riskless government bonds.
The government's total liabilities at the end of period t are

$$D_t = B_t + M_t.$$

By purchasing government bonds via open market operations, the central bank controls the nominal interest rate on government bonds.
The government's budget constraint is

$$B_t + M_t + P_t T_t = (1 + i_{t-1}) B_{t-1} + M_{t-1} + P_t G_t + P_t s(T_t).$$

Rearranging yields

$$D_t = (1 + i_{t-1}) D_{t-1} + P_t F_t - P_t T_t - i_{t-1} M_{t-1},$$

[4] See page 151 in Woodford (2003).

where $i_{t-1}M_{t-1}$ stands for the seigniorage revenues.

2.5 Transversality Conditions

The stochastic discount factor is defined by

$$Q_{t,t+j} = \mathbb{E}_t \prod_{k=0}^{j-1} \frac{1}{1+i_{t+k}}. \tag{2.21}$$

Combining Equations (2.15) and (2.21) yields

$$Q_{t,t+j} = \mathbb{E}_t[\beta^j \frac{u_C(C_{t+j}, \frac{M_{t+j}}{P_{t+j}}, \xi_{t+j})}{u_C(C_t, \frac{M_t}{P_t}, \xi_t)} \frac{P_t}{P_{t+j}}]. \tag{2.22}$$

Accordingly, the transversality condition for the government is[5]

$$\lim_{j\to\infty} \mathbb{E}_t[Q_{t,t+j} D_{t+j}] = 0. \tag{2.23}$$

We assume that the government's budget is balanced such that B_t does not grow at a rate that would violate the transversality condition.

The transversality condition for the household amounts to[6]

$$\lim_{j\to\infty} \mathbb{E}_t[Q_{t,t+j}((1+i_{t+j-1})B_{t+j-1} + M_{t+j-1})] = 0. \tag{2.24}$$

In addition, we rule out Ponzi schemes:[7]

$$\mathbb{E}_t \left(\sum_{j=0}^{\infty} Q_{t,t+j} (\int_0^1 N_{t+j}(i) W_{t+j}(i) di + \int_0^1 Z_{t+j}(i) di) \right) < \infty. \tag{2.25}$$

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[5] See page 72 in Woodford (2003).

[6] See page 70 in Woodford (2003).

[7] See pages 67-68 in Woodford (2003). The representative household's wealth at the end of period $t-1$ is $(1+i_{t-1})B_{t-1} + M_{t-1}$. In perfect financial markets, households can borrow against the present value of all future incomes $\sum_{j=0}^{\infty} Q_{t,t+j}(\int_0^1 N_{t+j}(i)W_{t+j}(i)di + \int_0^1 Z_{t+j}(i)di)$. The representative household's wealth satisfies the lower bound $(1+i_{t-1})B_{t-1} + M_{t-1} \geq -\sum_{j=0}^{\infty} Q_{t,t+j}(\int_0^1 N_{t+j}(i)W_{t+j}(i)di + \int_0^1 Z_{t+j}(i)di)$. To prevent households from unbounded consumption, the present value of all future incomes must converge to some positive value, which is described by Equation (2.25).

2.6 Utility Functions

We specify the household's utility function as follows:[8]

$$u(C_t, \frac{M_t}{P_t}, \xi_t) = \frac{C_t^{1-\sigma}(\xi_t^C)^\sigma}{1-\sigma} + \frac{(M_t/P_t)^{1-\nu}(\xi_t^M)^\nu}{1-\nu},$$

where σ, ν are relative risk aversion coefficients. Note that we assume a household utility with additive separability of C_t and $\frac{M_t}{P_t}$.

The disutility from working is specified as follows:

$$v(N_t(i), \xi_t) = \frac{N_t(i)^{1+\phi}(\xi_t^N)^{-\phi}}{1+\phi},$$

where $\phi > 0$.

With these specifications, the three central relationships (2.15), (2.16) and (2.17) take the following form:

$$\frac{1}{1+i_t} = \mathbb{E}_t[\beta(\frac{C_{t+1}}{C_t})^{-\sigma}(\frac{\xi_{t+1}^C}{\xi_t^C})^\sigma \frac{P_t}{P_{t+1}}], \tag{2.26}$$

$$\frac{i_t}{1+i_t} = C_t^\sigma(\frac{M_t}{P_t})^{-\nu}(\xi_t^M)^\nu(\xi_t^C)^{-\sigma}, \tag{2.27}$$

and

$$N_t = N_t(i) = \left(\frac{W_t(\xi_t^C)^\sigma}{P_t C_t^\sigma}\right)^{\frac{1}{\phi}} \xi_t^N. \tag{2.28}$$

Equation (2.22) can be written as

$$Q_{t,t+j} = \mathbb{E}_t[\beta^j(\frac{C_{t+j}}{C_t})^{-\sigma}(\frac{\xi_{t+j}^C}{\xi_t^C})^\sigma \frac{P_t}{P_{t+j}}]. \tag{2.29}$$

2.7 Log-linearization around the Steady State

In the absence of shocks, the system exhibits a unique steady state. For any variable O_t, we can express the percentage deviation[9] of O_t from its steady-state level $\bar{O}$ in terms of $\hat{o}_t = o_t - \bar{o}$, where $o_t = \ln O_t$ and $\bar{o} = \ln \bar{O}$.

[8] The utility function is essentially the same as in Eggertsson (2005). There are notational differences, as we use σ instead of σ^{-1} and ξ_t^C and ξ_t^N instead of u and q. Eggertsson assumes that the money holding term in the utility function is small and neglects it.

[9] Note that $\hat{o}_t = o_t - \bar{o} = \ln \frac{O_t}{\bar{O}} = \ln(1 + \frac{O_t - \bar{O}}{\bar{O}}) \approx \frac{O_t - \bar{O}}{\bar{O}}$ represents the percentage deviation.

Real marginal costs are given by

$$MC_t = \frac{W_t}{A_t P_t} = \frac{C_t^{\sigma} N_t^{\phi} (\xi_t^C)^{-\sigma} (\xi_t^N)^{-\phi}}{A_t}. \tag{2.30}$$

Taking logs leads to

$$mc_t = \sigma c_t + \phi n_t - a_t - \phi \xi_t^n - \sigma \xi_t^c, \tag{2.31}$$

where $\xi_t^n = \ln \xi_t^N$ and $\xi_t^c = \ln \xi_t^C$ and ξ^n and ξ^c are the respective steady-state levels.

Due to $\bar{mc} = \sigma \bar{c} + \phi \bar{n} - \bar{a} - \phi \xi^n - \sigma \xi^c$ in the steady state, we can write Equation (2.31) as

$$\hat{mc}_t = \sigma \hat{c}_t + \phi \hat{n}_t - \hat{a}_t - \phi \hat{\xi}_t^n - \sigma \hat{\xi}_t^c. \tag{2.32}$$

One caveat is in order. While linearizing the non-linear model helps solve analytically and helps understand the economic models, the linear approximations can be inaccurate and should be used with caution. For example, peculiar phenomena occur if the economic system deviates too far from the steady state. One example is shown in Appendix B.1.1.

2.8 Phillips Curve

We follow Rotemberg (1982) and assume each firm i bears a cost $\gamma(\frac{P_t(i)}{P_{t-1}(i)})$ when it changes prices from period $t-1$ to period t, where $\gamma'(0) = 0$ and $\gamma''(0) > 0$. The property $\gamma'(0) = 0$ implies that the cost of price changes is only of second order in the neighborhood of zero inflation.

Firm i's profit in period t is

$$Z_t(i) = (1+\chi)(\frac{P_t(i)}{P_t})^{-\theta} Y_t P_t(i) - N_t(i) W_t(i) - P_t \gamma(\frac{P_t(i)}{P_{t-1}(i)}),$$

where χ is a subsidy financed by the lump-sum taxes.

Each firm chooses the sequence of prices $P_{t+j}(i)$ to maximize

$$\sum_{j=0}^{\infty} Q_{t,t+j} Z_{t+j}(i). \tag{2.33}$$

The expected value of this expression can be rewritten as follows:

$$\begin{aligned}
&\mathbb{E}_t \sum_{j=0}^{\infty} Q_{t,t+j} Z_{t+j}(i) \\
&= \mathbb{E}_t \sum_{j=0}^{\infty} Q_{t,t+j}[(1+\chi)(\frac{P_{t+j}(i)}{P_{t+j}})^{-\theta} Y_{t+j} P_{t+j}(i) - N_{t+j}(i) W_{t+j}(i) - P_{t+j}\gamma(\frac{P_{t+j}(i)}{P_{t+j-1}(i)})] \\
&= \mathbb{E}_t \sum_{j=0}^{\infty} Q_{t,t+j}[(1+\chi)(\frac{P_{t+j}(i)}{P_{t+j}})^{-\theta} Y_{t+j} P_{t+j}(i) - \frac{Y_{t+j}(i)}{A_{t+j}} \frac{P_{t+j} v_{N(i)}(N_{t+j}(i), \xi_{t+j})}{u_C(C_{t+j}, \frac{M_{t+j}}{P_{t+j}}, \xi_{t+j})} - P_{t+j}\gamma(\frac{P_{t+j}(i)}{P_{t+j-1}(i)})].
\end{aligned} \tag{2.34}$$

The first-order condition with respect to $P_t(i)$, for given P_t and Y_t, is given by

$$\begin{aligned}
&(1-\theta)(1+\chi) Y_t (\frac{P_t}{P_t(i)})^{\theta} + \theta \frac{Y_t}{A_t} (\frac{P_t}{P_t(i)})^{\theta} \frac{P_t v_{N(i)}(N_t(i), \xi_t)}{P_t(i) u_C(C_t, \frac{M_t}{P_t}, \xi_t)} \\
&- \frac{P_t}{P_{t-1}(i)} \gamma'(\frac{P_t(i)}{P_{t-1}(i)}) + \mathbb{E}_t[\frac{\beta u_C(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, \xi_{t+1})}{u_C(C_t, \frac{M_t}{P_t}, \xi_t)} \frac{P_t P_{t+1}(i)}{P_t(i)^2} \gamma'(\frac{P_{t+1}(i)}{P_t(i)})] = 0
\end{aligned} \tag{2.35}$$

Every firm adjusts its price in each period. As all the firms face the same optimization problem, we have $P_t(i) = P_t(j) = P_t$, $Y_t(i) = Y_t(j) = Y_t$, $W_t(i) = W_t(j) = W_t$ and $N_t(i) = N_t(j) = N_t \; \forall i, j$.

Taking into account $P_t(i) = P_t$ and multiplying both sides of Equation (2.35) by $u_C(C_t, \frac{M_t}{P_t}, \xi_t)$, yields

$$\begin{aligned}
&(1-\theta)(1+\chi) Y_t u_C(C_t, \frac{M_t}{P_t}, \xi_t) + \theta \frac{Y_t}{A_t} v_{N(i)}(N_t(i), \xi_t) \\
&- \frac{P_t}{P_{t-1}} \gamma'(\frac{P_t}{P_{t-1}}) u_C(C_t, \frac{M_t}{P_t}, \xi_t) + \mathbb{E}_t[\beta u_C(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, \xi_{t+1}) \frac{P_{t+1}}{P_t} \gamma'(\frac{P_{t+1}}{P_t})] = 0,
\end{aligned} \tag{2.36}$$

which is the “New Keynesian" Phillips Curve as in Eggertsson (2005).

Further following Eggertsson (2005), the cost of price adjustment is assumed to take the form of $\gamma(\pi_t) = \gamma_1(\Pi_t - 1)^2 = \gamma_1 \pi_t^2$, where $\Pi_t = \frac{P_t}{P_{t-1}}$ and $\pi_t = \Pi_t - 1$.

Using $Y_t(i) = A_t N_t(i)$ and $\gamma(\pi_t) = \gamma_1 \pi_t^2$, the Phillips Curve becomes

$$\begin{aligned}
&(1-\theta)(1+\chi) Y_t C_t^{-\sigma} (\xi_t^C)^{\sigma} + \theta (\frac{Y_t}{A_t})^{1+\phi} (\xi_t^N)^{-\phi} \\
&- 2\gamma_1 \frac{P_t}{P_{t-1}} (\frac{P_t}{P_{t-1}} - 1) C_t^{-\sigma} (\xi_t^C)^{\sigma} + \mathbb{E}_t[\beta 2 \gamma_1 C_{t+1}^{-\sigma} (\xi_{t+1}^C)^{\sigma} \frac{P_{t+1}}{P_t} (\frac{P_{t+1}}{P_t} - 1)] = 0.
\end{aligned} \tag{2.37}$$

Equation (2.37) can be rewritten in exponential form[10],

$$(1-\theta)(1+\chi)e^{y_t-\sigma c_t+\sigma\xi_t^c}+\theta e^{(1+\phi)(y_t-a_t)-\phi\xi_t^n}-2\gamma_1 e^{2\pi_t-\sigma c_t+\sigma\xi_t^c}$$
$$+2\gamma_1 e^{\pi_t-\sigma c_t+\sigma\xi_t^c}+\beta 2\gamma_1 e^{-\sigma\mathbb{E}_t c_{t+1}+\sigma\mathbb{E}_t\xi_{t+1}^c+2\mathbb{E}_t\pi_{t+1}}-\beta 2\gamma_1 e^{-\sigma\mathbb{E}_t c_{t+1}+\sigma\mathbb{E}_t\xi_{t+1}^c+\mathbb{E}_t\pi_{t+1}}=0, \quad (2.38)$$

where $c_t = \ln C_t$, $y_t = \ln Y_t$, $a_t = \ln A_t$ and $c_{t+1} = \ln C_{t+1}$.

The first-order Taylor expansion yields

$$(1-\theta)(1+\chi)e^{\bar{y}-\sigma\bar{c}+\sigma\xi^c}(1+\hat{y}_t-\sigma\hat{c}_t+\sigma\hat{\xi}_t^c)+\theta e^{(1+\phi)(\bar{y}-\bar{a})-\phi\xi^n}(1+(1+\phi)(\hat{y}_t-\hat{a}_t)-\phi\hat{\xi}_t^n)$$
$$-2\gamma_1 e^{2\bar{\pi}-\sigma\bar{c}+\sigma\xi^c}(1+2\pi_t-\sigma\hat{c}_t+\sigma\hat{\xi}_t^c)+2\gamma_1 e^{\bar{\pi}-\sigma\bar{c}+\sigma\xi^c}(1+\pi_t-\sigma\hat{c}_t+\sigma\hat{\xi}_t^c)$$
$$+\beta 2\gamma_1 e^{-\sigma\bar{c}+\sigma\xi^c+2\bar{\pi}}(1-\sigma\mathbb{E}_t\hat{c}_{t+1}+\sigma\mathbb{E}_t\hat{\xi}_{t+1}^c+2\mathbb{E}_t\pi_{t+1})$$
$$-\beta 2\gamma_1 e^{-\sigma\bar{c}+\sigma\xi^c+\bar{\pi}}(1-\sigma\mathbb{E}_t\hat{c}_{t+1}+\sigma\mathbb{E}_t\hat{\xi}_{t+1}^c+\mathbb{E}_t\pi_{t+1})=0. \quad (2.39)$$

Due to our assumption of a zero-inflation steady state ($\bar{\pi}=0$), Equation (2.39) simplifies to

$$(1-\theta)(1+\chi)e^{\bar{y}-\sigma\bar{c}+\sigma\xi^c}(1+\hat{y}_t-\sigma\hat{c}_t+\sigma\hat{\xi}_t^c)+\theta e^{(1+\phi)(\bar{y}-\bar{a})-\phi\xi^n}(1+(1+\phi)(\hat{y}_t-\hat{a}_t)-\phi\hat{\xi}_t^n)$$
$$-2\gamma_1 e^{-\sigma\bar{c}+\sigma\xi^c}\pi_t+\beta 2\gamma_1 e^{-\sigma\bar{c}+\sigma\xi^c}\mathbb{E}_t\pi_{t+1}=0. \quad (2.40)$$

To ensure an efficient level of steady state output, as in Woodford (2003), we assume $1+\chi=\frac{\theta}{\theta-1}$.

Rearranging terms yields

$$\begin{aligned}&1+\hat{y}_t-\sigma\hat{c}_t+\sigma\hat{\xi}_t^c-e^{\sigma\bar{c}+\phi\bar{y}-(1+\phi)\bar{a}-\phi\xi^n-\sigma\xi^c}(1+(1+\phi)(\hat{y}_t-\hat{a}_t)-\phi\hat{\xi}_t^n)\\&+\frac{2\gamma_1 e^{-\bar{y}}}{\theta}\pi_t-\frac{2\gamma_1\beta e^{-\bar{y}}}{\theta}\mathbb{E}_t\pi_{t+1}=0.\end{aligned} \quad (2.41)$$

Without price-changing costs, the first-order condition of firms' maximization problem is

$$1=\frac{C_t{}^\sigma N_t^\phi(\xi_t^C)^{-\sigma}(\xi_t^N)^{-\phi}}{A_t}=\frac{C_t{}^\sigma Y_t^\phi(\xi_t^C)^{-\sigma}(\xi_t^N)^{-\phi}}{A_t^{1+\phi}}.$$

In the steady state, we have

$$1=\frac{\bar{C}^\sigma\bar{Y}^\phi(\xi^C)^{-\sigma}(\xi^N)^{-\phi}}{\bar{A}^{1+\phi}}.$$

[10] $\Pi_t = 1+\pi_t$ implies $\pi_t \approx \ln\Pi_t$, due to $\Pi_t = e^{\ln\Pi_t} \approx 1+\ln\Pi_t$.

Therefore,

$$\sigma\bar{c} + \phi\bar{y} - (1+\phi)\bar{a} - \phi\xi^n - \sigma\xi^c = 0.$$

Subtracting[11]

$$\bar{Y} = \bar{C} + F$$

from

$$Y_t = C_t + F + \gamma(\pi_t)$$

yields

$$Y_t - \bar{Y} = C_t - \bar{C} + \gamma(\pi_t),$$

where we have used that $\gamma(\pi_t)$ is zero in the steady state.

Using the definitions of $\hat{y}_t$ and $\hat{c}_t$ results in

$$\bar{Y}(e^{\hat{y}_t} - 1) = \bar{C}(e^{\hat{c}_t} - 1) + \gamma(\pi_t).$$

Neglecting terms of order 2 and higher gives

$$\bar{Y}\hat{y}_t = \bar{C}\hat{c}_t,$$

as price changing costs $\gamma(\pi_t)$ have only second-order effects around the steady state. By normalizing $\bar{Y} = 1$, we obtain

$$\hat{y}_t = \bar{C}\hat{c}_t.$$

By using $\hat{y}_t = \bar{C}\hat{c}_t$, we simplify Equation (2.41) as follows:

$$(\tilde{\sigma} + \phi)(\hat{y}_t - \hat{y}_t^p) = \frac{2\gamma_1}{\theta}\pi_t - \frac{2\gamma_1\beta}{\theta}\mathbb{E}_t\pi_{t+1}, \tag{2.42}$$

where $y_t^p = \frac{1}{\tilde{\sigma}+\phi}[(1+\phi)a_t + \phi\xi_t^n + \sigma\xi_t^c]$ is the (log) potential output in the absence of price stickness and $\tilde{\sigma} = \frac{\sigma}{\bar{C}}$.

Finally, we obtain the Phillips Curve

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} + \kappa x_t, \tag{2.43}$$

where $\kappa = \frac{(\tilde{\sigma}+\phi)\theta}{2\gamma_1}$ and $x_t = \hat{y}_t - \hat{y}_t^p$ is the (log) output gap.

[11] As we concentrate on monetary policy, we assume constant fiscal policy, i.e. $G_t = G$ and $T_t = T$. Thus, the government's total spending is constant, i.e. $F_t = F$.

2.9 IS Equation

Taking the logs of the IS Equation (2.26)[12] leads to

$$-\ln(1+i_t) = \ln\beta - \sigma\mathbb{E}_t \ln C_{t+1} + \sigma \ln C_t - \mathbb{E}_t\pi_{t+1} + \sigma(\mathbb{E}_t\xi^c_{t+1} - \xi^c_t). \qquad (2.44)$$

We can rewrite the above equation as

$$c_t = \mathbb{E}_t c_{t+1} - \frac{1}{\sigma}(i_t - \mathbb{E}_t\pi_{t+1}) + \rho_t, \qquad (2.45)$$

where $\rho_t = -\frac{\ln\beta}{\sigma} - (\mathbb{E}_t\xi^c_{t+1} - \xi^c_t)$ is the demand shock.

We can rewrite Equation (2.45) as

$$\hat{c}_t = \mathbb{E}_t\hat{c}_{t+1} - \frac{1}{\sigma}[i_t - \mathbb{E}_t\pi_{t+1} - (\sigma\hat{\xi}^c_t - \sigma\mathbb{E}_t\hat{\xi}^c_{t+1} - \ln\beta)]$$

and by using $\hat{y}_t = \bar{C}\hat{c}_t$ as

$$\hat{y}_t = \mathbb{E}_t\hat{y}_{t+1} - \frac{1}{\tilde{\sigma}}[i_t - \mathbb{E}_t\pi_{t+1} - (\sigma\hat{\xi}^c_t - \sigma\mathbb{E}_t\hat{\xi}^c_{t+1} - \ln\beta)].$$

We define the natural real interest rate by

$$\begin{aligned} r_t &= \tilde{\sigma}(\mathbb{E}_t\hat{y}^p_{t+1} - \hat{y}^p_t) + \sigma\hat{\xi}^c_t - \sigma\mathbb{E}_t\hat{\xi}^c_{t+1} - \ln\beta \\ &= \frac{\tilde{\sigma}}{\tilde{\sigma}+\phi}\mathbb{E}_t[(1+\phi)\hat{a}_{t+1} + \phi\hat{\xi}^n_{t+1} - \phi\bar{C}\hat{\xi}^c_{t+1}] - \frac{\tilde{\sigma}}{\tilde{\sigma}+\phi}\mathbb{E}_t[(1+\phi)\hat{a}_t + \phi\hat{\xi}^n_t - \phi\bar{C}\hat{\xi}^c_t] - \ln\beta. \end{aligned} \qquad (2.46)$$

We note that r_t only depends on technology and preference parameters shocks and that it is thus exogenous.

Notice that Equation (2.44) implies that the steady-state level of nominal interest rate is $\bar{i} = \frac{1}{\beta} - 1$.

We can rewrite Equation (2.45) in terms of the output gap,

$$x_t = \mathbb{E}_t x_{t+1} - \frac{1}{\tilde{\sigma}}(i_t - \mathbb{E}_t\pi_{t+1} - r_t), \qquad (2.47)$$

which is the usual expression for the IS Equation.

[12] See e.g. page 244 in Walsh (2003) and page 23 in Woodford (2003).

2.10 LM Curve

We consider the log-linearized money demand equation when the nominal interest i_t is around its steady-state level $\bar{i}$ with $i_t > 0$[13].
Using Equation (2.27) and taking logs, we obtain

$$\ln \frac{i_t}{1+i_t} = \sigma c_t - \nu m_t + \nu p_t + \nu \xi_t^m - \sigma \xi_t^c. \tag{2.48}$$

As we focus on circumstances with comparatively low nominal interest rate, we can write

$$\ln \frac{i_t}{1+i_t} = \ln i_t - \ln(1+i_t) \approx \ln i_t.$$

Using this expression and rearranging Equation (2.48) yields

$$m_t = p_t + \frac{\sigma}{\nu} c_t - \frac{1}{\nu} \ln i_t + \xi_t^m - \frac{\sigma}{\nu} \xi_t^c. \tag{2.49}$$

In the steady state, the equation reads

$$\bar{m} = \bar{p} + \frac{\sigma}{\nu} \bar{c} - \frac{1}{\nu} \ln \bar{i} + \xi^m - \frac{\sigma}{\nu} \xi^c. \tag{2.50}$$

Subtracting Equation (2.50) from Equation (2.49) yields

$$\hat{m}_t = \hat{p}_t + \frac{\tilde{\sigma}}{\nu} \hat{y}_t - \frac{1}{\nu \bar{i}} \hat{i}_t + \hat{\xi}_t^m - \frac{\sigma}{\nu} \hat{\xi}_t^c, \tag{2.51}$$

where $\hat{i}_t = i_t - \bar{i}$ is the deviation of the nominal interest rate from its steady-state level.
We note that money holding is positively associated with the price level, the output level and money demand shocks, and that it negatively depends on the nominal interest rate and consumption shocks.
When the nominal interest rate is non-zero and satisfies Equation (2.51), it can be used as an instrument for monetary policy. The zero lower bound on nominal interest rates is binding if

$$\hat{m}_t \geq \hat{p}_t + \frac{\tilde{\sigma}}{\nu} \hat{y}_t + \hat{\xi}_t^m - \frac{\sigma}{\nu} \hat{\xi}_t^c. \tag{2.52}$$

2.11 Intertemporal Social Losses

We now derive intertemporal social losses.

[13] Money demand is indeterminate when $i_t = 0$, as shown in Equation (2.52).

The government maximizes the household's intertemporal utility function. The household's utility in period t is

$$U_t = u(C_t, \frac{M_t}{P_t}, \xi_t) - v(N_t, \xi_t).$$

The utility from consumption and money holding is given by

$$u(C_t, M_t^r, \xi_t) = \frac{C_t^{1-\sigma}(\xi_t^C)^\sigma}{1-\sigma} + \frac{(M_t^r)^{1-\nu}(\xi_t^M)^\nu}{1-\nu}, \tag{2.53}$$

where $M_t^r = M_t/P_t$ is the real money holding.

The utility from real money holding is assumed to be negligible — cashless limit, see Woodford (1997) — and can thus be neglected.

Due to market clearing $Y_t = C_t + F + \gamma(\pi_t)$, Equation (2.53) can be rewritten as

$$u(C_t, M_t^r, \xi_t) = \frac{(Y_t - F - \gamma(\pi_t))^{1-\sigma}(\xi_t^C)^\sigma}{1-\sigma},$$

where $\xi^C = \bar{C}$, $\bar{\pi} = 0$ and $\bar{Y} = \bar{C} + F$ in the steady state.

The disutility from labor supply is

$$v(N_t, \xi_t) = \frac{N_t^{1+\phi}(\xi_t^N)^{-\phi}}{1+\phi} = \frac{Y_t^{1+\phi}(\xi_t^N)^{-\phi}}{A_t^{1+\phi}(1+\phi)},$$

where $\xi^N = \bar{N} = \bar{Y}/\bar{A}$ in the steady state.

The Taylor expansions of the representative household's utility from consumption and money holding up to the second order yield

$$\begin{aligned} u =& \bar{u} + u_C(\hat{Y}_t - \gamma'(\bar{\pi})\pi_t) + u_{\xi^C}\hat{\xi}_t^C \\ &+ \frac{1}{2}u_{CC}\hat{Y}_t^2 - u_{CC}\gamma'(\bar{\pi})\hat{Y}_t\pi_t + \frac{1}{2}u_{CC}\gamma'(\bar{\pi})^2\pi_t^2 - \frac{1}{2}u_C\gamma''(\bar{\pi})\pi_t^2 \\ &+ \frac{1}{2}u_{\xi^C\xi^C}(\hat{\xi}_t^C)^2 + u_{\xi^C C}\hat{\xi}_t^C\hat{Y}_t - u_{\xi^C C}\hat{\xi}_t^C\gamma'(\bar{\pi})\pi_t. \end{aligned} \tag{2.54}$$

Similarly, the second-order approximation of the representative household's disutility from labor supply is

$$\begin{aligned} v =& \bar{v} + v_Y\hat{Y}_t + v_A\hat{A}_t + v_{\xi^N}\hat{\xi}_t^N \\ &+ \frac{1}{2}v_{YY}\hat{Y}_t^2 + \frac{1}{2}v_{AA}\hat{A}_t^2 + v_{Y\xi^N}\hat{Y}_t\hat{\xi}_t^N + v_{A\xi^N}\hat{A}_t\hat{\xi}_t^N + v_{YA}\hat{Y}_t\hat{A}_t \\ &+ \frac{1}{2}v_{\xi^N\xi^N}(\hat{\xi}_t^N)^2. \end{aligned} \tag{2.55}$$

In the steady state, we have

$$u_C = 1,$$
$$u_{\xi^C} = \frac{\sigma}{1-\sigma},$$
$$u_{CC} = -\frac{\sigma}{\bar{C}},$$
$$u_{C\xi^C} = u_{\xi^C C} = \frac{\sigma}{\bar{C}},$$
$$u_{\xi^C\xi^C} = -\frac{\sigma}{\bar{C}},$$
$$v_Y = \frac{1}{\bar{A}},$$
$$v_A = -\frac{\bar{Y}}{\bar{A}^2},$$
$$v_{\xi^N} = -\frac{\phi}{1+\phi},$$
$$v_{YY} = \frac{\phi}{\bar{A}\bar{Y}},$$
$$v_{AA} = \frac{(\phi+2)\bar{Y}}{\bar{A}^3},$$
$$v_{YA} = v_{AY} = -\frac{(\phi+1)}{\bar{A}^2},$$
$$v_{Y\xi^N} = v_{\xi^N Y} = -\frac{\phi}{\bar{Y}},$$
$$v_{\xi^N A} = v_{A\xi^N} = \frac{\phi}{\bar{A}},$$

and

$$v_{\xi^N\xi^N} = \frac{\phi\bar{A}}{\bar{Y}}.$$

Thus we can rewrite Equations (2.54) and (2.55) as

$$u = \bar{u} + \frac{\sigma}{1-\sigma}\hat{\xi}_t^C - \frac{1}{2}\frac{\sigma}{\bar{C}}(\hat{\xi}_t^C)^2 - \frac{1}{2}\frac{\sigma}{\bar{C}}\hat{Y}_t^2 + (1 + \frac{\sigma}{\bar{C}}\hat{\xi}_t^C)\hat{Y}_t - \gamma_1\pi_t^2, \tag{2.56}$$

and

$$\begin{aligned} v =& \bar{v} - \frac{\phi}{1+\phi}\hat{\xi}_t^N + \frac{1}{\bar{A}}\hat{Y}_t - \frac{\bar{Y}}{\bar{A}^2}\hat{A}_t + \frac{1}{2}\frac{\phi\bar{A}}{\bar{Y}}(\hat{\xi}_t^N)^2 + \frac{1}{2}\frac{\phi}{\bar{A}\bar{Y}}\hat{Y}_t^2 + \frac{1}{2}\frac{(\phi+2)\bar{Y}}{\bar{A}^3}\hat{A}_t^2 \\ & - (1+\phi)\frac{1}{\bar{A}^2}\hat{Y}_t\hat{A}_t - \frac{\phi}{\bar{Y}}\hat{Y}_t\hat{\xi}_t^N + \frac{\phi}{\bar{A}}\hat{A}_t\hat{\xi}_t^N. \end{aligned} \tag{2.57}$$

We normalize the steady state of technology level[14] $\bar{A} = 1$. Recall that the steady-state level of output has been normalized to one, i.e. that $\bar{Y} = 1$.
We use the second-order Taylor expansions of the following expressions:[15]

$$\hat{Y}_t = Y_t - \bar{Y} = Y_t - 1 = e^{\hat{y}_t} - 1 \approx \hat{y}_t + \frac{1}{2}\hat{y}_t^2,$$

$$\hat{Y}_t^2 \approx \bar{Y}^2(\hat{y}_t + \frac{1}{2}\hat{y}_t^2)^2 \approx \hat{y}_t^2,$$

$$\hat{A}_t \approx \hat{a}_t + \frac{1}{2}\hat{a}_t^2,$$

$$\hat{A}_t^2 \approx \hat{a}_t^2,$$

$$\hat{\xi}_t^C \approx \xi^C(\hat{\xi}_t^c + \frac{1}{2}(\hat{\xi}_t^c)^2),$$

$$(\hat{\xi}_t^C)^2 \approx (\xi^C)^2(\hat{\xi}_t^c)^2,$$

$$\hat{\xi}_t^N \approx \xi^N(\hat{\xi}_t^n + \frac{1}{2}(\hat{\xi}_t^n)^2),$$

and

$$(\hat{\xi}_t^N)^2 \approx (\xi^N)^2(\hat{\xi}_t^n)^2.$$

With these expressions and the normalizations, Equations (2.56) and (2.57) can be rewritten as

$$u = \bar{u} + \frac{\sigma}{1-\sigma}\bar{C}\hat{\xi}_t^c + \frac{1}{2}\frac{\sigma^2\bar{C}}{1-\sigma}(\hat{\xi}_t^c)^2 + \frac{1}{2}\frac{\bar{C}-\sigma}{\bar{C}}\hat{y}_t^2 + (1+\sigma\hat{\xi}_t^c)\hat{y}_t - \gamma_1\pi_t^2$$

and

$$v = \bar{v} - \frac{\phi}{1+\phi}\hat{\xi}_t^n + \frac{1}{2}\frac{\phi^2}{1+\phi}(\hat{\xi}_t^n)^2 + \hat{y}_t + \frac{1+\phi}{2}\hat{y}_t^2 - \hat{a}_t + \frac{1+\phi}{2}\hat{a}_t^2 - (1+\phi)\hat{y}_t\hat{a}_t - \phi\hat{y}_t\hat{\xi}_t^n + \phi\hat{a}_t\hat{\xi}_t^n,$$

with $\bar{u} = \frac{\bar{C}}{1-\sigma}$ and $\bar{v} = \frac{1}{1+\phi}$.
Taking the difference and keeping only the first two orders yields

$$u - v = -\gamma_1\pi_t^2 - \frac{1}{2}(\phi+\tilde{\sigma})(\hat{y}_t - \hat{y}_t^p)^2 - \frac{1}{2}\frac{(\tilde{\sigma}-1)(1+\phi)}{\phi+\tilde{\sigma}}(\hat{a}_t - \hat{a}_t^n)^2 - \frac{1}{2}\frac{1-\bar{C}}{(\sigma-1)(\tilde{\sigma}-1)}(\sigma\hat{\xi}_t^c+1)^2, \tag{2.58}$$

where $a_t^n = \frac{\tilde{\sigma}+\phi}{(1+\phi)(\tilde{\sigma}-1)} + \frac{\sigma}{\tilde{\sigma}-1}\xi_t^c - \frac{\phi}{1+\phi}\xi_t^n$ and the (log) potential output $\hat{y}_t^p = \frac{1}{\tilde{\sigma}+\phi}[(1+\phi)\hat{a}_t + \phi\hat{\xi}_t^n + \sigma\hat{\xi}_t^c]$.
A remark about technology is in order. a_t^n implies that in the case of a negative labor

[14] In the steady state, $1 = \frac{\bar{C}^\sigma\bar{Y}^\phi(\xi^C)^{-\sigma}(\xi^N)^{-\phi}}{\bar{A}^{1+\phi}}$ implies $\bar{A} = 1$.
[15] Note that for any positive variable O_t with steady-state level $\bar{O}$, we have $\hat{O}_t = O_t - \bar{O}$ and $o_t = \ln O_t - \ln\bar{O} = o_t - \bar{o}$.

supply shock or of a positive consumption shock, an increase of the production efficiency by firms would compensate this shock. Increase in productivity could be modeled as the outcome of private applied research investment or public investment in the form of R&D subsidies or basic research[16].

In this theis, A_t evolves exogenously and thus, the losses from technology movements $(\hat{a}_t - \hat{a}_t^n)^2$ are exogenous. That's why they will be neglected in our analysis.

Using $x_t = \hat{y}_t - \hat{y}_t^p$, the intertemporal social losses become

$$\begin{aligned} &-\frac{1}{2}\sum_{t=0}^{\infty}\beta^t[(\tilde{\sigma}+\phi)x_t^2 + 2\gamma_1\pi_t^2] \\ &= -\gamma_1\sum_{t=0}^{\infty}\beta^t[\pi_t^2 + \lambda x_t^2] \\ &= -\gamma_1\sum_{t=0}^{\infty}\beta^t l_t, \end{aligned} \tag{2.59}$$

where $\lambda = \frac{\tilde{\sigma}+\phi}{2\gamma_1} = \frac{\kappa}{\theta}$ and $l_t = \pi_t^2 + \lambda x_t^2$.

2.12 Microfoundation of the Central Banker's Objective Function[17]

In this section we consider the objective function of the central bank. We assume, as is standard, that the economy is populated by a continuum of identical infinitely-lived households. In addition, there is an individual central banker. Hence, the central banker's individual consumption choices have no consequences for aggregate output and consumption.

We derive the central banker's intertemporal social losses under incentive contracts. The central banker aims at achieving a high individual utility. Moreover, he is altruistic and is also interested in furthering the well-being of the other agents. More precisely, the central banker faces one of two wage schemes. If no incentive contract is in effect, he receives a fixed wage $\overline{w} \geq 0$. By contrast, if an incentive contract was signed, the central banker is paid according to the deviation of his actual choice of nominal interest rate from the forecast $w_t^{CB} = \zeta(\tilde{i}_t) \geq 0$, where w_t^{CB} is the real wage paid to the central banker and $\tilde{i}_t = i_t - i_t^f$ is the deviation of the interest-rate choice from the forecast made at the time when the contract is signed. We focus on functions $\zeta(\,\cdot\,)$ with a global maximum, $\bar{w}^{CB}$, at $\tilde{i}_t = 0$, which satisfy $\zeta'(0) = 0$, $\zeta'(\tilde{i}_t > 0) < 0$ and $\zeta''(0) < 0$. Hence, the central

[16] In the latter case, for instance, the government could subsidize firms to do applied research. The government's budget constraint would amount to $F_t = G_t + s(T_t) + I_t$, where I_t is the amount of money given to firms to invest in applied research.

[17] This microfoundation has been used in Gersbach et al. (2015)).

banker faces wage reductions that increase with the size of the deviation from his earlier announcements. The central banker's wage is financed through a lump-sum tax. We note that payment to the central banker is negligible at the aggregate level, so the lump-sum tax necessary to finance his wage does not affect the household's budget constraint.

We make the assumption that the central banker is excluded from trading in financial markets. The main motivation for this assumption is that the central banker should be prevented from hedging against the variations of his income.[18]

For simplicity, we assume that the central banker is infinitely-lived and has the same individual utility from consumption and the same discount factor β as the households. The central banker's utility from consumption is

$$u(C_t^{CB}) = \frac{(C_t^{CB})^{1-\sigma} - 1}{1-\sigma} = \frac{\zeta(\tilde{i}_t)^{1-\sigma} - 1}{1-\sigma}.$$

We evaluate this expression in the steady state with $\tilde{i}$ equal to 0. Thus the second-order Taylor approximation delivers

$$u(C_t^{CB}) \approx \frac{\zeta(0)^{1-\sigma} - 1}{1-\sigma} + \zeta(0)^{-\sigma}\zeta'(0)\tilde{i}_t + \frac{1}{2}\frac{\partial}{\partial \tilde{i}_t}\left(\zeta(\tilde{i}_t)^{-\sigma}\zeta'(\tilde{i}_t)\right)|_{\tilde{i}_t=0}\tilde{i}_t^2.$$

Since $\zeta'(0) = 0$, we can rewrite the approximation as

$$u(C_t^{CB}) \approx \frac{\zeta(0)^{1-\sigma} - 1}{1-\sigma} + \frac{1}{2}\zeta(0)^{-\sigma}\zeta''(0)\tilde{i}_t^2.$$

The first term is constant. The constant utility term can be neglected when we compute the behavior of central bankers. However, the constant utility term and thus the fixed wage $\bar{w}^{CB}$ are important to satisfy the central bankers' participation constraints. Wage $\bar{w}^{CB}$ has to be set at levels at which central bankers are at least as well off as with other occupations—e.g. being a household. We assume that $\bar{w}^{CB}$ is set at levels at which the participation constraint is fulfilled.

As mentioned earlier, the central banker is also altruistic towards the households. Specifically, the overall loss of the central banker in period t is

$$\alpha l_t - u(C_t^{CB}) = \frac{\alpha}{2}(\pi_t^2 + \lambda x_t^2 - 2\frac{\zeta(0)^{1-\sigma} - 1}{\alpha(1-\sigma)} - \frac{\zeta(0)^{-\sigma}\zeta''(0)}{\alpha}\tilde{i}_t^2), \tag{2.60}$$

with α being the weight of altruism. We scale the central banker's overall loss by $\frac{1}{\alpha}$ and

[18] This is in line with actual practices, as central bankers have to adhere to procedures for the management of their personal assets that avoid any conflict of interest (see Swiss National Bank, Bankrat (2012) and European Central Bank, Banking Supervision (2014)). Under incentive contracts, prohibiting the use of hedging instruments would be particularly important.

deduct the constant term. The resulting loss function is denoted by l_t^{CB} and given by

$$l_t^{CB} = \frac{1}{2}(\pi_t^2 + \lambda x_t^2 - \frac{\zeta(0)^{-\sigma}\zeta''(0)}{\alpha}\tilde{i}_t^2). \tag{2.61}$$

We set[19] $b = -\frac{\zeta(0)^{-\sigma}\zeta''(0)}{\alpha}$ and obtain

$$l_t^{CB} = \frac{1}{2}(\pi_t^2 + \lambda x_t^2 + b\tilde{i}_t^2). \tag{2.62}$$

We note that the sensitivity of the wage scheme with regard to the precision of forecasts, $\zeta''(0)$, enters weight b of the deviation of the interest-rate forecast from actual policy choice in the central banker's loss function.

[19] We note that the extreme case $b = \infty$ implies $\zeta''(0) = -\infty$.

2.13 List of Variables and Notations

Table 2.1: List of Variables and Notations (1)

Variables	Description
ξ_t	a vector of shocks
$\xi_t^C, \xi_t^M, \xi_t^N, \xi_t^G, \xi_t^O$	shocks on consumption, money holding,
	labor supply, government spending and other factors
$\sigma, \tilde{\sigma}, \nu$	relative risk-aversion coefficients of consumption and real money holdings
ρ_t	demand shock
ϕ	relative risk-aversion coefficient of labor supply
λ	relative weight of output gap objective w.r.t. inflation objective
b	intensity of incentives
y_t^p	potential output in the absence of price rigidity
r_t	natural real interest rate
κ	coefficient of Phillips Curve
γ_1	coefficient of price adjustment cost
$Q_{t,t+j}$	stochastic discount factor used by financial market
Households	
β	households' discount factor
$C_t(i)$	consumption of variety i
M_t^r	real money holding
$u(\cdot)$	households' utility from consumption and money holding
$v(\cdot)$	households' disutility from labor supply
$g(\cdot)$	households' utility from government spending
U_t	households' total utility
C_t	aggregate consumption
Ω_t	households' expenditure level

Table 2.2: List of Variables and Notations (2)

Variables	Description
Firms	
i	firm and variety index
θ	elasticity of substitution between differentiated goods
A_t	technology level
$N_t(i)$, $P_t(i)$, $Y_t(i)$, $Z_t(i)$	labor employed, price of variety i, production, profit in firm i
$\gamma(\cdot)$	cost of price adjustment
W_t	prevailing wage in period t
N_t	total labor employed
P_t	aggregate price
MC_t	real marginal cost
χ	subsidy on firms' production
Government	
M_t, B_t, D_t	monetary base, stock of government bonds, government total debt
G_t, F_t	government's real consumption and total consumption
T_t, $s(\cdot)$	taxes, tax-collecting costs
i_t, i_t^M	nominal interest rates on government bonds and on monetary base
Central Bank	
Π_t	gross inflation rate
π_t	inflation rate
x_t	output gap
l_t	social loss in period t
$\bar{w}$	central banker's fixed wage
w_t^{CB}	central banker's real wage
$\bar{w}^{CB}$	central banker's maximal wage
$\zeta(\cdot)$	central banker's wage function
$\tilde{i}_t$	deviation of central banker's interest-rate choice from its forecast
C_t^{CB}	central banker's consumption in period t
α	central banker's weight of altruism
l_t^{CB}	central banker's loss in period t

3 Simple Forward Guidance Contracts*

We start with simple renewable FGCs (Forward Guidance Contracts). In this chapter, we study the properties of such contracts and characterize the contracts that the government would offer repeatedly. This chapter is organized as follows: In the next section we present the model. To assess the potential of and challenges to FGCs, the standard discretionary solution without FGCs is presented as a benchmark in Section 3.2. In Section 3.3 we establish the properties and the welfare implications of optimal FGCs. In Section 3.4, we discuss which type of FGCs can be used in the presence of uncertainty about natural real-interest rate shocks. A discussion and our conclusions are presented in Section 3.5.

3.1 A General Framework

Our model combines FGCs with the standard New Keynesian Framework to examine the zero-bound problem. To model the zero-bound problem, we follow Eggertsson and Woodford (2003) and Eggertsson (2003). Time is discrete and indexed by $t = 0, 1, 2, ...$. The IS curve, derived in Section 2.9, is described by

$$x_t = \mathbb{E}_t[x_{t+1}] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}] - r_t), \tag{3.1}$$

where x_t denotes the (log) output gap in period t and $\mathbb{E}_t[\pi_{t+1}]$ denotes the inflation rate in $t+1$ expected in t. Parameter σ satisfies[1] $\sigma > 0$, i_t is the nominal interest rate, and r_t the natural real interest rate.

Following Eggertsson (2003), we consider two possible realizations of r_t, that correspond to two different states $s \in \{L, H\}$. With a slight abuse of notation, we write r_L and r_H for these realizations. We assume $r_H > 0$ and $r_L < 0$, which ensures that the zero lower bound typically binds in state L but not in state H. In the following, we will say that the economy is in a "downturn" if the state is L. Similarly, we will use the term "normal

* This chapter is based on joint research with Hans Gersbach and Volker Hahn and was published as a CESifo working paper (see Gersbach et al. (2015)).

[1] For simplicity, we use σ to replace $\tilde{\sigma}$ in equation (2.47).

times" to describe an economy in state H.

Like Eggertsson (2003), we consider a situation where the economy is initially in a downturn, i.e. $s = L$. In each period $t = 1, 2, ...$, the state will change to $s = H$ with constant probability $1 - \delta$ $(0 < \delta < 1)$ and then remain in this state forever. With probability δ, the economy remains in the downturn. The Phillips curve, derived in Section 2.8, is

$$\pi_t = \kappa x_t + \beta \mathbb{E}_t[\pi_{t+1}], \tag{3.2}$$

with $\kappa > 0$ and β $(0 < \beta < 1)$ as the common discount factor.

The instantaneous social loss function, derived in Section 2.11, is

$$l_t = \frac{1}{2}\left(\pi_t^2 + \lambda x_t^2\right), \tag{3.3}$$

where $\lambda > 0$. Future losses are discounted by the factor β.

In Section 2.12 we provide a foundation of the central banker's utility function when the government offers him a wage contract composed of a fixed wage and a variable component increasing with the accuracy of the interest-rate forecast and hence decreasing with $(i_t - i_t^f)^2$. The parameter b—chosen by the government acting as contract designer—measures the intensity of incentives provided by the FGC.[2]

As explained in more detail in Section 2.12, we assume that the central banker shares the private agents' objectives and thus faces the loss function (3.3) in each period. In addition, he may face an FGC characterized by parameter b, which implies that the central banker incurs utility losses $b(i_t - i_t^f)^2$ when the interest rate he has chosen, i_t, differs from the level stipulated in the contract, i_t^f. We assume in the following that the level of interest rates stipulated in the contract[3] is zero, i.e. $i_t^f = 0$. First, this is broadly in line with current forward guidance practices of central banks in different countries. Second, it is straightforward to show that zero is the optimal non-negative level for interest-rate forecasts when the economy is in the downturn.

For the moment we will focus on simple renewable FGCs that may be chosen by the government and affect the central banker's incentives in the subsequent period. In particular, we will consider two scenarios. In the first scenario, we will examine the implications of FGCs under the assumption that the low realization of the natural real interest rate, r_L, is known when contract parameter b is determined. Later, we will also examine a second scenario where r_L is unknown when b is selected.

[2] For the framework of Krippner and Thornton (2012), the parameter b could be related to the number of overnight indexed swap (OIS) contracts purchased by central bank, as the capital losses the central bank would incur if the interest rate is raised before OIS contracts expire are linear in the number of contracts.

[3] In Chapter 4 we analyze FGCs in which the forecast is not part of the contract and is chosen by the central banker himself.

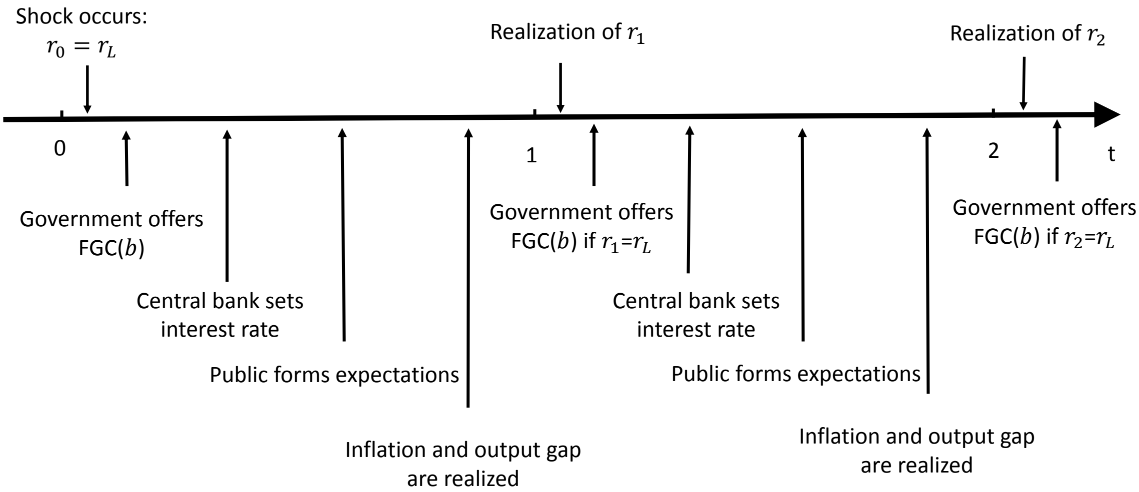

Figure 3.1: The sequence of events.

More precisely, we assume in the first scenario that the government chooses b at an ex-ante stage. In all periods, it can only offer contracts with this parameter. In a particular period $t = 0, 1, 2, ...$, the sequence is as follows: First, the current state $s \in \{H, L\}$ is realized and becomes common knowledge. Second, the government decides whether to sign a new FGC with given parameter b (henceforth FGC(b)), which will be effective in period $t + 1$. Third, the private sector forms its expectations about inflation and output in period $t + 1$. Also, the central banker selects the nominal interest rate i_t to minimize his losses, subject to (3.1) and (3.2). The central banker's loss function in period t is influenced by a possible FGC signed in period $t - 1$. More precisely, it is

$$l_t^{CB} = \frac{1}{2}\left(\pi_t^2 + \lambda x_t^2\right) + \frac{1}{2} b i_t^2$$

if an FGC was signed in period $t - 1$ and

$$l_t^{CB} = \frac{1}{2}\left(\pi_t^2 + \lambda x_t^2\right)$$

otherwise. We assume that in the initial period $t = 0$, an FGC is effective.[4] Figure 3.1 shows the sequence of events.

[4] This assumption is immaterial to our findings.

In Section 3.4 we consider the second scenario. In particular, we study FGCs in a situation with uncertainty about parameter r_L when contracts are designed. The only difference with the first scenario is that b is chosen before the exact value of r_L becomes known.

3.2 Discretion without Forward Guidance Contracts

To have a benchmark for assessing the potential and the limitations of FGCs, we briefly summarize in this section the standard discretionary solution in the absence of FGCs. In the following we focus on Markov equilibria, i.e. all economic variables depend only on the current state of the economy $s \in \{H, L\}$.

In each period, the central bank discretionarily chooses the nominal interest rate as its policy instrument, taking both its own future behavior and the public's expectations as given. In a Markov equilibrium there are only two possible realizations for inflation, the output gap, and the nominal interest rate. We use π_L^D, x_L^D, i_L^D for the corresponding values in a downturn and π_H^D, x_H^D, and i_H^D for normal times, where the superscript D stands for "discretionary".

It is easy to compute the values of inflation and the output gap in normal times. When the natural real interest rate has returned to the positive value r_H, i.e. in period t when $s = H$, optimal policy involves $i_H^D = r_H$. Therefore we obtain $x_H^D = 0$ and $\pi_H^D = 0$. Computing the equilibrium in the downturn is somewhat more involved. During the downturn, the zero lower bound is binding because of $r_L < 0$. Hence, in periods when $s = L$, we obtain $i_L^D = 0$. We note that $\mathbb{E}_t[\pi_{t+1}] = \delta\pi_L^D + (1-\delta)\pi_H^D = \delta\pi_L^D$ in a downturn, where we have used $\pi_H^D = 0$, and we also note that the probability of the state remaining at $s = L$ is δ. Analogously, we observe $\mathbb{E}_t[x_{t+1}] = \delta x_L^D$.

Inserting these expressions into (3.1) and (3.2) and solving for π_L^D and x_L^D yields

$$\pi_L^D = \frac{\kappa}{\sigma(1-\delta)(1-\beta\delta) - \delta\kappa} r_L, \tag{3.4}$$

$$x_L^D = \frac{1-\beta\delta}{\sigma(1-\delta)(1-\beta\delta) - \delta\kappa} r_L. \tag{3.5}$$

Henceforth we assume that δ is sufficiently small for the denominator in the above equations to be strictly positive.

Assumption 3.1
Parameter δ satisfies

$$\sigma(1-\delta)(1-\beta\delta) - \delta\kappa > 0. \tag{3.6}$$

Assumption 3.1 imposes an upper bound on δ, i.e. the probability of remaining in a downturn cannot be too large. For the parameter values in Table 3.1 below, this assumption is fulfilled for all $\delta < 0.68$.[5] Together with $r_L < 0$, this assumption ensures that π_L^D and x_L^D are negative in a downturn.[6] Throughout the paper we illustrate the properties of the economy using the following set of parameters:[7]

Table 3.1: Parameter values.

$\beta = 0.99$
$\lambda = 0.03$
$\kappa = 0.3$
$\sigma = 2$
$r_H = 0.02$

3.3 Forward Guidance Contracts

In this section we analyze how the possibility of signing FGCs affects the equilibrium. We assume that both the government and the central bank act under discretion. More specifically, upon observing the current state s, the government decides whether to sign a new contract, taking as given the central bank's decisions both in the current period and in all future periods, together with the possible existence of a contract for the current period. Then the central bank chooses its instrument subject to a possible FGC, taking its own future decisions and the government's future behavior as given. We consider a Markov equilibrium, i.e. an equilibrium where the decision-makers' choices depend solely on payoff-relevant state variables, i.e. on $s \in \{H, L\}$ and the possibility that a contract was signed in the previous period.

For the moment we assume that the government will always choose an FGC in a downturn and no contract in normal times. Later we will see that this behavior is indeed optimal for the government. We need to consider $2 \times 2 = 4$ different constellations because there are two different states $s \in \{H, L\}$ and there may be either an active contract (C) or no active contract (N). The corresponding levels of inflation will be denoted as π_H^C, π_L^C, π_H^N, and π_L^N. Analogous notation will be used for the different possible values of the output gap.

In normal times and in the absence of an FGC, it is obvious that $\pi_H^N = x_H^N = 0$ holds. Next we turn to the constellation where $s = H$ and a contract was signed in the previous

[5] Henceforth, we adopt $\delta = 0.5$ for the numerical examples.

[6] For a discussion of sign reversals in the initial responses of inflation and output when the duration of an interest-rate peg is extended, see Carlstrom et al. (2012).

[7] The values $\beta = 0.99$, $\lambda = 0.03$, and $\kappa = 0.3$ are taken from Gersbach and Hahn (2014). The values $\sigma = 2$ and $r_H = 0.02$ are taken from Eggertsson (2006).

period. Given the fact that in the next period, the output gap will be x_H^N and inflation will amount to π_H^N, π_H^C and x_H^C can be determined with the help of (3.1) and (3.2) as follows:

$$\begin{aligned}\pi_H^C &= \beta\pi_H^N + \kappa x_H^C,\\ x_H^C &= -\frac{1}{\sigma}(i_H^C - \pi_H^N - r_H) + x_H^N.\end{aligned}$$

Using $\pi_H^N = x_H^N = 0$, these equations simplify to

$$\pi_H^C = \kappa x_H^C, \tag{3.7}$$
$$x_H^C = -\frac{1}{\sigma}(i_H^C - r_H). \tag{3.8}$$

Minimizing $\frac{1}{2}\left(\pi_t^2 + \lambda x_t^2\right) + \frac{1}{2}bi_t^2$ subject to (3.7) and (3.8) yields

$$\pi_H^C = \frac{b\sigma}{\lambda + \kappa^2 + b\sigma^2}\kappa r_H = \kappa f(b), \tag{3.9}$$
$$x_H^C = \frac{b\sigma}{\lambda + \kappa^2 + b\sigma^2} r_H = f(b), \tag{3.10}$$
$$i_H^C = \frac{\lambda + \kappa^2}{\lambda + \kappa^2 + b\sigma^2} r_H, \tag{3.11}$$

where we have introduced

$$f(b) := \frac{b\sigma}{\lambda + \kappa^2 + b\sigma^2} r_H. \tag{3.12}$$

We note that $f(b)$ is a monotonically increasing function with $f(0) = 0$ and $\lim_{b\to\infty} f(b) = r_H/\sigma$.

Equations (3.9) and (3.11) are useful in understanding why FGCs are potentially welfare-improving. With the help of (3.11), we observe that the nominal interest rate in the first period after the downturn is a decreasing function of b and that it is lower than the level that would prevail in the absence of an FGC, r_H. Hence FGCs enable the central bank to commit to expansionary monetary policy once the economy has left the downturn. Note that inflation will be higher if an FGC is present in state H, which is shown by the fact that π_H^C is an increasing function of b.[8] Figure 3.2 shows how, under an FGC, the nominal interest rate in state H decreases when the incentive intensity b increases. In turn, the inflation and output gap increase with b in state H. This is plausible because increases in b raise the relative importance of achieving small interest rates compared to the other objectives in the central bank's loss function, namely inflation and output stabilization.

Finally, we examine inflation and output in a downturn. Because there is a constant prob-

[8] We observe that for $b = 0$, (3.9)-(3.11) entail the values of inflation, the output gap, and the nominal interest rate from the standard discretionary solution examined in Section 4.1.

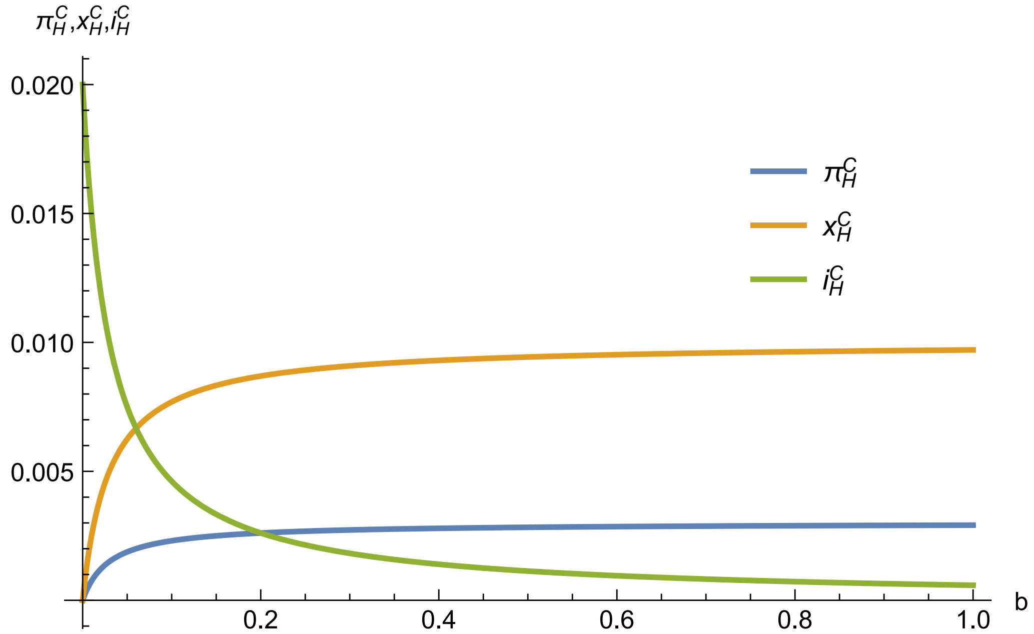

Figure 3.2: Inflation, output gap, and interest rate as a function of the value of b in state H with an active contract.

ability δ of remaining in state L, expectations of inflation and output are

$$\mathbb{E}_t[\pi_{t+1}] = \delta\pi_L^C + (1-\delta)\pi_H^C, \tag{3.13}$$
$$\mathbb{E}_t[x_{t+1}] = \delta x_L^C + (1-\delta)x_H^C. \tag{3.14}$$

We are now in a position to compute inflation and the output gap in the downturn. For the moment we assume that the zero lower bound is binding in the downturn, i.e. $i_L^C = 0$. Later we will identify the range of values of b for which this is actually the case. Further, we will show that the government's optimal choice of b always lies within this range. Using (3.1), (3.2), (3.9), (3.10), (3.12)-(3.14), and $i_L^C = 0$ yields

$$\pi_L^C = Af(b) + \pi_L^D, \tag{3.15}$$
$$x_L^C = Bf(b) + x_L^D, \tag{3.16}$$

where

$$A := \frac{\kappa(1-\delta)\left(\sigma(1+\beta(1-\delta))+\kappa\right)}{\sigma(1-\delta)(1-\beta\delta)-\delta\kappa}, \tag{3.17}$$
$$B := \frac{(1-\delta)\left(\sigma(1-\beta\delta)+\kappa\right)}{\sigma(1-\delta)(1-\beta\delta)-\delta\kappa}, \tag{3.18}$$

and π_L^D and x_L^D are defined in (3.4) and (3.5). Recall that, in line with Assumption 3.1,

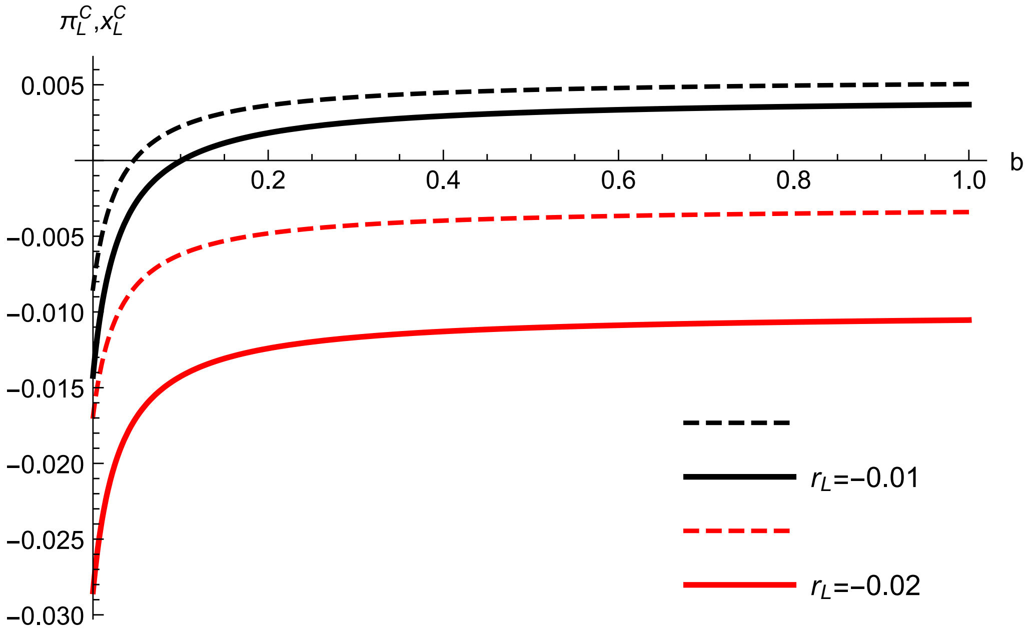

Figure 3.3: Inflation (dashed curves) and output gap (solid curves) as a function of the value of b for different shock sizes.

the values π_L^D and x_L^D, which would prevail without FGCs, are strictly negative. Moreover, Assumption 3.1 entails $A > 0$ and $B > 0$. Together with $f'(b) > 0 \; \forall b \geq 0$, this implies that π_L^C and x_L^C strictly increase with b. Hence, for small b, FGCs can cushion the harmful consequences of a downturn on the output gap and also mitigate the ensuing deflation. These beneficial effects are possible because FGCs enable the central bank to commit to loose monetary policy after the downturn (see (3.9)-(3.11)). This commitment to expansionary policy after the downturn raises inflation expectations during the downturn (see (3.13)) and thereby enables the central bank to implement a lower real interest rate $i_t - \mathbb{E}_t[\pi_{t+1}] = -\mathbb{E}_t[\pi_{t+1}]$ when the nominal interest rate is constrained by the zero lower bound. Figure 3.3 illustrates how, in state L, inflation and the output gap increase with incentive intensity b under an FGC.

Finally, we need to examine the circumstances in which our assumption that the zero lower bound is binding in a downturn is fulfilled under FGCs. The following lemma, which is proved in Appendix A.1, establishes a sufficient condition for the zero lower bound to be binding:

Lemma 3.1

If

$$f(b) \leq \frac{\kappa^2 + \lambda(1-\beta\delta)}{\kappa(\kappa A + \lambda B)} |\pi_L^D| =: \hat{f}, \tag{3.19}$$

then the zero lower bound is binding in a downturn with FGCs.

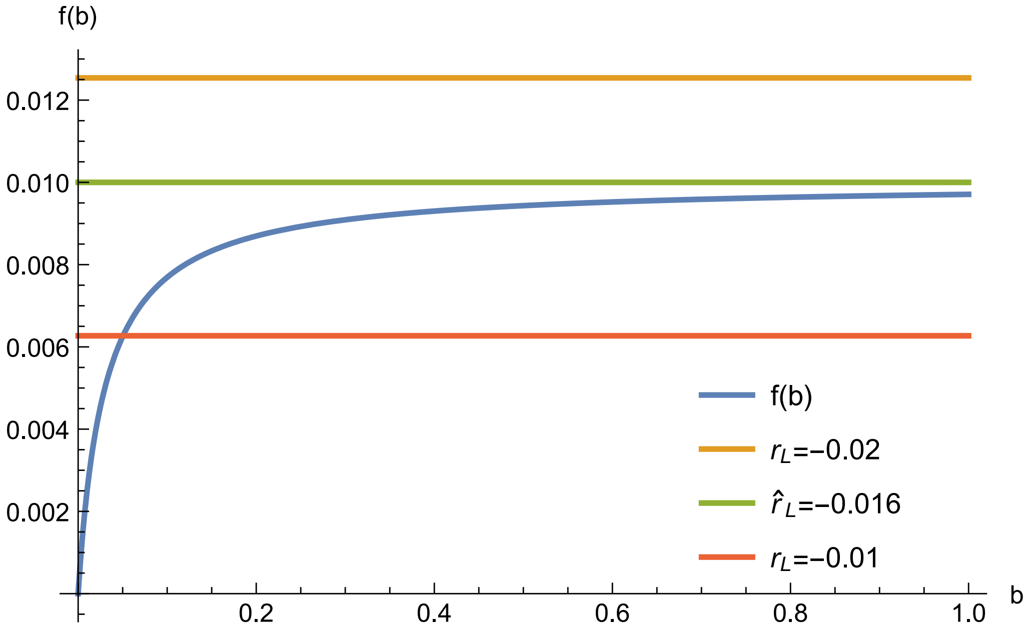

Figure 3.4: The function $f(b)$ and three horizontal curves representing the right-hand side of (3.19) for different shock sizes.

As (i) $f(b)$ monotonically increases with b, (ii) $f(0) = 0$, and (iii) the right-hand side of the condition in the lemma is positive and does not depend on b, the lemma defines a critical value of b, henceforth denoted by $\hat{b}$, such that the zero lower bound is binding in state L for all values of b below this critical value. Note that this value will be infinite if the right-hand side of (3.19) is at least as large as $\lim_{b\to\infty} f(b) = r_H/\sigma$.
We thus obtain the following corollary:

Corollary 3.1
The zero lower bound is binding regardless of the value of b *when* $r_L \leq \hat{r}_L$*, where*

$$\hat{r}_L := -\frac{(\kappa A + \lambda B)[\sigma(1-\delta)(1-\beta\delta) - \delta\kappa]}{\sigma[\kappa^2 + \lambda(1-\beta\delta)]} r_H.$$

We note that when the shock is severe, i.e. $r_L \leq \hat{r}_L$, the inflationary expectation induced by the FGC is not large enough to lift the optimal nominal interest rate above zero, even if the value of b is set at an extremely high level.
Figure 3.4 depicts the left-hand side and the right-hand side of (3.19) from Lemma 3.1 for different values of r_L. In the case $r_L = -0.01$, a large value for b induces positive inflation and output gap, as already shown in Figure 3.3. Hence, (3.19) is only satisfied for low values of b. In the case $r_L = -0.02$, inflation and output gap in downturns are negative regardless of the value of b, as shown in Figure 3.3. Correspondingly, (3.19) is always satisfied. Thus, in such a case, the zero lower bound is binding for all values of b.

In the following we restrict our attention to values of b that satisfy the condition in the lemma. The justification for this assumption is that the government would never find it optimal to select a value of b that would violate (3.19), which will be demonstrated in the next section.

3.3.1 Optimal Contracts with Commitment to Contracting

In this section we derive optimal FGCs and thus determine the socially optimal value of b. In doing so, we continue to assume that the government always offers an FGC in downturns and no contract in normal times. Later we will consider FGCs in the absence of such commitments.

First we observe that in equilibrium per-period social losses can take only three different values:

$$l_L^C = \frac{1}{2}\left(\pi_L^C\right)^2 + \frac{1}{2}\lambda\left(x_L^C\right)^2 = \frac{1}{2}\left(Af(b) + \pi_L^D\right)^2 + \frac{1}{2}\lambda\left(Bf(b) + x_L^D\right)^2, \quad (3.20)$$

$$l_H^C = \frac{1}{2}\left(\pi_H^C\right)^2 + \frac{1}{2}\lambda\left(x_H^C\right)^2 = \frac{1}{2}(\kappa^2 + \lambda)\left(x_H^C\right)^2 = \frac{1}{2}(\kappa^2 + \lambda)\left(f(b)\right)^2, \quad (3.21)$$

$$l_H^N = \frac{1}{2}\left(\pi_H^N\right)^2 + \frac{1}{2}\lambda\left(x_H^N\right)^2 = 0, \quad (3.22)$$

where we have used that $\pi_H^N = x_H^N = 0$, (3.9), (3.10), (3.15), and (3.16). Social losses expected in period 0 are given by

$$V_L(C) = \sum_{t=0}^{\infty} \beta^t \delta^t l_L^C + \sum_{t=1}^{\infty} \beta^t \delta^{t-1}(1-\delta) l_H^C, \quad (3.23)$$

where the subscript L stands for the current state of the economy and C stands for the fact that a (C)ontract was signed in the previous period. In (3.23) we have utilized $l_H^N = 0$ as well as the fact that (a) the probability of the economy being in a downturn is δ^t in all periods t with $t \geq 0$ and (b) the probability that the economy has just left the downturn and hence an FGC is still effective is $\delta^{t-1}(1-\delta)$ in all periods t with $t \geq 1$. It is straightforward to rewrite (3.23) as

$$V_L(C) = \frac{1}{1-\beta\delta}\left[l_L^C + \beta(1-\delta) l_H^C\right]. \quad (3.24)$$

Together with (3.20) and (3.21), (3.24) can be used to explain the tradeoff involved with FGCs. First, l_H^C is an increasing function of b, which is a consequence of the facts that $\pi_H^C = \kappa f(b)$ (see (3.9)), $x_H^C = f(b)$ (see (3.10)), $f(0) = 0$, and $f'(b) > 0 \;\forall b \geq 0$. The interpretation of this observation is that FGCs induce expansionary policy for one period once the downturn has ended. This is socially costly ex post. Second, l_L^C is a monotonically decreasing function for small b. Hence FGCs induce welfare gains in the downturn. This follows from the observation that the commitment to expansionary policy after the downturn increases inflation expectations during the downturn and thereby enables the central bank to implement lower real interest rates when the economy is stuck at the zero lower bound.

The socially optimal value of b balances these costs and benefits. In Appendix A.2 we prove the following lemma:

Lemma 3.2

Suppose the government always offers an FGC in state L and never offers a contract in state H. Then the optimal value of b, b^*, can be determined in the following way:

1. *If $r_H/\sigma > f^*$, b^* is given by $f(b^*) = f^*$, where*

$$f^* := \frac{A + \lambda B \frac{1-\beta\delta}{\kappa}}{A^2 + \lambda B^2 + \beta(1-\delta)(\lambda + \kappa^2)} |\pi_L^D|. \tag{3.25}$$

   At the optimal value of b, the zero lower bound is binding in equilibrium.

2. *If $r_H/\sigma \leq f^*$, social losses decrease strictly with b $\forall b \geq 0$. In this case the zero lower bound is binding $\forall b \geq 0$.*

It is instructive to conduct comparative statics with respect to $|r_L|$. For this purpose, observe that f^* is a monotonically increasing function of $|r_L|$ because $|\pi_L^D|$ is a monotonically increasing function of $|r_L|$ (see (3.4)). As a result, the optimal value of b, b^*, which is given by $f(b^*) = f^*$ for $r_H/\sigma > f^*$, increases with $|r_L|$. This is plausible, as a higher value of $|r_L|$ corresponds to a larger shock and thus calls for stronger incentives. For $|r_L| \to 0$, the optimal value of b converges to zero. By comparing f^* (equation (3.25)) and $\hat{f}$ (equation (3.19)) and rearranging terms, it turns out that $f^* < \hat{f}$, where $\hat{f}$ is the critical value, given in Lemma 3.1, below which the zero lower bound is binding. Thus, the zero lower bound is binding with an optimal FGC(b^*).

Lemma 3.2 also defines a critical value of r_L^c below which it is optimal to apply extremely harsh FGCs.

Corollary 3.2

The optimal value of b is infinite when $r_L \leq r_L^c$, where

$$r_L^c := -\frac{[A^2 + \lambda B^2 + \beta(1-\delta)(\lambda + \kappa^2)][\sigma(1-\delta)(1-\beta\delta) - \delta\kappa]}{\sigma[\kappa A + \lambda B(1-\beta\delta)]} r_H.$$

3.3.2 Optimal Contracts without Commitment to Contracting

Up to now we have simply assumed that the government will behave in a certain way. It remains to show that this behavior is indeed optimal when the government decides in each period whether to offer the FGC or not. The next lemma, which is proved in Appendix A.3, identifies a respective condition.

Lemma 3.3
The assumed behavior of the government, i.e. always signing an FGC with $b = b^*$ in a downturn and refraining from signing a contract in normal times, is optimal if $f^* \leq 2\tilde{f}$, where

$$\tilde{f} := \frac{A - P + \lambda(B - Q)\frac{1-\beta\delta}{\kappa}}{A^2 - P^2 + \lambda B^2 - \lambda Q^2 + \beta(1-\delta)(\kappa^2 + \lambda)} |\pi_L^D|. \tag{3.26}$$

P and Q are constants and given in Appendix A.3. Figure 3.5 shows that $f^* \leq 2\tilde{f}$ is fulfilled in the range of δ that satisfies Assumption 3.1 at parameter values specified in Table 3.1.

The lemma reveals that it is conceivable that for the optimal value of b, b^* identified in Lemma 3.2, the government would not find it optimal to offer the contract. This may occur because the government takes its own future behavior and the behavior of private agents as given. This distinguishes the government's decision problem in a particular period from the problem in the ex-ante stage, where the government can choose b for all future periods.

More specifically, when weighing up the costs and benefits of signing an FGC in a particular period, the government will fully take the costs into account that would materialize in the next period, provided that the state were H. However, the government only considers a fraction of the benefits. This can be seen when we look at the well-known representation of the New Keynesian Phillips curve, where current inflation is proportional to the expected discounted sum of all future output gaps

$$\pi_t = \kappa \mathbb{E}_t \left[\sum_{i=0}^{\infty} \beta^i x_{t+i} \right], \tag{3.27}$$

which directly follows from iterating (3.2). When the government signs an FGC in period t, it only takes into account the effect this contract has for $\mathbb{E}_t[x_{t+1}]$. Because the government takes its own future behavior as given, it does not consider those benefits of FGCs that result from the contracts' influence on output gaps farther away in the future, i.e. $\mathbb{E}_t[x_{t+i}]\ \forall i \geq 2$.

Despite the difficulty that a contract with b^* may not be offered in equilibrium by the government, it is straightforward to determine the optimal value of b for the case where

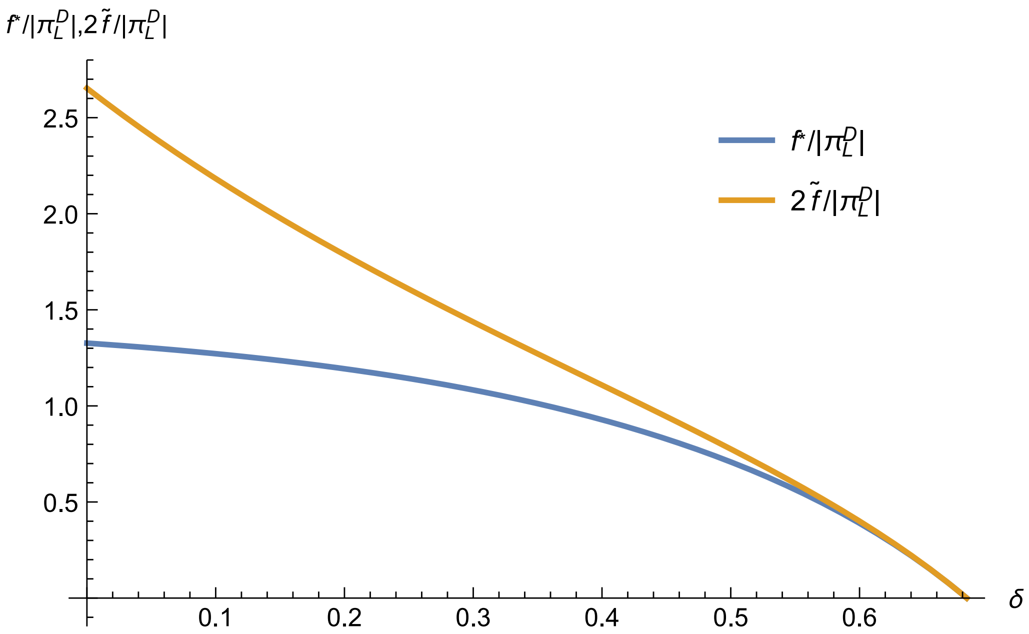

Figure 3.5: The values of f^* and $2\tilde{f}$, scaled by $|\pi_L^D|$, for the range of δ satisfying Assumption 3.1.

the government optimally decides in each period whether to sign a contract for the next period or not.

Proposition 3.1
Suppose the government only offers an FGC in each period if this is profitable. Then the optimal level of b*,* b^{**}*, can be determined in the following way:*

1. *For* $2\tilde{f} \geq f^*$ *and* $f^* < r_H/\sigma$*, the optimal level of* b *is given by* $f(b^{**}) = f^*$*.*

2. *For* $2\tilde{f} < f^*$ *and* $2\tilde{f} < r_H/\sigma$*, the optimal level of* b *is given by* $f(b^{**}) = 2\tilde{f}$*.*

3. *For* $2\tilde{f} \geq r_H/\sigma$ *and* $f^* \geq r_H/\sigma$*, we obtain* $b^{**} = \infty$*.*

To close this section, we present in Figure 3.6 the discounted social losses with the use of an FGC expected in period 0 (see (3.24)). The expected social loss with an optimal FGC(b^*) stays below the one in the discretionary case for all values of r_L. When we compare the ratio of social losses under FGC with the social losses under discretion, we observe that a considerable welfare gain can be achieved with such contracts, as this ratio attains nearly 0.03 for $r_L \in (r_L^c, 0)$ and is still around 0.25 for a large negative natural real interest-rate shock. Hence we obtain quite favorable tradeoffs between the efficacy of FGCs at the zero lower bound and the reduced flexibility in reacting to future events. Figure 3.6 also shows that the social loss with FGC($b^* = \infty$) starts to increase

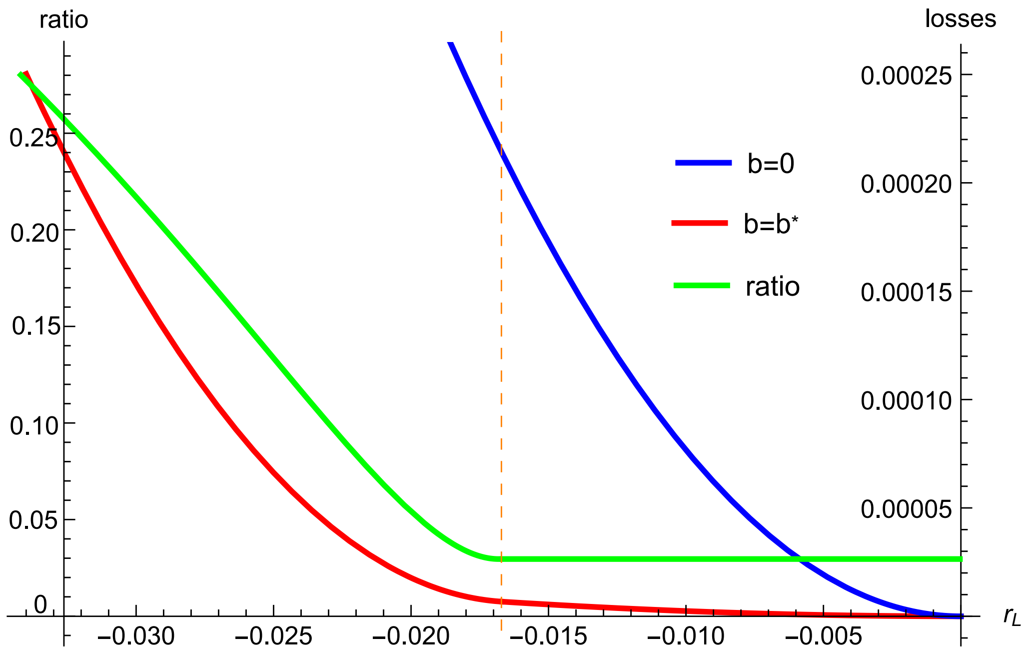

Figure 3.6: Discounted social losses with optimal FGC(b^*), in the discretionary case (right axis), and the ratio of these two discounted social losses (left axis), as functions of r_L (dashed curve represents $r_L = r_L^c$).

considerably when $r_L < r_L^c$.[9]

3.4 Forward Guidance Contracts under Uncertainty

In this section we analyze the second scenario outlined in Section 3.1, asking whether FGCs would also be desirable if r_L were unknown at the point in time when the value of b is chosen. For this purpose, we assume that r_L is randomly distributed with commonly-known prior distribution. A further assumption we make is that the value of r_L becomes known in the period when the downturn occurs.

First we observe that the possibility of FGCs can never lead to lower expected levels of welfare—provided that b is chosen optimally ex ante—because it would always be possible to select FGCs with $b = 0$, which would result in a scenario equivalent to the benchmark case without FGCs.

Second, we show that FGCs actually lead to strict increases in welfare. In particular, we show that b can be chosen in such a way that FGCs will improve welfare for all possible realizations of r_L.

[9] In such circumstances, welfare could be further improved by longer-term FGCs, which we discuss in Chapter 4.

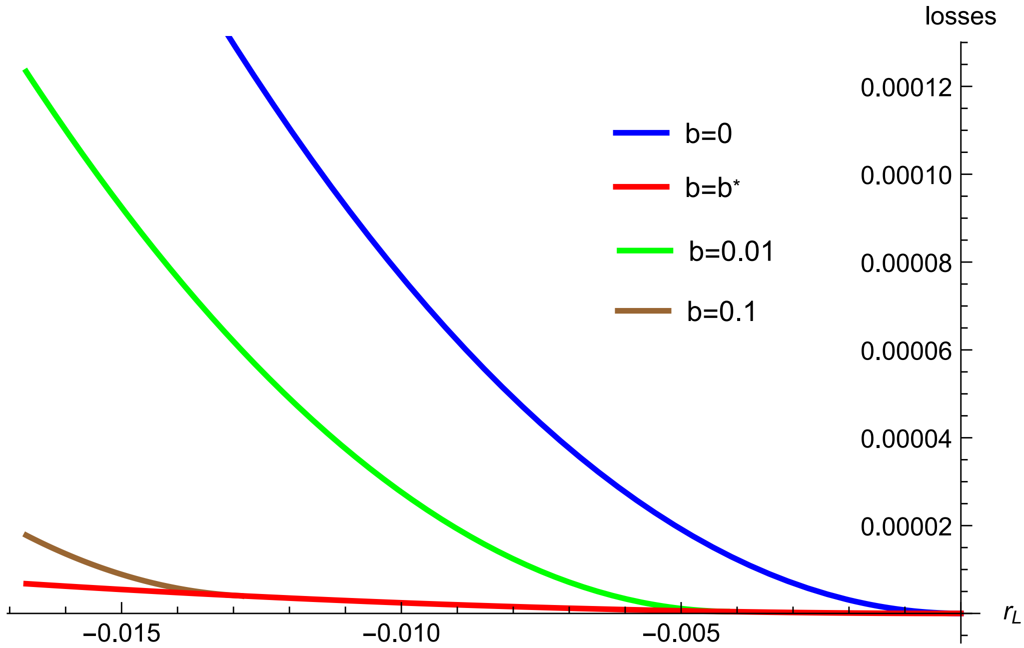

Figure 3.7: Discounted social losses under FGC, as a function of r_L, for different values of b.

Proposition 3.2

Suppose that r_L is randomly distributed with a maximum possible realization $\bar{r}_L < 0$. Then it is possible to select a value of b such that (i) for each realization of r_L, the government offers the contract, and (ii) welfare conditional on this value of r_L is strictly higher than in the benchmark case.

As the proposition implies that for all realizations of r_L welfare is higher under FGCs, the welfare level expected before the realization of r_L becomes known is also higher under FGCs than in the benchmark case without FGCs.

In Figure 3.7, the blue and red curves represent the discounted social losses in the benchmark case and with an optimal FGC for each realization of r_L, respectively. The green curve represents the discounted social loss with $\bar{r}_L = -0.0042$ and the corresponding optimal $b = 0.01$. The brown curve represents the discounted social loss with $\bar{r}_L = -0.0129$ and the corresponding optimal $b = 0.1$. Intuitively, all of these curves representing the discounted social losses with a fixed value of b are tangent to the red curve representing the discounted social loss with the optimal b. This figure also demonstrates that choosing the value of b that is optimal for $\bar{r}_L$ improves social welfare for all $r_L \leq \bar{r}_L$. However, as the value of b chosen ex ante is not the optimal one for the realization of r_L below the upper bound $\bar{r}_L$, the social loss can be unnecessarily high when r_L is significantly lower than $\bar{r}_L$.

When the maximum possible realization of the negative natural real interest rate is $r_L = 0$, one can still construct FGCs that will improve welfare for a wide range of natural real interest-rate shocks.[10] Let us choose a value $\tilde{b}$ which is the optimal value of b for some value $r_L = \tilde{r}_L$. As is demonstrated in Proposition 3.2, it is socially desirable to sign FGC($\tilde{b}$) for all the realizations $r_L \leq \tilde{r}_L$. Henceforth, we focus on the remaining range $r_L \in (\tilde{r}_L, 0)$.

In Appendix A.5 we prove the following lemma:

Lemma 3.4
Given some value $\tilde{b}$ selected in period -1, the zero lower bound is binding, and it is socially desirable to offer the FGC($\tilde{b}$) when $r_L \leq a\tilde{r}_L$, where $a \in (0,1)$ is given in (A.34) in the proof.

Given the value $\tilde{b}$ that is optimal for $\tilde{r}_L$, the central bank would still set the nominal interest rate at zero when $r_L \in (\tilde{r}_L, a\tilde{r}_L)$. Intuitively, it is still socially desirable to offer the FGC($\tilde{b}$), as the induced inflation expectation is not unnecessarily large. If the size of the shock is small, i.e. $r_L > a\tilde{r}_L$, the induced inflation expectation in downturns that stems from FGC($\tilde{b}$) is unduly large. Then the central bank will set a positive interest rate to suppress the inflation boom in downturns.

We obtain the following proposition, proved in Appendix A.6:

Proposition 3.3
Given some value $\tilde{b}$ selected in period -1, there exists an $\tilde{r}_L^c$ such that FGC($\tilde{b}$) improves social welfare for all $r_L < \tilde{r}_L^c$, where $\tilde{r}_L^c > a\tilde{r}_L$ and $\tilde{r}_L^c$ is given in the proof.

Figure 3.8 shows how the threshold value $\tilde{r}_L^c$, below which the government offers the FGC($\tilde{b}$), decreases as $\tilde{b}$ increases. Hence, if the government wants to ensure welfare gains for a very large range of realizations of negative natural real interest-rate shocks, $\tilde{b}$ has to be set at moderate levels. As an example, consider $\tilde{b} = 0.1$. In Figure 3.9, we show social losses under such a contract (dark line), under discretion (blue line) and when the parameter b can be tailored to the precise realization of shocks as in Proposition 3.1 (beige line). The critical threshold is $\tilde{r}_L^c \approx -0.004$, and FGCs lead to lower social losses for all values below that. For natural real interest rates above $\tilde{r}_L^c$, FGCs involve higher losses than under discretion. In these cases, the value of $\tilde{b} = 0.1$ is too high, given the comparably small size of the shock, and thereby induces a too expansionary monetary policy in the future.

[10] If we assumed a particular distribution of natural real interest-rate shocks, we could, of course, calculate the optimal FGC that improves welfare in expectation. For example, suppose that r_L is distributed uniformly in $[\underline{r}_L, 0]$. Then there exists an optimal value of $b > 0$ that minimizes expected welfare. Typically, the intensity of incentives for such exercises is moderate as long as $\underline{r}_L$ is not extremely low.

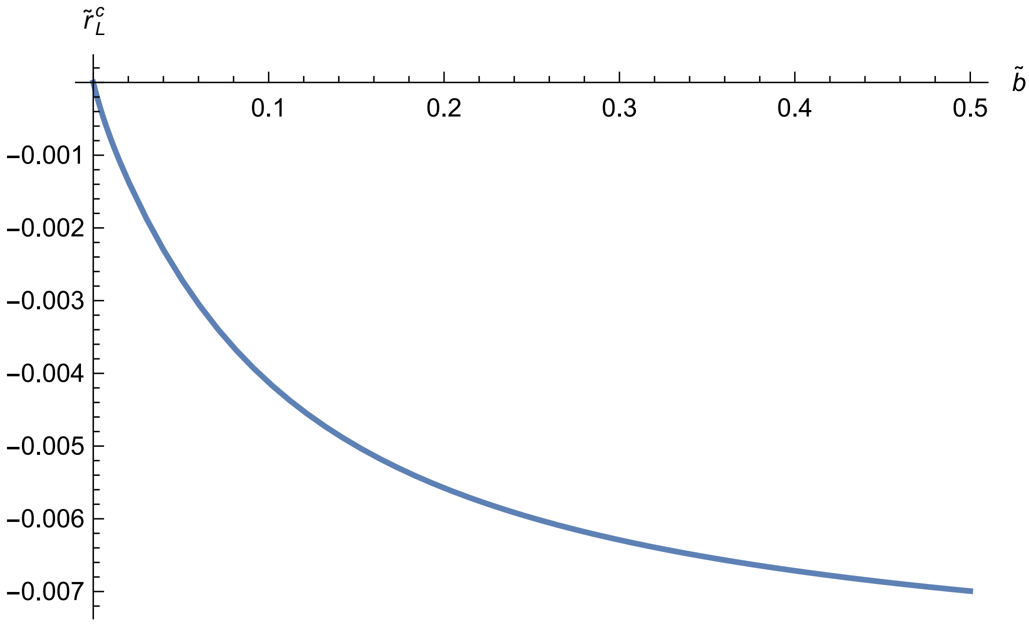

Figure 3.8: $\tilde{r}_L^c$ as a function of $\tilde{b}$.

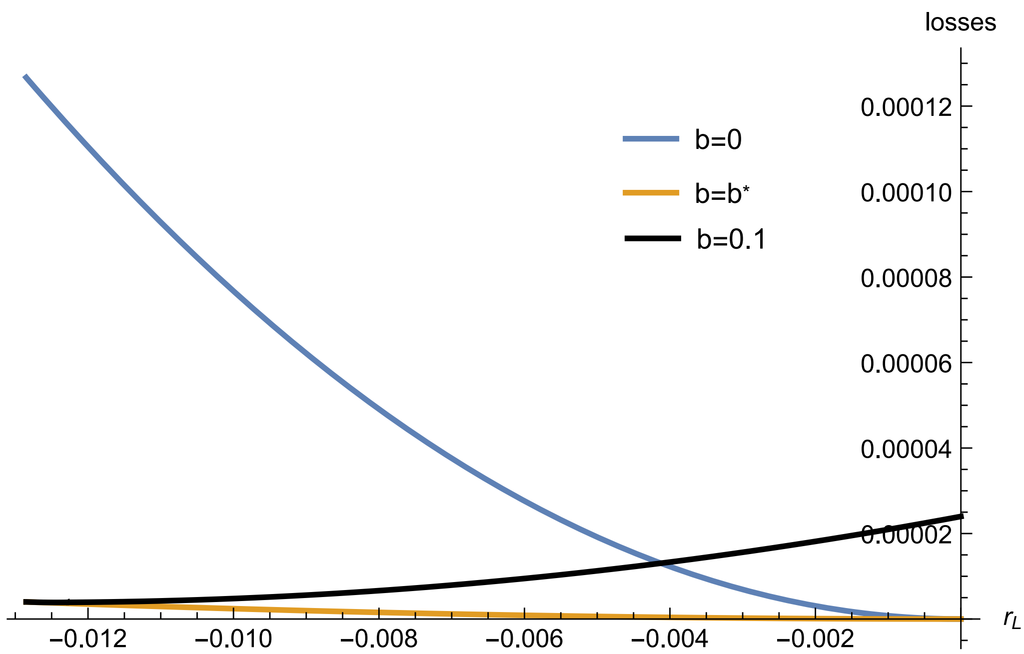

Figure 3.9: Discounted social losses with an FGC with $b = \tilde{b} = 0.1$, under discretion and when b is tailored to the realization of the shock.

3.5 Discussion and Conclusion

Forward guidance aims at influencing the public's expectations, an objective that has a long tradition in monetary policy. Today the reliance on forward guidance has become a central aspect of monetary policy, often associated with the belief that forward guidance can provide a stimulus when economies are mired in longer downturns. We have explored a simple contractual tool that makes forward guidance more effective when the economy is at the zero bound.

Such FGCs strike a balance between Odyssian policy commitments and the need to react to new developments. We have confined ourselves to very simple contracts, written either after a downturn or in normal times. Numerous extensions of our research could be pursued. For example, one could take into account the fact that it takes time to learn the magnitude of the shock, so the government necessarily has to sign such contracts under a veil of ignorance, which, in turn, may call for moderate intensity of incentives.

3.6 List of Variables and Notations

Table 3.2: List of Variables and Notations

Variables	Description
π_t, x_t, i_t	inflation, output gap, nominal interest rate in period t
β	households' discount factor
κ	coefficient in Phillips curve
σ	relative risk-aversion coefficient of consumption
δ	the probability of the economy being trapped in the downturn in each period
λ	relative weight of output-gap objective with respect to inflation objective
i_t^f	central banker's forecast of interest rate in period t
b	intensity of incentives provided by FGCs
$s \in L, H$	low and high states
r_t, r_H, r_L	natural real interest rate in period t, in states H and L
C, N	an active contract exists, no active contract exists
π_s^D, x_s^D, i_s^D	inflation, output gap, and interest rate in state s in discretion
π_s^C, x_s^C, i_s^C	inflation, output gap, and interest rate in state s with an active contract
π_s^N, x_s^N, i_s^N	inflation, output gap, and interest rate in state s without active contract
	π_s^N, x_s^N, i_s^N are equivalent to π_s^D, x_s^D, i_s^D
l_t, l_t^{CB}	social loss function and the central banker's loss function
l_s^C, l_s^N	social loss functions with and without an active contract in state s
$V_s(C), V_s(N)$	expected discounted intertemporal social losses in state s with and without contract
$f(b)$	a function of b
$\hat{f}, \hat{b}$	threshold values below which the zero lower bound is binding
f^*, b^*, b^{**}	optimal designs of the contract
$\hat{r}_L$	threshold value below which the zero lower bound is binding regardless of the value of b
r_L^c	threshold value below which the optimal value of b is infinitely large
$\tilde{f}$	threshold value regarding the government's behavior
A, B, P, Q	constants
$\bar{r}_L$	the maximum possible realization of r_L in uncertainty scenario
$\tilde{b}, \tilde{r}_L, \tilde{r}_L^c, a$	values in uncertainty scenario

4 Longer-term Forward Guidance Contracts[†]

So far, we have focused on simple renewable FGCs signed in one period and taking effect in the next. In this chapter, we explore a simple alternative: contracts that become effective immediately after signing (and remain effective for several further periods), i.e. we study FGCs that are effective for k periods, denoted by FGC (b,k), $(b>0, k\geq 1)$.
The setup studied in this chapter differs from the one in Chapter 3 in two important respects. First, the interest-rate forecast is not part of the contract and is chosen by the central banker after the contract has been signed. Second, the setup in this chapter allows the construction of analytically tractable, longer-term contracts that might be needed when the natural real interest-rate shock is extremely severe.
We assume that a representative central banker shares the objectives of private agents and thus faces the same social loss function captured by the term $-0.5(\pi_t^2+\lambda x_t^2)$ in each period. However, in addition, he faces an FGC (b,k), which implies that he faces utility losses $b(i_t - i^f_{[0,k-1]})^2$ when he deviates during the contract term from his forecast $i^f_{[0,k-1]}$ made in period 0.
Thus, the loss function of the central banker in periods $t\in[0,k-1]$ is

$$L_{[0,k-1]} = -\frac{1}{2}\mathbb{E}_0\sum_{t=0}^{k-1}\beta^t[\pi_t^2+\lambda x_t^2+b(i_t - i^f_{[0,k-1]})^2], \tag{4.1}$$

where $i^f_{[0,k-1]}$ is the interest rate forecast made by the central bank in period 0 for k periods.
We consider two scenarios. In the first scenario, we examine the implications of a forward guidance contract $(b,2)$ (FGC $(b,2)$) when the size of the shock is known when the contract is signed. We will determine the optimal two-period FGC, i.e. the optimal intensity of incentives b that minimizes social losses. In the second scenario, we study ex-ante FGCs, i.e. cost-minimized FGCs that minimize expected losses when there is uncertainty about the parameter r_L when the contracts are signed. In this more realistic scenario, FGCs should perform well for different types of negative shocks. When shocks are particularly severe in terms of size and duration for instance, FGCs should make it

[†] Part of this chapter is summarized as part of CESifo working paper No.5375 (Gersbach et al. (2015)).

credible to keep nominal interest rates low, or at zero, for longer periods. The opposite should occur when shocks are small and the economy recovers quickly.

We first consider the simplest contract, i.e. one that only applies to the period in which it is written. Such a contract would never be signed in normal times. In the downturn, the central banker would also set zero interest rate in the absence of an FGC. Hence, such a contract would only replicate the discretionary solution, which we will study in the next section.

4.1 Benchmark Solution

In order to assess the potential and limitations of FGCs, we now summarize the well-known benchmark—discretionary policy. The formal details are given in Appendix B.1.[1]

We first summarize the behavior of a central banker that discretionarily chooses the nominal interest rate as its policy instrument in each period. In the downturn, a discretionary central banker will set interest rates to zero, as this is the maximal possible move to stabilize the shock. When the natural real interest rate has returned to a positive value, setting the nominal interest rate at the same value as the natural real interest rate minimizes social losses in a particular period. Hence, in normal times, $i_t = r_H$, $x_t = 0$ and $\pi_t = 0$. In downturns, $i_t = 0$.

Using Equations (3.1) and (3.2), the dynamics of the inflation and output gap during the downturn can be rewritten as[2]

$$\mathbb{E}_t \boldsymbol{Q_{t+1}} \equiv \mathbb{E}_t \begin{pmatrix} \pi_{t+1} \\ x_{t+1} \end{pmatrix} = \boldsymbol{O Q_t} - \frac{1}{\sigma} \boldsymbol{r_t}, \tag{4.2}$$

where $\boldsymbol{O} = \begin{pmatrix} \frac{1}{\beta} & -\frac{\kappa}{\beta} \\ -\frac{1}{\sigma\beta} & 1 + \frac{\kappa}{\sigma\beta} \end{pmatrix}$ and $\boldsymbol{r_t} = \begin{pmatrix} 0 \\ r_t \end{pmatrix}$.

The inflation and output gap are constant in the downturn and are given by

$$\boldsymbol{Q_D^d} = \frac{1}{h(\delta)} \begin{pmatrix} \kappa & 1 - \delta\beta \end{pmatrix}^T r_L, \tag{4.3}$$

where $h(\delta) = \sigma\beta\delta^2 - (\sigma + \kappa + \sigma\beta)\delta + \sigma > 0$, as set in Assumption 3.1.

[1] Another well-known benchmark—commitment policy—is also provided in Appendix B.1.
[2] For notational simplicity, we henceforth use the matrix expression.

4.2 Two-period Forward Guidance Contract

In this section, we will focus on two-period FGCs which can be renewed as long as the central bank faces a zero bound problem.

We note that two-period contracts, written at the beginning of a period, are contracts of the shortest length that can generate an impact through forward guidance.[3] We will illustrate in Section 4.4, that long-term contracts can generate high welfare when the natural real interest rate shock is large.

We assume for the moment that the interest-rate forecast chosen by the central bank is zero, i.e. $i_t^f = i_{t+1}^f = 0$. We will later argue that this is indeed the optimal choice.

We will use the stochastic recovery mode (see Eggertsson (2003)).

Definition 4.1 (Stochastic recovery)
In the stochastic recovery mode, the natural real interest rate returns to r_H with probability $1 - \delta \in (0, 1]$ in each period.

Throughout this chapter, we assume that δ is not too large and thus that the probability that the economy recovers is not too low. As in Eggertsson (2003) and Carlstrom et al. (2012), and as discussed in Appendix B.1, the approximation methods used to obtain the IS Equation and the Phillips Curve do not work well when the economy is expected to remain in a downturn for a very long time.

The variable $s \in \{d, n\}$ denotes the state of the economy in a particular period, where d stands for downturn and n represents normal time. If $s = d$, $r_t = r_L$ and if $s = n$, $r_t = r_H$.

4.2.1 The Sequence of Events

The detailed sequence of events is shown in Figure 4.1. In the beginning of each period, the natural real interest rate is realized (either r_L or r_H) and is common knowledge. After that, either an FGC exists or the government decides whether or not to (re-)sign the (same) FGC. In the next step, the central banker chooses the nominal interest rate. In the end of each period, expectations of inflations and output gaps are formed, and inflation and output gap are realized.

In period 0, for instance, after the realization of r_L, the government signs an FGC $(b, 2)$ that makes the central banker's remuneration contingent on the precision of his forecast.

[3] Renewable short-term contracts are attractive as they can reap most—if not all—possible welfare gains from FGCs for small and moderate negative natural real interest-rate shocks. They constrain the central bank as little as possible and thus involve the lowest risk in case of unforeseen events.

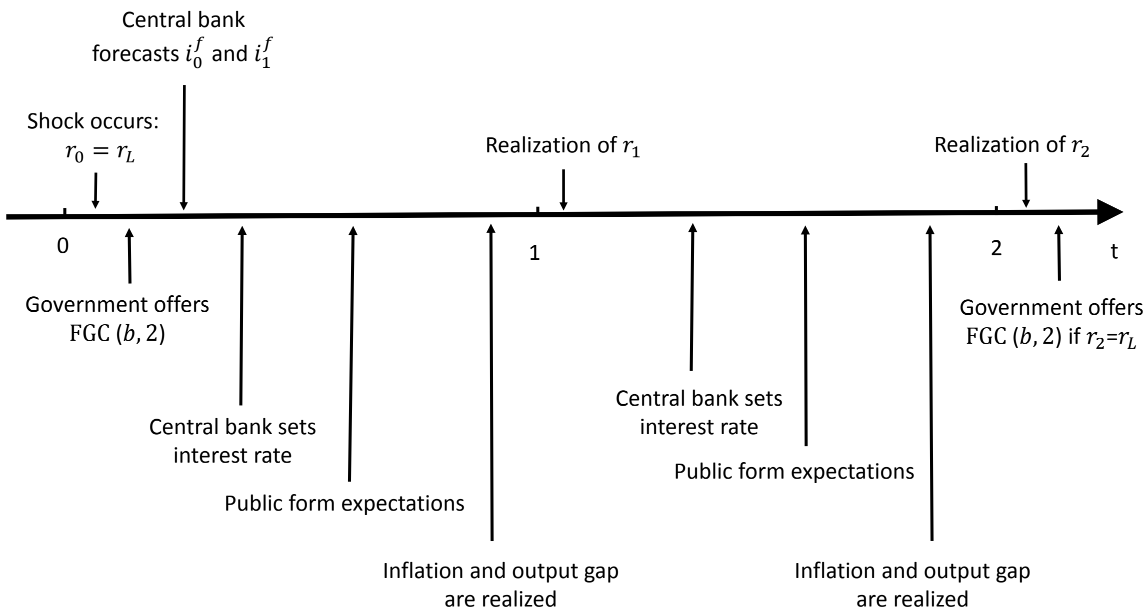

Figure 4.1: The sequence of events.

Then the central banker forecasts interest rates in periods 0 and 1. After that, the interest rate, i_0^d, in period 0 is set, and the inflation π_0^d and output gap x_0^d are realized.

In period $t = 1$, the central banker sets i_1^n if the economy has recovered. The corresponding inflation and output gap are π_1^n and x_1^n. If the economy is still in a downturn, the central banker sets i_1^d, and the corresponding inflation and output gap are π_1^d and x_1^d.

In period 2, the economy has either recovered and the central bank chooses the discretionary solution leading to zero inflation and zero output gap, or the natural real interest rate is still negative. In the latter case, the FGC $(b, 2)$ is re-signed and thus the central banker continues to select interest rates under FGCs.[4]

We observe that two-period contracts FGC $(b, 2)$ are the shortest contract under which forward guidance can have an impact. Indeed, a contract FGC $(b, 1)$ that only applies to the period in which it is written only replicates the discretionary solution. If the economy is in the downturn, the central banker would set zero interest rate even without such an FGC, and thus FGC $(b, 1)$ has no impact on economic variables. FGC $(b, 1)$ has no impact on the central banker's action in normal times either, since the contract, written in a downturn, ends in the same period. Thus, signing an FGC $(b, 1)$ is equivalent to the discretionary situation.

[4] In this section, we only explore the FGC $(b, 2)$ when the government commits to contracting. An exercise similar to the one in Subsection 3.3.2 can also be performed for this type of contract.

4.2.2 Evolution of the Economy

In this section, we derive the evolution of the economy for a given FGC $(b, 2)$ and thus for a given value of b.

Period 0

We start the analysis by expressing the IS Equation (3.1) and Phillips Curve (3.2) in matrix form as follows

$$\boldsymbol{Q_t} = \mathbb{E}_t \boldsymbol{O^{-1}} \boldsymbol{Q_{t+1}} + \frac{r_t - i_t}{\sigma} \begin{pmatrix} \kappa \\ 1 \end{pmatrix}, \tag{4.4}$$

where $\boldsymbol{O^{-1}} = \begin{pmatrix} \beta + \frac{\kappa}{\sigma} & \kappa \\ \frac{1}{\sigma} & 1 \end{pmatrix}$ and $\boldsymbol{Q_t} = \begin{pmatrix} \pi_t \\ x_t \end{pmatrix}$.

In period 0, a negative shock on the natural real interest rate occurs with $r_0 = r_L$ and the 2-period FGC is operating. Equation (4.4) becomes

$$\boldsymbol{Q_0^d} = \mathbb{E}_0 \boldsymbol{O^{-1}} \boldsymbol{Q_1} + \frac{r_L - i_0^d}{\sigma} \begin{pmatrix} \kappa \\ 1 \end{pmatrix}, \tag{4.5}$$

where i_0^d is the chosen nominal interest rate.

Recovery in Period 1

In period 1, the natural real interest rate reverts back to r_H with probability $1 - \delta$. In this case, in period 2—one period later—the contract ends and the central bank becomes discretionary. Thus, $\pi_t = x_t = 0, \forall t \geq 2$. In period 1 with $\mathbb{E}_1 \boldsymbol{Q_2} = 0$, the central banker maximizes

$$L_1 = -\frac{1}{2}[\pi_1^2 + \lambda x_1^2 + b(i_1^n)^2] \tag{4.6}$$

$$s.t.$$

$$\pi_1 = \kappa x_1,$$

$$x_1 = -\frac{1}{\sigma}(i_1^n - r_H),$$

$$i_1^n \geq 0.$$

The objective can be rewritten as

$$L_1 = -\frac{1}{2}[\frac{\lambda + \kappa^2}{\sigma^2}(i_1^n - r_H)^2 + b(i_1^n)^2]. \tag{4.7}$$

We note that the restriction $i_1^n \geq 0$ is not binding. Calculating the first-order condition and solving for i_1^n yields

Lemma 4.1

$$i_1^n = \frac{\lambda + \kappa^2}{\lambda + \kappa^2 + b\sigma^2} r_H. \tag{4.8}$$

We note how different values of b impact the chosen interest rate. If costs of deviation are very small (i.e. if b is small), i_1^n is close to r_H and thus to the discretionary solution. If b is very large, the central banker sets a very low interest rate.
We combine Equations (4.4) and (4.8) and obtain the inflation and output gap

$$\boldsymbol{Q_1^n} \equiv \begin{pmatrix} \pi_1^n \\ x_1^n \end{pmatrix} = \frac{b\sigma}{\lambda + \kappa^2 + b\sigma^2} \begin{pmatrix} \kappa \\ 1 \end{pmatrix} r_H, \tag{4.9}$$

where $\boldsymbol{Q_1^n}$ represents the inflation and output gap levels in period 1 when the economy has recovered.

No Recovery in Period 1

In period 1, with probability δ, the natural real interest rate is still low, i.e. $r_1 = r_L$. In this situation, Equation (4.4) becomes

$$\boldsymbol{Q_1^d} = \mathbb{E}_1 \boldsymbol{O^{-1}} \boldsymbol{Q_2} + \frac{r_L - i_1^d}{\sigma} \begin{pmatrix} \kappa \\ 1 \end{pmatrix}, \tag{4.10}$$

where $\boldsymbol{Q_1^d}$ represents the inflation and output gap levels and i_1^d is the interest rate the central banker chooses in the downturn.
From the perspective of period 1, two situations can occur in period 2. First, in period 2, the natural real interest rate is still r_L with probability δ. Since the FGC $(b, 2)$ signed in period 0 expires in this period, and the economic situation in period 2 is the same as in period 0, the FGC $(b, 2)$ is introduced again. As in period 2, the central bank faces the same economic situation and the same contract as in period 0, the inflation and output gap in period 2 are the same as the ones in period 0.
Second, in period 2, the natural real interest rate bounces back to the high natural real interest rate with probability $1 - \delta$. Then there is no need for the government to re-sign the FGC $(b, 2)$. Thus, the central banker discretionarily chooses his policy and sets $i_2^n = r_H$. Therefore, inflation and output gap are zero in period 2 and in all subsequent periods.
From the perspective of period 1, when the economy is still in the downturn, the expected

inflation and output gap for period 2 are thus

$$\mathbb{E}_1 \boldsymbol{Q_2} = (1-\delta)\boldsymbol{Q_2^n} + \delta \boldsymbol{Q_2^d} = (1-\delta)\boldsymbol{0} + \delta \boldsymbol{Q_0^d} = \delta \boldsymbol{Q_0^d}. \tag{4.11}$$

Equations (4.5), (4.9), (4.10) and (4.11) express the economic outcomes in periods 0 and 1 explicitly as functions of the parameters. In Appendix B.2, we derive $\boldsymbol{Q_0^d}$ and $\boldsymbol{Q_1^d}$ as follows:

Lemma 4.2

$$\boldsymbol{Q_0^d} = \frac{1}{f(\delta)} \left(\boldsymbol{O_0} \begin{pmatrix} \frac{b\sigma^2}{\lambda+\kappa^2+b\sigma^2}(1-\delta) r_H \\ r_L \end{pmatrix} - \boldsymbol{\tilde{O}_0} \begin{pmatrix} i_0^d \\ i_1^d \end{pmatrix} \right), \tag{4.12}$$

$$\boldsymbol{Q_1^d} = \frac{1}{f(\delta)} \left(\boldsymbol{O_1} \begin{pmatrix} \frac{b\sigma}{\lambda+\kappa^2+b\sigma^2}\delta(1-\delta) r_H \\ r_L \end{pmatrix} - \boldsymbol{\tilde{O}_1} \begin{pmatrix} i_0^d \\ i_1^d \end{pmatrix} \right), \tag{4.13}$$

where the function $f(\delta)$ and the 2×2 matrices $\boldsymbol{O_0}$, $\boldsymbol{\tilde{O}_0}$, $\boldsymbol{O_1}$ and $\boldsymbol{\tilde{O}_1}$ are defined in the proof of the lemma, and solely depend on the parameters of the model.

We observe that changes of b—through its impact on the choice of the interest rate once the economy recovers—affect the inflation and output gap during the entire downturn. Hence, FGCs can affect expected losses.

4.2.3 Optimal Contracts

In this section, we derive optimal FGCs and thus determine the socially optimal value of b. We proceed in three steps. We first calculate the expected intertemporal social losses, evaluated in period 0. In the second step, we establish that zero interest rate during downturns continues to be a binding constraint with FGCs. Finally, in the third step, we establish the optimal value of b and thus the optimal FGC $(b, 2)$.

In the first step, we combine the results in the previous subsections and obtain the expected social losses

$$\begin{aligned} \mathbb{E}_0 L_{[0,\infty]}^{k=2} &= -0.5(\boldsymbol{\Lambda Q_0^d})^{\boldsymbol{T}} \boldsymbol{\Lambda Q_0^d} - 0.5\beta[(1-\delta)(\boldsymbol{\Lambda Q_1^n})^{\boldsymbol{T}} \boldsymbol{\Lambda Q_1^n} + \delta(\boldsymbol{\Lambda Q_1^d})^{\boldsymbol{T}} \boldsymbol{\Lambda Q_1^d}] \\ &\quad - 0.5\beta^2\delta^2\{(\boldsymbol{\Lambda Q_0^d})^{\boldsymbol{T}} \boldsymbol{\Lambda Q_0^d} + \beta[(1-\delta)(\boldsymbol{\Lambda Q_1^n})^{\boldsymbol{T}} \boldsymbol{\Lambda Q_1^n} + \delta(\boldsymbol{\Lambda Q_1^d})^{\boldsymbol{T}} \boldsymbol{\Lambda Q_1^d}]\} \\ &\quad - \cdots \\ &= -0.5 \frac{(\boldsymbol{\Lambda Q_0^d})^{\boldsymbol{T}} \boldsymbol{\Lambda Q_0^d} + \beta[(1-\delta)(\boldsymbol{\Lambda Q_1^n})^{\boldsymbol{T}} \boldsymbol{\Lambda Q_1^n} + \delta(\boldsymbol{\Lambda Q_1^d})^{\boldsymbol{T}} \boldsymbol{\Lambda Q_1^d}]}{1-\beta^2\delta^2}, \end{aligned} \tag{4.14}$$

where $\boldsymbol{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}$ and $\boldsymbol{Q_0^d}$, $\boldsymbol{Q_1^d}$ and $\boldsymbol{Q_1^n}$ are given in Equations (4.12), (4.13) and (4.9).

We next establish that the zero interest rate is still a binding constraint when FGCs are used.

Proposition 4.1

(i) There exists a threshold value $\hat{r}_L < 0$, such that the central bank sets $i_0^d = i_1^d = 0$ for any value of b in an FGC $(b, 2)$ if $r_L \leq \hat{r}_L$.

(ii) The central bank sets $i_0^d = i_1^d = 0$ under an optimal FGC $(b, 2)$ for any $r_L < 0$.

The proof of Proposition 4.1 is given in Appendix B.3. In the proof of this Proposition, we also provide an explicit formula for the threshold values $\hat{r}_L$. The intuition for Proposition 4.1 is straightforward. If the negative interest rate shock is sufficiently severe, the zero bound is a constraint for the central bank for any FGC $(b, 2)$. Any boom and inflation that can be created by an FGC $(b, 2)$ once the shock has died out is insufficient[5] to lift the economy in the downturn to the level at which the central bank optimally starts to move away from the zero interest rate. In contrast, if the natural real interest rate shock is moderate or small, optimal levels of b are set at a sufficiently low level such that after the return to normal times, the induced boom and inflation do not cause the central bank to already start moderating the economy in the downturn. Too high values of b would cause inefficiently large booms and inflation.

The threshold value $\hat{r}_L$ depends on the parameters. In Figure 4.2, we show how $\hat{r}_L$ depends on the recovery probability δ for the baseline calibration.

Finally, in the third step, we derive the optimal value of b in FGC $(b, 2)$.

Proposition 4.2

There exists a unique optimal FGC $(b, 2)$ characterized by

$$b = \begin{cases} 0, & \text{if } r_L \geq 0, \\ \frac{\lambda+\kappa^2}{\sigma^2} \frac{1}{\frac{r_L^c}{r_L} - 1}, & \text{if } r_L \in (r_L^c, 0), \\ \infty, & \text{if } r_L \leq r_L^c, \end{cases} \tag{4.15}$$

where $r_L^c < 0$ is a critical value of the natural real interest rate shock whose explicit formula is given in the proof in Appendix B.4.

Proposition 4.2 shows that as soon as the natural real interest rate shock becomes negative,

[5] In this case, an FGC $(b, k > 2)$ might be optimal. The FGCs of longer term are studied in Section 4.4.

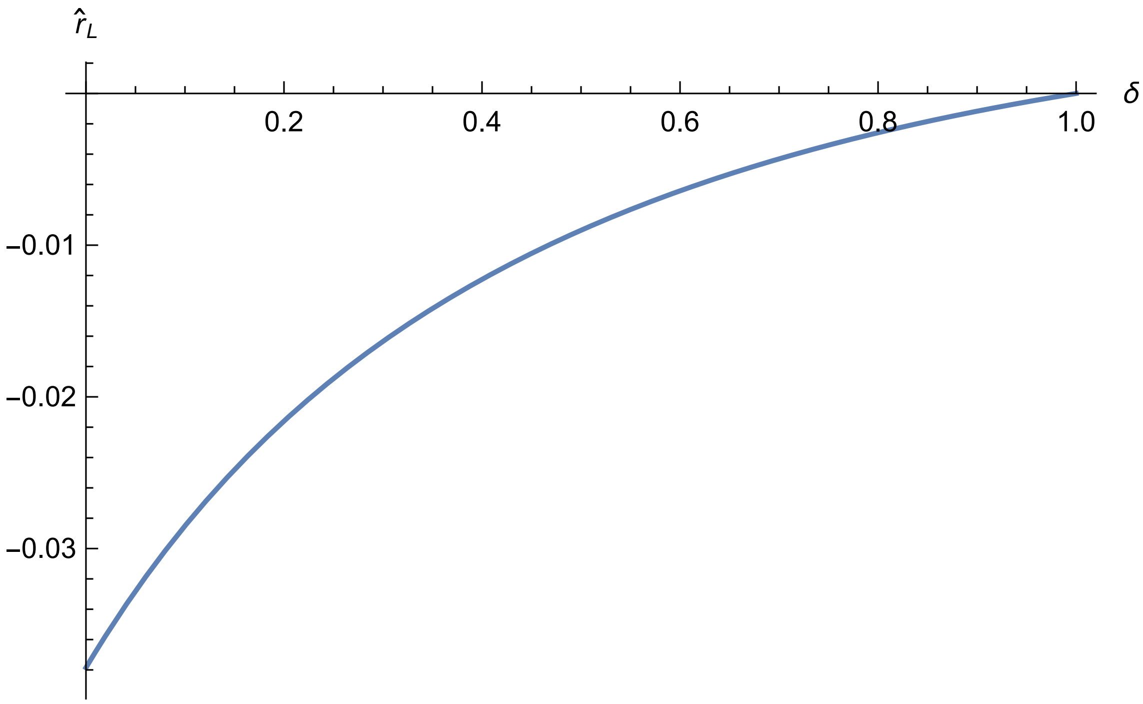

Figure 4.2: The threshold value $\hat{r}_L$ with respect to δ.

it is optimal to use FGCs with small values of b if the shock is small and large values if the shock is severe.

Figure 4.3 demonstrates the optimal b with respect to the size of the shock r_L for the baseline calibration. The critical level r_L^c depends on the parameters. In Figure 4.4, we show how r_L^c depends on the recovery probability δ for the baseline calibration.

4.2.4 Examples

In this section, we illustrate the proceeding results by examples using our baseline calibration. We first assume $r_L = -0.03$. Thus, $r_L \leq \hat{r}_L = -0.009$ and $r_L \leq r_L^c = -0.0092$. According to Proposition 4.1, the central banker sets $i_0^d = i_1^d = 0$. According to Proposition 4.2, the corresponding optimal b is infinity.

Figures 4.5–4.7 display the dynamics of inflation and the output gap in period 0 and 1 as a function of b. The negative values of inflation and output gap in the downturn confirm that setting $i_0^d = 0$ and $i_1^d = 0$ is optimal, and thus that the zero bound is binding, regardless of the value of b.

Figures 4.5 and 4.6 show that a higher value of b induces less deflation and decline of output in the downturn, caused by greater inflation and output in period 1 when the economy is out of the downturn, as shown in Figure 4.7. However, the impact on π_1^n and x_1^n is comparatively small. This rationalizes that the optimal value of b is extremely large (infinity) which is illustrated by Figure 4.8.

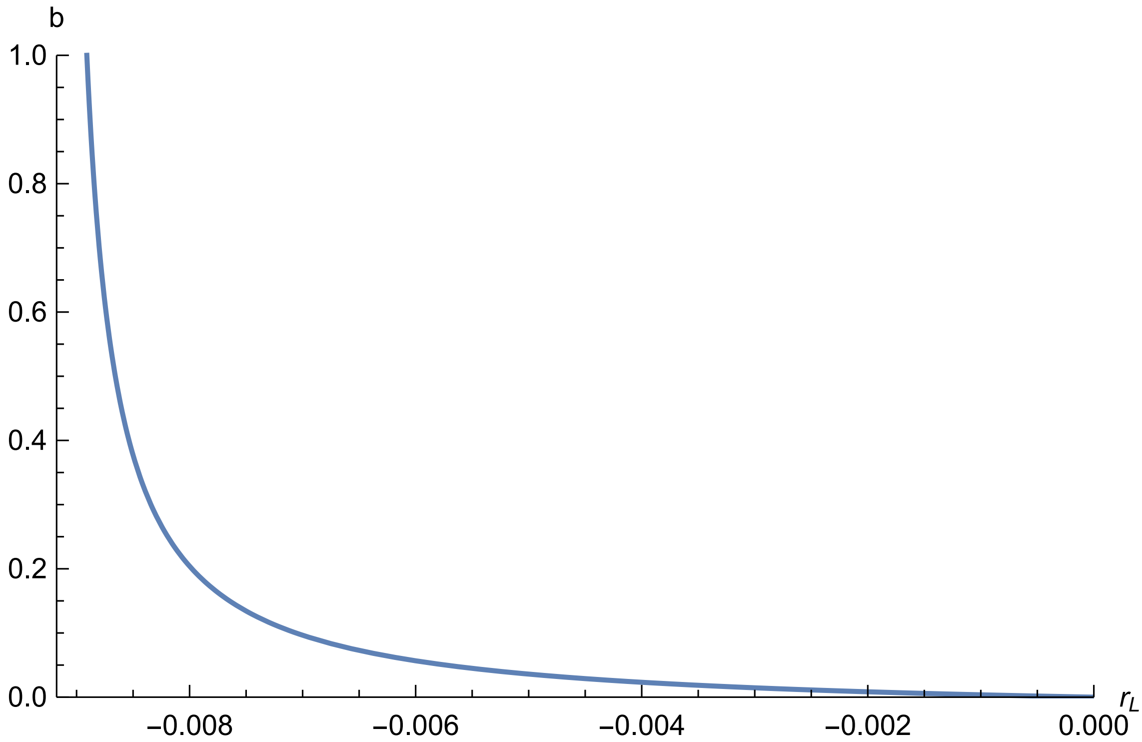

Figure 4.3: Optimal b with respect to the size of the shock r_L.

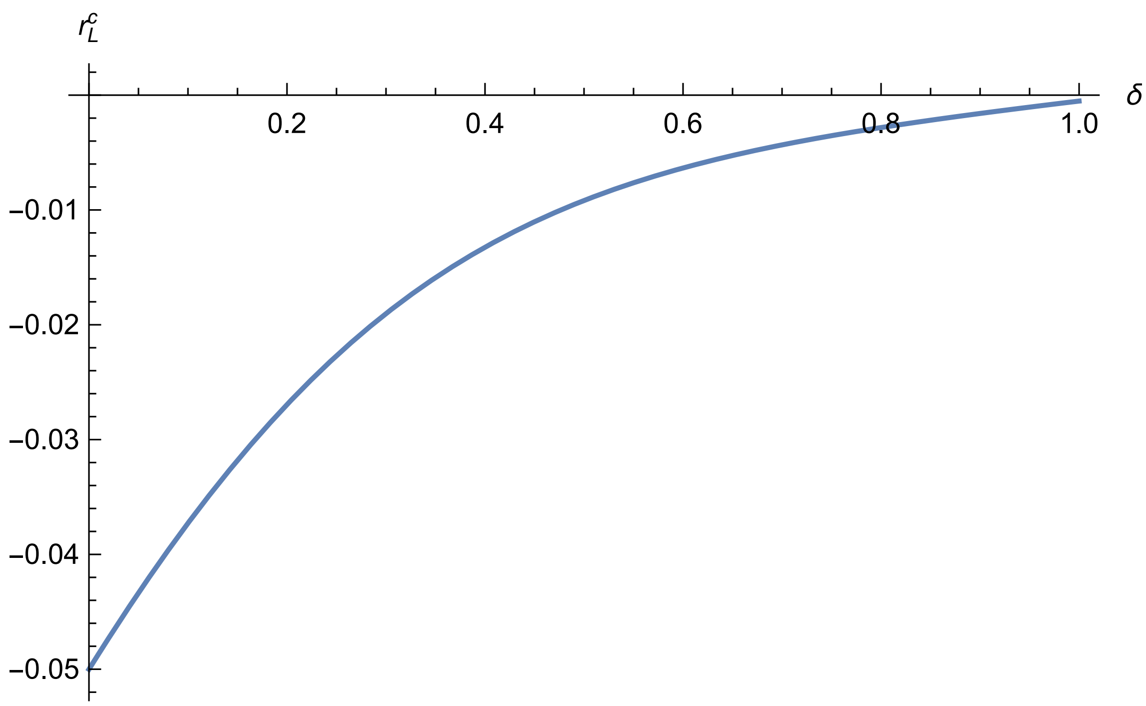

Figure 4.4: The critical value r_L^c with respect to δ.

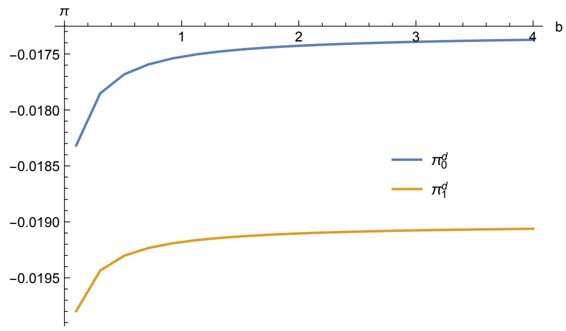

Figure 4.5: The inflation rates in period 0 and in period 1 w.r.t. b when the economy is in the downturn.

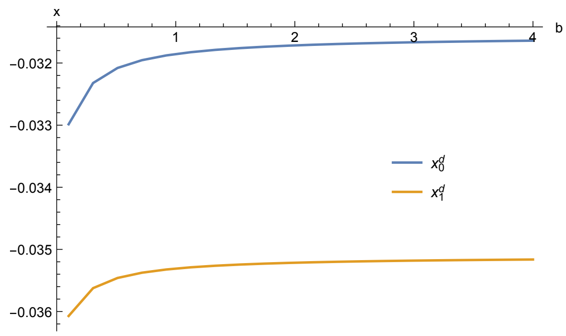

Figure 4.6: The output gaps in period 0 and in period 1 w.r.t. b when the economy is in the downturn.

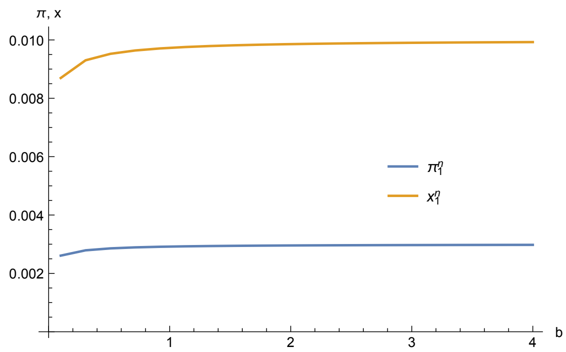

Figure 4.7: The inflation and output gap in period 1 w.r.t. b when the economy recovers in period 1.

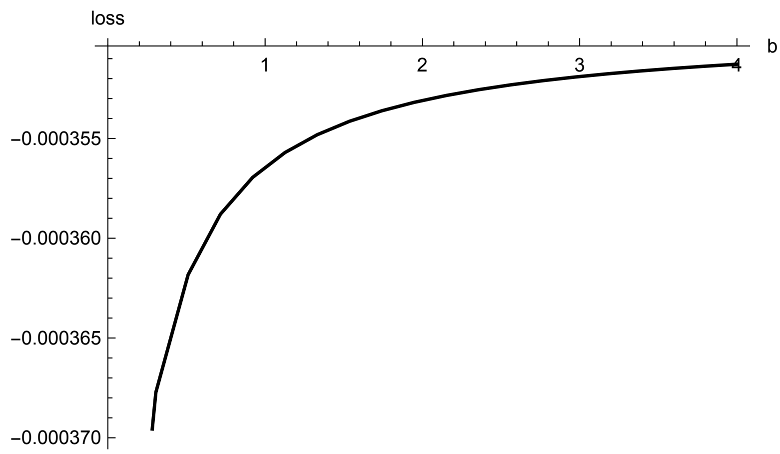

Figure 4.8: The intertemporal social losses of two-period FGC w.r.t. b.

We next investigate the scenario, when $r_L \in (r_L^c, 0)$. In our calibration, we assume $r_L = -0.005$. Equation (4.15) implies the optimal b is 0.036. Figures 4.9–4.11 show the dynamics of inflation and output gap in period 0 and 1 as a function of b.

Figures 4.9 and 4.10 show that large values of b induce positive values of inflation and output gap, while Figure 4.11 demonstrates how larger values of b cause greater inflation and output gap in period 1, when the economy is out of the downturn. Figure 4.12 illustrates that the social loss is minimized at $b = 0.036$.

Finally, we examine how the outcomes depend on the size of the shock in more detail. When $r_L \in (r_L^c, 0)$ and b is optimally chosen, Equations (4.12), (4.13) and (4.9) can be written as

$$\boldsymbol{Q_0^d} = \frac{1}{f(\delta)} \boldsymbol{O_0} \begin{pmatrix} (1-\delta)\frac{r_L}{r_L^c} r_H \\ r_L \end{pmatrix}, \tag{4.16}$$

$$\boldsymbol{Q_1^d} = \frac{1}{f(\delta)} \boldsymbol{O_1} \begin{pmatrix} \frac{\delta(1-\delta)}{\sigma} \frac{r_L}{r_L^c} r_H \\ r_L \end{pmatrix}, \tag{4.17}$$

and

$$\boldsymbol{Q_1^n} = \frac{1}{\sigma} \begin{pmatrix} \kappa \\ 1 \end{pmatrix} \frac{r_L}{r_L^c} r_H. \tag{4.18}$$

Figure 4.13 shows the inflation and output gap in the downturn when b is chosen optimally. We note that the output gap is negative while inflation π_0^d is almost zero[6].

Figures 4.14 and 4.15 display the dynamics of inflation and output gap in function of r_L and δ, respectively, in period 1 when the economy recovers to the normal time. These two figures demonstrate that the larger the probability of staying in the downturn or the larger the size of the shock, the higher is the optimal inflation in the first period in normal time.

4.2.5 Interest Rate Forecasts in Downturns

So far, we have assumed that the central bank makes zero interest rate forecasts in downturns. However, given the choices of b described in Proposition 4.2, making positive interest rate forecasts in downturns would only increase losses for central bankers, since $i_0^d = i_1^d = 0$ is optimal. In turn, by the same logic as in Proposition 4.1, setting b at levels that would induce positive interest rate forecast cannot be optimal.

4.3 Ex Ante Forward Guidance Contracts

In this section, we explore optimal FGCs that are written under a veil of uncertainty about the precise nature of the natural real interest rate shock. This is a plausible scenario, as

[6] A refined figure of π_0^d reveals that it is always below 0.0004, but positive.

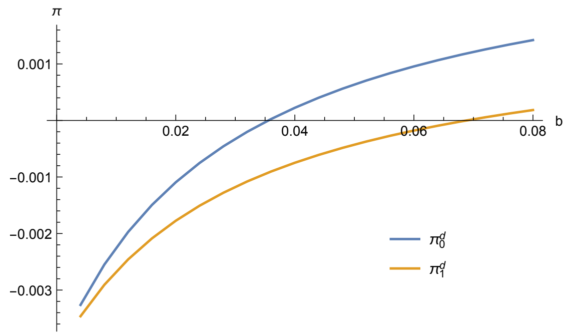

Figure 4.9: The inflation rates in period 0 and in period 1 w.r.t. b when the economy is in a downturn.

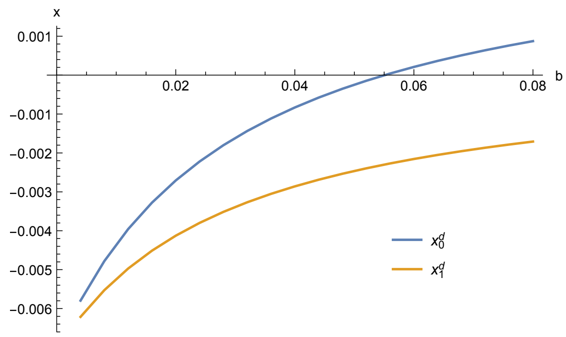

Figure 4.10: The output gaps in period 0 and in period 1 w.r.t. b when the economy is in a downturn.

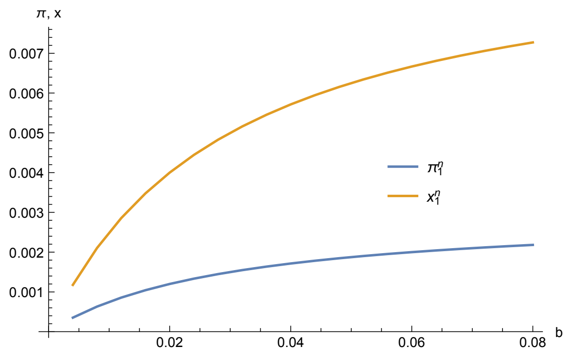

Figure 4.11: The inflation and output gap in period 1 w.r.t. b when the economy recovers in period 1.

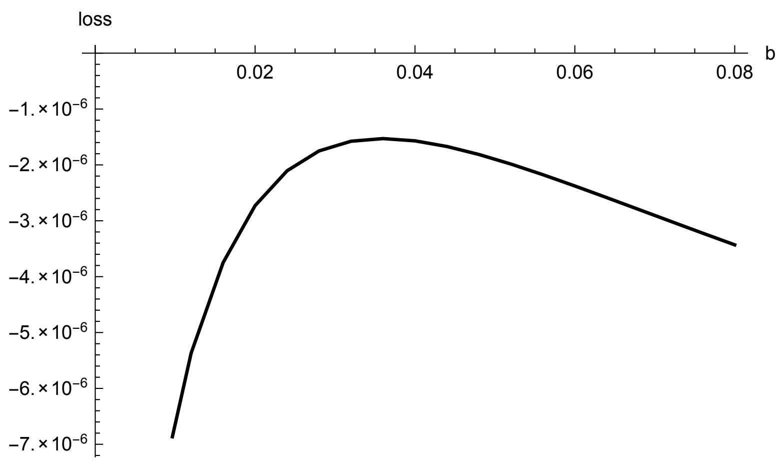

Figure 4.12: The intertemporal social losses of a two-period FGC w.r.t. b.

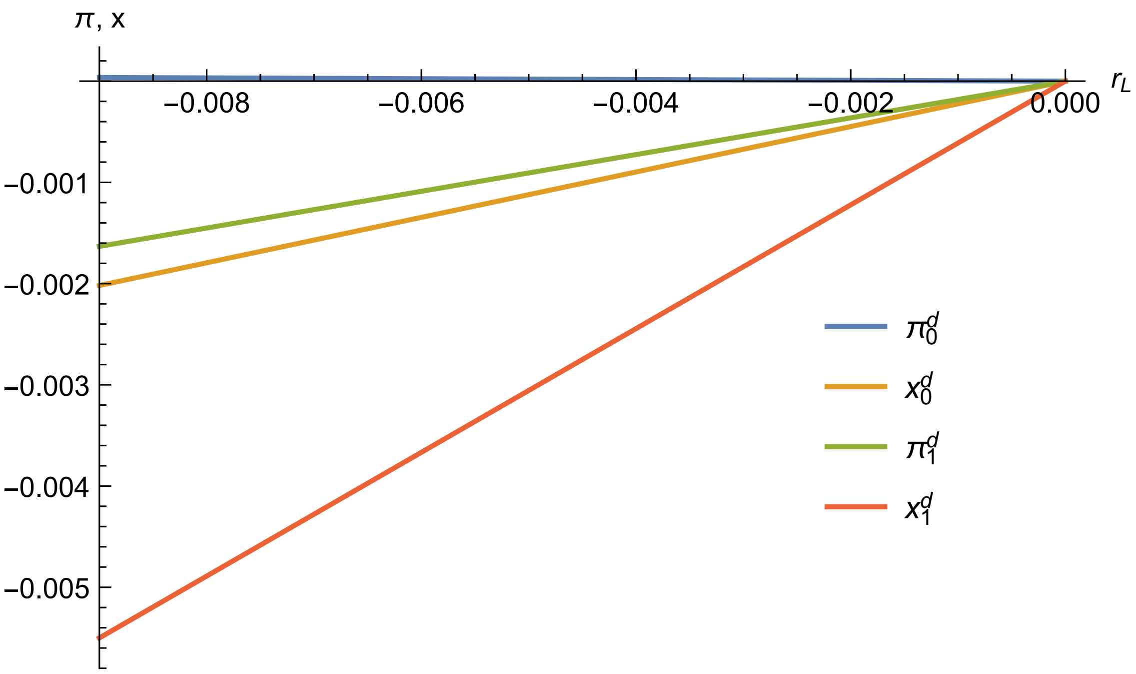

Figure 4.13: The evolution of the inflation and output gap with optimal value of b in the downturn.

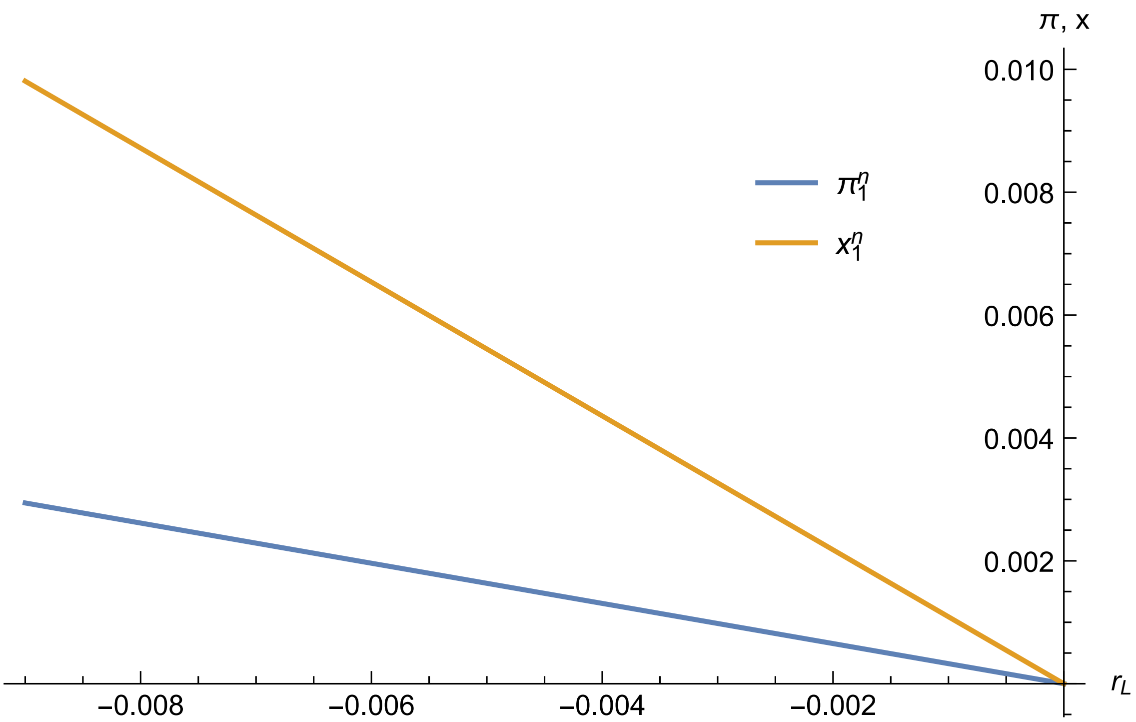

Figure 4.14: The evolution of the inflation rate and output gap with optimal value of b in the normal time.

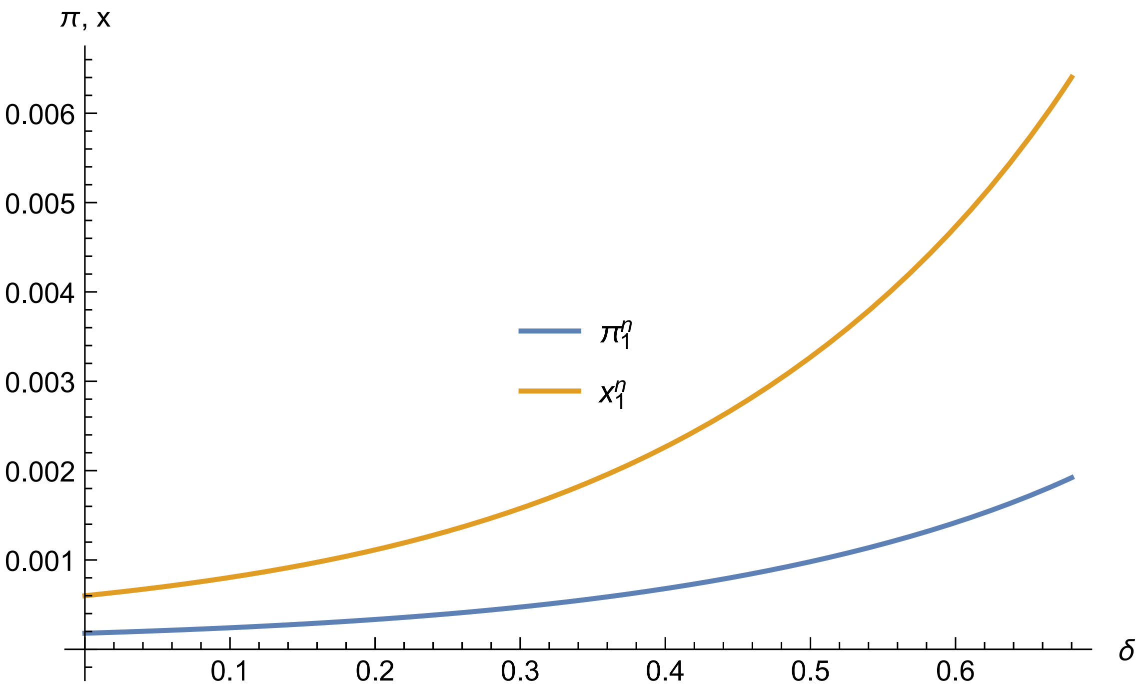

Figure 4.15: The evolution of the inflation rate and output gap with optimal value of b in the normal time with $r_L = -0.003$.

FGCs cannot be adjusted at a high frequency and may have to be written in normal times. Moreover, negative natural real interest rate shocks are difficult to identify instantaneously when they occur. Hence, it is important that FGCs have positive effects in normal times or in situations where negative natural real interest rate shocks with different sizes can occur once the contract has been written.

The timing of events is as follows. At the end of period -1, the government designs an FGC^{ex} $(b, 2)$. In period 0, a shock to natural real interest rate occurs. More specifically, the natural real interest rate drops to r_{L1} with probability ω_1, to r_{L2} with probability ω_2 and to r_{L3} with probability $\omega_3 = 1 - \omega_1 - \omega_2$, where $r_{L1} \leq r_L^c < r_{L2} < 0 \leq r_{L3} \leq r_H$. With probability $\omega_1 + \omega_2$, the natural real interest rate is negative. In this case, the economy risks a deflation spiral even after reducing the nominal interest rate to the zero lower bound. To avoid this bad outcome, the government signs an FGC^{ex} $(b, 2)$ with the central bank at the very end of period -1.

In period 2, with probability $1 - \delta$, the natural real interest rate recovers to r_H. In this case, no FGC is re-signed and the central banker sets $i_2^n = r_H$. Therefore, $\boldsymbol{Q_2^n} = \mathbf{0}$.

With probability δ, the natural real interest rate is still at a low level. We distinguish three cases.

Case 1: With probability ω_1, $r_L = r_{L1}$.

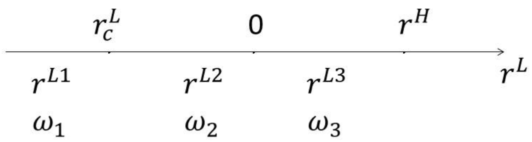

Figure 4.16: The distribution of the natural real interest rate after the shock.

The optimal FGC ($b^{d1} = \infty, 2$) is signed in period 2[7]. According to Proposition 4.1, the central bank sets a zero nominal interest rate in the downturn. Thus, Equation (4.12) implies

$$\boldsymbol{Q_2^{d1}} = \frac{1}{f(\delta)}\boldsymbol{O_0}\begin{pmatrix} (1-\delta)r_H \\ r_{L1} \end{pmatrix}. \tag{4.19}$$

The expectations formed in period 1 are

$$\mathbb{E}_1\boldsymbol{Q_2} = (1-\delta)\boldsymbol{Q_2^n} + \delta\boldsymbol{Q_2^{d1}} = \delta\boldsymbol{Q_2^{d1}}. \tag{4.20}$$

Equation (4.4) implies

$$\boldsymbol{Q_1^{d1}} = \mathbb{E}_1\boldsymbol{O^{-1}Q_2} + \frac{r_{L1}}{\sigma}\begin{pmatrix} \kappa \\ 1 \end{pmatrix}, \tag{4.21}$$

where the formal proof[8] of $i_1^{d1} = 0$ is provided in Lemma 4.3.

[7] We use the superscripts $d1$, $d2$ and $d3$ to indicate the variables in the downturn in Cases 1, 2 and 3, respectively.

[8] Note that Proposition 4.1 might not be appropriate here. In Proposition 4.1, FGC $(b, 2)$ is effective whenever the economy is in the downturn. However, in the present circumstances, different FGCs $(b, 2)$ are signed in periods -1 and 2.

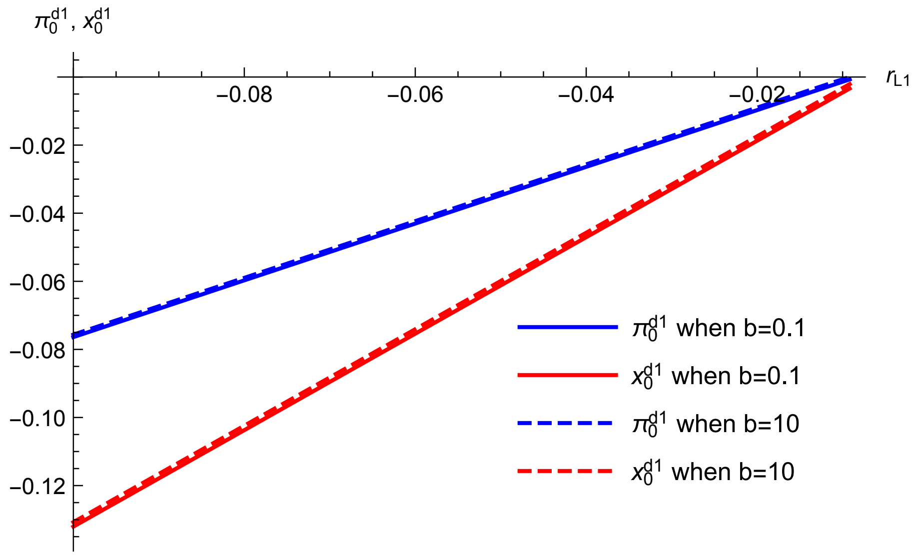

Figure 4.17: The values of π_0^{d1} and x_0^{d1} in function of $r_{L1} \in [-0.1, r_L^c]$ for different values of b, where $r_L^c = -0.009$ in our calibration.

Combining Equations (4.19), (4.21) and (4.20) yields

$$\begin{aligned} \boldsymbol{Q_1^{d1}} &= \delta \boldsymbol{O^{-1}} \boldsymbol{Q_2^{d1}} + \frac{r_{L1}}{\sigma} \begin{pmatrix} \kappa \\ 1 \end{pmatrix} \\ &= \frac{\delta}{f(\delta)} \boldsymbol{O^{-1}} \boldsymbol{O_0} \begin{pmatrix} (1-\delta) r_H \\ r_{L1} \end{pmatrix} + \frac{r_{L1}}{\sigma} \begin{pmatrix} \kappa \\ 1 \end{pmatrix}. \end{aligned} \tag{4.22}$$

If the economy recovers, the inflation and output gap are given in Equation (4.9). Thus, the expectations in period 0 are

$$\mathbb{E}_0 \boldsymbol{Q_1} = (1-\delta) \boldsymbol{Q_1^n} + \delta \boldsymbol{Q_1^{d1}}. \tag{4.23}$$

The inflation and output gap in period 0 are determined by

$$\boldsymbol{Q_0^{d1}} = \mathbb{E}_0 \boldsymbol{O^{-1}} \boldsymbol{Q_1} + \frac{r_{L1}}{\sigma} \begin{pmatrix} \kappa \\ 1 \end{pmatrix}, \tag{4.24}$$

where $i_0^{d1} = 0$ is proved in Lemma 4.4.

Figure 4.17 demonstrates that the zero lower bound on the nominal interest rate is binding, due to the negative inflation and output gap in period 0. It also demonstrates that increasing the value of b has little impact on the inflation and output gap in period 0.

The value of b has no impact on $\boldsymbol{Q_1^{d1}}$, since the central banker would set $i_1^{d1} = 0$ irrespective of the value of b. The value of b does not affect the central banker's choice of setting $i_0^{d1} = 0$, but it does have an impact on $\boldsymbol{Q_0^{d1}}$ via the expectations, as is shown in Equations

(4.9) and (4.23). A higher value of b induces larger $\boldsymbol{Q_1^n}$, which ultimately induces larger, but still negative, $\boldsymbol{Q_0^{d1}}$.

Case 2: With probability ω_2, $r_L = r_{L2}$.
In period 2, if the economy is still in the downturn, the FGC ($b^{d2}, 2$) is signed, where the optimal b is given by Equation (4.15)

$$b^{d2} = -\frac{(\lambda + \kappa^2)\Delta_1 g(\delta)}{\sigma\Delta_2 r_H + \sigma^2\Delta_1 g(\delta) r_{L2}} r_{L2}. \tag{4.25}$$

Equation (4.16) implies

$$\boldsymbol{Q_2^{d2}} = \frac{1}{f(\delta)\Delta_2}\boldsymbol{O_0}\begin{pmatrix} -(1-\delta)\sigma\Delta_1 g(\delta) \\ \Delta_2 \end{pmatrix} r_{L2}. \tag{4.26}$$

Thus,

$$\mathbb{E}_1\boldsymbol{Q_2} = (1-\delta)\boldsymbol{Q_2^n} + \delta\boldsymbol{Q_2^{d2}} = \delta\boldsymbol{Q_2^{d2}}. \tag{4.27}$$

In period 1, if the economy is still in the downturn, Equation (4.4) implies

$$\boldsymbol{Q_1^{d2}} = \mathbb{E}_1\boldsymbol{O^{-1}Q_2} + \frac{r_{L2}}{\sigma}\begin{pmatrix} \kappa \\ 1 \end{pmatrix}, \tag{4.28}$$

where $i_1^{d2} = 0$ is proved in Lemma 4.3.
Combining Equations (4.26), (4.27) and (4.28) yields

$$\begin{aligned} \boldsymbol{Q_1^{d2}} &= \delta\boldsymbol{O^{-1}Q_2^{d2}} + \frac{r_{L2}}{\sigma}\begin{pmatrix} \kappa \\ 1 \end{pmatrix} \\ &= \frac{\delta}{f(\delta)\Delta_2}\boldsymbol{O^{-1}O_0}\begin{pmatrix} -(1-\delta)\sigma\Delta_1 g(\delta) \\ \Delta_2 \end{pmatrix} r_{L2} + \frac{r_{L2}}{\sigma}\begin{pmatrix} \kappa \\ 1 \end{pmatrix}. \end{aligned} \tag{4.29}$$

If the economy recovers, the inflation and output gap are given in Equation (4.9).
Thus,

$$\mathbb{E}_0\boldsymbol{Q_1} = (1-\delta)\boldsymbol{Q_1^n} + \delta\boldsymbol{Q_1^{d2}}. \tag{4.30}$$

The inflation and output gap in period 0 are determined by

$$\boldsymbol{Q_0^{d2}} = \mathbb{E}_0\boldsymbol{O^{-1}Q_1} + \frac{r_{L2} - i_0^{d2}}{\sigma}\begin{pmatrix} \kappa \\ 1 \end{pmatrix}. \tag{4.31}$$

In conclusion, the value of b has no impact on $\boldsymbol{Q_1^{d2}}$, since the central banker would set $i_1^{d2} = 0$ irrespective of the value of b. However, the value of b does affect the central banker's choice of i_0^{d2}, as is proved in Lemma 4.4. In addition, it has impact on $\boldsymbol{Q_1^n}$, which influences the expectations in period 0.

Lemma 4.3

With an FGC^{ex} $(b, 2)$ signed in period -1 and anticipating that the optimal renewable FGC $(b, 2)$ would be signed, if necessary, the central banker would set $i_1^{d1} = i_1^{d2} = 0$ for any value of b.

The formal proof of Lemma 4.3 is given in Appendix B.5.

Case 3: With probability $1 - \omega_1 - \omega_2$, $r_L = r_{L3}$

Since $r_{L3} > 0$, conventional monetary policy suffices to handle this situation. Thus, no FGC is signed in period 2, and the central banker sets $i_2^{d3} = r_{L3}$ in the downturn and $i_2^n = r_H$ if out of the downturn. Therefore,

$$\boldsymbol{Q_2^{d3}} = \boldsymbol{0}. \tag{4.32}$$

Thus, the expectations in period 1 are

$$\mathbb{E}_1\boldsymbol{Q_2} = (1-\delta)\boldsymbol{Q_2^n} + \delta\boldsymbol{Q_2^{d3}} = \boldsymbol{0}. \tag{4.33}$$

In period 1, if the economy is in the downturn, as in Equations (4.8) and (4.9), the optimal interest rate set by the central banker is

$$i_1^{d3} = \frac{\lambda + \kappa^2}{\lambda + \kappa^2 + b\sigma^2} r_{L3}. \tag{4.34}$$

The inflation and output gap are

$$\boldsymbol{Q_1^{d3}} = \frac{b\sigma}{\lambda + \kappa^2 + b\sigma^2} \begin{pmatrix} \kappa \\ 1 \end{pmatrix} r_{L3}. \tag{4.35}$$

If the economy recovers, the inflation and output gap are given in Equation (4.9).

Thus,

$$\begin{aligned} \mathbb{E}_0\boldsymbol{Q_1} &= (1-\delta)\boldsymbol{Q_1^n} + \delta\boldsymbol{Q_1^{d3}} \\ &= \frac{b\sigma[(1-\delta)r_H + \delta r_{L3}]}{\lambda + \kappa^2 + b\sigma^2} \begin{pmatrix} \kappa \\ 1 \end{pmatrix}. \end{aligned} \tag{4.36}$$

The inflation and output gap in period 0 are determined by

$$\boldsymbol{Q_0^{d3}} = \mathbb{E}_0 \boldsymbol{O^{-1}Q_1} + \frac{r_{L3} - i_0^{d3}}{\sigma} \begin{pmatrix} \kappa \\ 1 \end{pmatrix}. \tag{4.37}$$

Lemma 4.4

With an FGC^{ex} $(b, 2)$ signed in period -1 and in anticipation that the optimal renewable FGC $(b, 2)$ would be signed, if necessary, the central banker would set $i_0^{d1} = 0$ for any value of b and $r_{L1} \leq r_L^c$; while the values of i_0^{d2} and i_0^{d3} depend on the size of the shock and the value of b. If the zero lower bound is not binding, the optimal nominal interest rate in period 0 in Case 2 is

$$\begin{aligned} i_0^{d2} =& [(\kappa^2 + \lambda)(\kappa + \sigma) + \kappa^2 \sigma \beta] \frac{\sigma(1-\delta)b}{(\lambda + \kappa^2 + b\sigma^2)^2} r_H + \delta \frac{[\lambda + \kappa(\kappa + \tilde{\sigma\beta})]\pi_1^{d2} + \sigma(\lambda + \kappa^2) x_1^{d2}}{\lambda + \kappa^2 + b\sigma^2} \\ &+ \frac{\lambda + \kappa^2}{\lambda + \kappa^2 + b\sigma^2} r_{L2}, \end{aligned} \tag{4.38}$$

otherwise, $i_0^{d2} = 0$.

The optimal nominal interest rate in period 0 in Case 3 is

$$i_0^{d3} = [(\kappa^2 + \lambda)(\kappa + \sigma) + \kappa^2 \sigma \beta] \frac{\sigma b[(1-\delta) r_H + \delta r_{L3}]}{(\lambda + \kappa^2 + b\sigma^2)^2} r_H + \frac{\lambda + \kappa^2}{\lambda + \kappa^2 + b\sigma^2} r_{L3} > 0. \tag{4.39}$$

The formal proof of Lemma 4.4 is given in Appendix B.6.

Figure 4.18 displays the optimal nominal interest rate for different values of b when $r_{L2} \in (r_L^c, 0)$. Figure 4.19 displays the optimal nominal interest rate for different values of b when $r_{L3} > 0$.

Lemma 4.5

Signing an ex-ante FGC of one period impedes the economy.

With an FGC $(b, 1)$, the central banker would set $i_0^{d1} = i_0^{d2} = 0$ in Cases 1 and 2. However, the central banker would set $i_0^{d1} = i_0^{d2} = 0$ even without the FGC $(b, 1)$, while in Case 3, FGC $(b, 1)$ would constrain the central banker in setting $i_0^{d3} = r_{L3}$ optimally. Thus, FGC $(b, 1)$ is inappropriate and impedes the economy.

In period -1, the government chooses the optimal value of b to minimize the expected

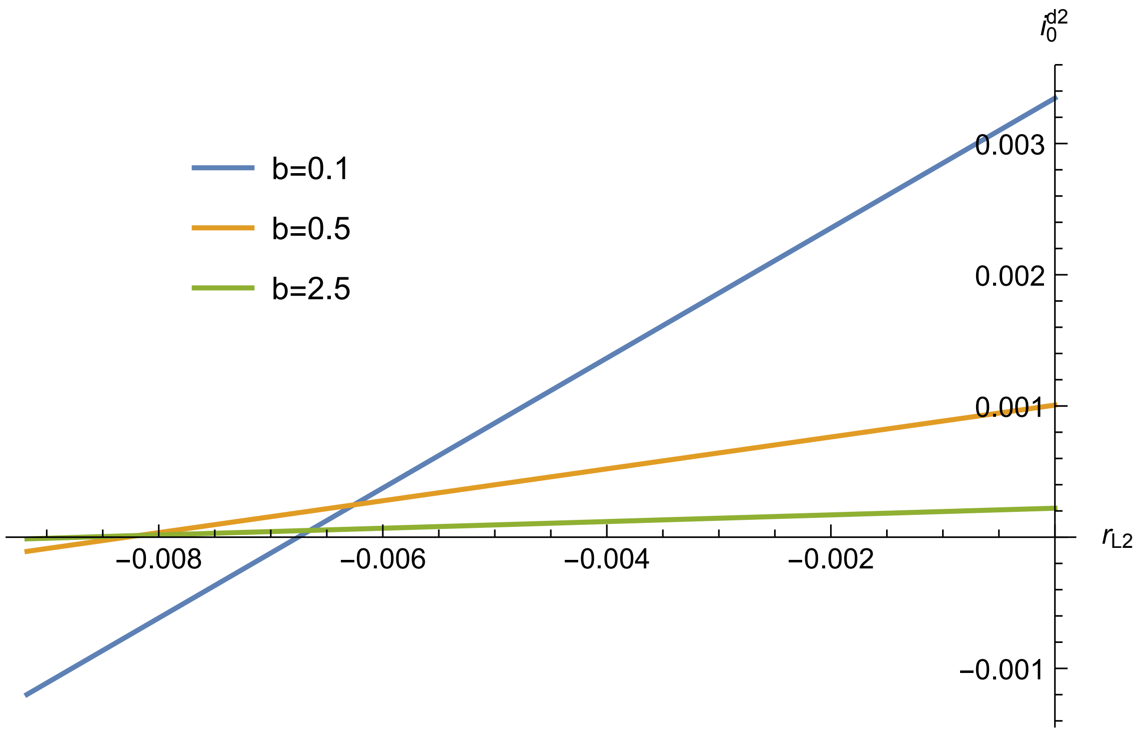

Figure 4.18: The optimal nominal interest rate in function of r_{L2} for different values of b.

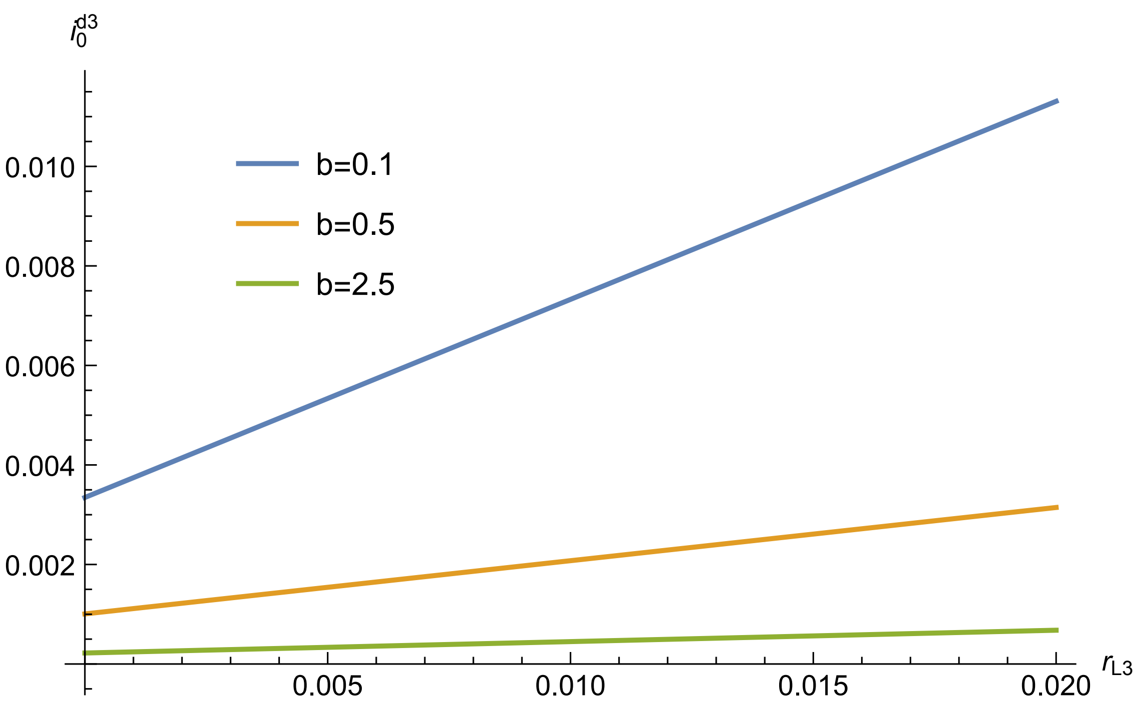

Figure 4.19: The optimal nominal interest rate in function of r_{L3} for different values of b.

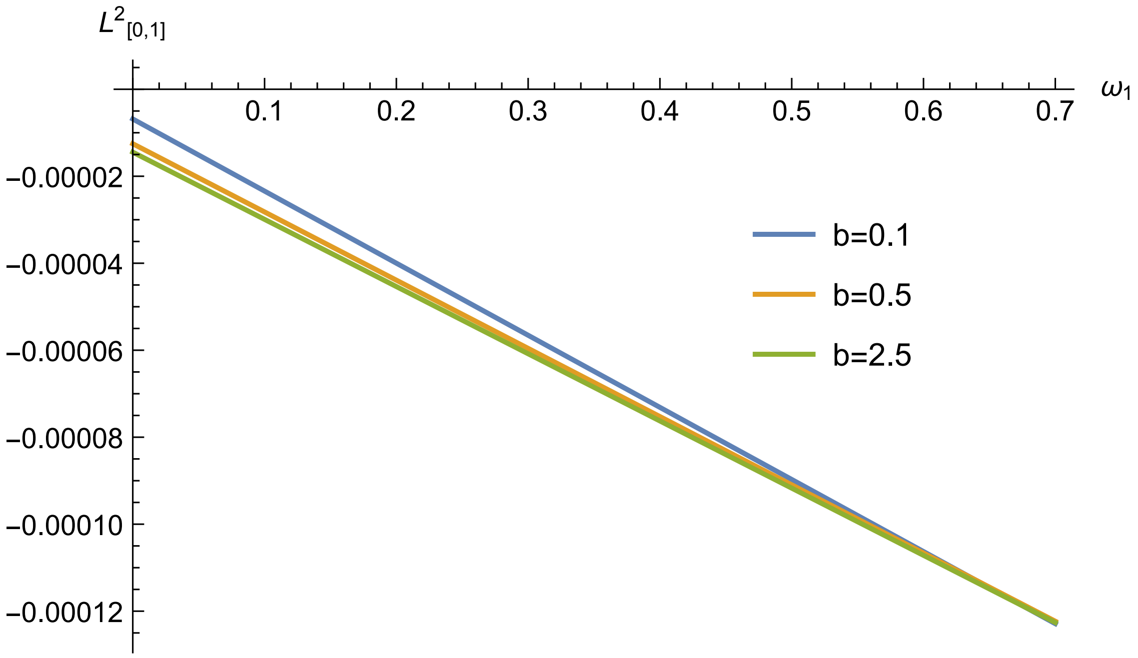

Figure 4.20: The evolution of the expected intertemporal social losses in periods 0 and 1 in function of ω_1, for different values of b.

social losses in periods 0 and 1

$$\begin{aligned}\mathbb{E}_0 L^{k=2}_{[0,1]} = & -0.5[\omega_1(\boldsymbol{\Lambda Q_0^{d1}})^{\boldsymbol{T}}\boldsymbol{\Lambda Q_0^{d1}} + \omega_2(\boldsymbol{\Lambda Q_0^{d2}})^{\boldsymbol{T}}\boldsymbol{\Lambda Q_0^{d2}} + (1-\omega_1-\omega_2)(\boldsymbol{\Lambda Q_0^{d3}})^{\boldsymbol{T}}\boldsymbol{\Lambda Q_0^{d3}}] \\ & - 0.5\beta(1-\delta)(\boldsymbol{\Lambda Q_1^{n}})^{\boldsymbol{T}}\boldsymbol{\Lambda Q_1^{n}} \\ & - 0.5\beta\delta[\omega_1(\boldsymbol{\Lambda Q_1^{d1}})^{\boldsymbol{T}}\boldsymbol{\Lambda Q_1^{d1}} + \omega_2(\boldsymbol{\Lambda Q_1^{d2}})^{\boldsymbol{T}}\boldsymbol{\Lambda Q_1^{d2}} + (1-\omega_1-\omega_2)(\boldsymbol{\Lambda Q_1^{d3}})^{\boldsymbol{T}}\boldsymbol{\Lambda Q_1^{d3}}].\end{aligned} \tag{4.40}$$

In our calibration[9], we assume that $r_{L1} = -0.025$, $r_{L2} = -0.005$, $r_{L3} = 0.015$ and $\omega_3 = 0.3$. The corresponding optimal values of b are ∞, 0.036 and 0 in Cases 1, 2 and 3, respectively.

Figure 4.20 shows that the expected intertemporal social losses in periods 0 and 1 decrease with ω_1. Larger b is only optimal when ω_1 is large.

Figure 4.21 displays the optimal value of b in function of ω_1. Figure 4.22 demonstrates that the expected intertemporal social losses with the optimal b in periods 0 and 1 are reduced, compared to the ones with little credibility ($b = 0.001$) and the ones without FGCs[10]. The larger the value of ω_1, the more the economy benefits from FGCs.

The upshot of the considerations and calibrations of ex-ante FGCs is that optimal values

[9] Note that $i_0^{d2} > 0$ as long as $b \geq 0.04$. So we distinguish two cases: $b < 0.04$ and $b \geq 0.04$.

[10] The great discrepancy between the ones with FGC^{ex} ($b = 0.001, 2$) and the ones without FGCs is stems from the fact that from period 2 on, optimal FGCs are signed in the former case, if necessary, while in the latter case, no FGCs are signed.

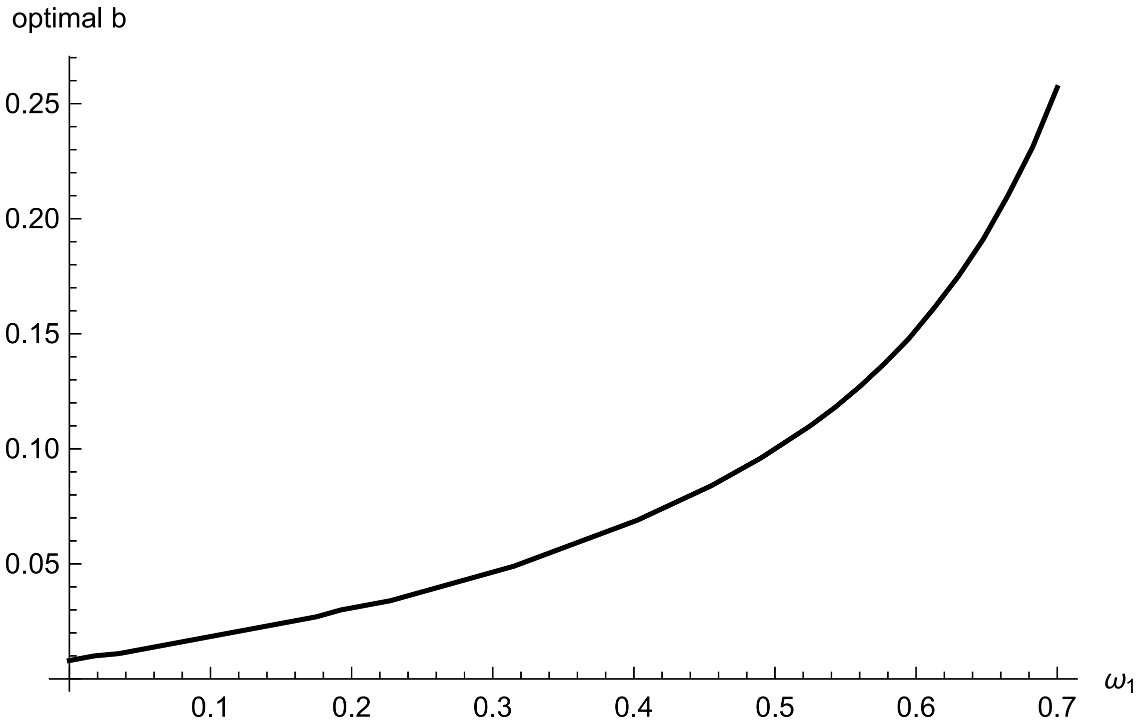

Figure 4.21: The optimal values of b in function of ω_1.

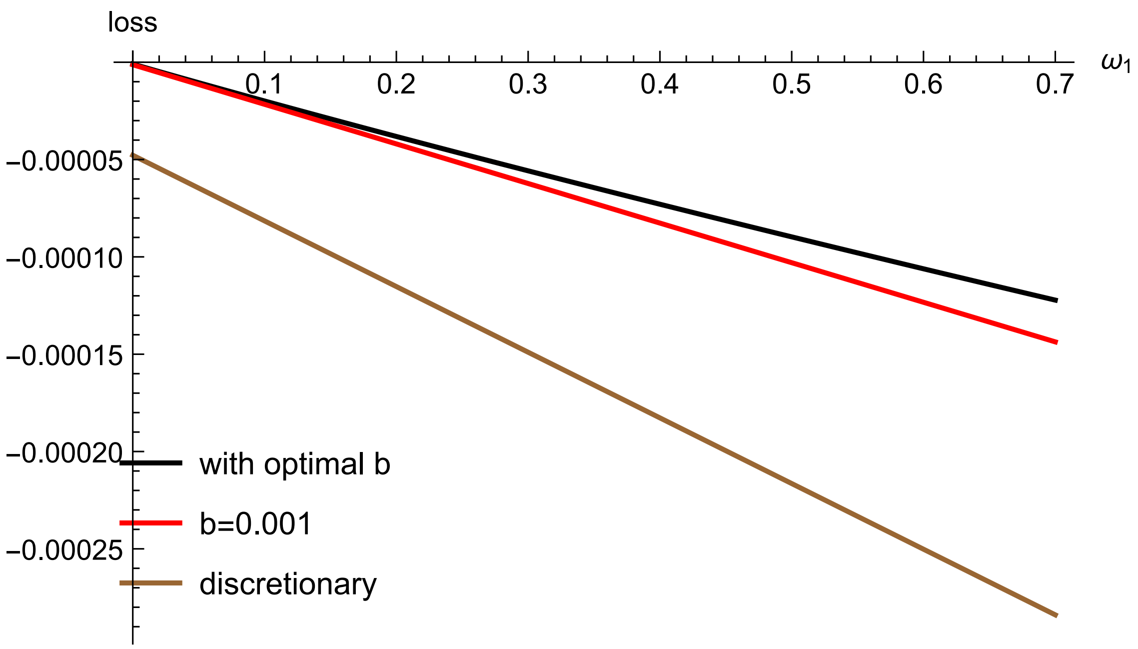

Figure 4.22: The expected intertemporal social losses in periods 0 and 1 with the optimal b, with $b = 0.001$ and without FGCs, in function of ω_1.

of b are positive, but only at moderate levels. The reason is that circumstances in which extremely high values of b and circumstances in which a zero value of b are optimal have to be balanced. Moreover, when such contracts are written in normal times, the likelihood of a severe negative shock to the natural real interest rate appears to be moderate.

4.4 Longer-term Forward Guidance Contracts

In this section, we examine whether FGCs with longer term could further improve welfare. We first consider the dynamics of inflation and output gap when the natural real interest rate reverts to r_H while the contract is still binding.

Assume that the economy recovers to the normal time l periods before the FGC ends, where l is an integer and $l \in [1, k-1]$.

In period k, the contract ends and the central banker becomes discretionary. Thus, $\pi_k = x_k = 0$.

Assume that m is an integer and $m \in [1, l]$.

In period $k-m$, the central bank maximizes

$$L_{k-m} = -\frac{1}{2}[(\pi^n_{k-m})^2 + \lambda(x^n_{k-m})^2 + b(i^n_{k-m})^2] \tag{4.41}$$

$$s.t.$$

$$\pi^n_{k-m} = \kappa x^n_{k-m} + \beta\pi^n_{k-m+1}, \tag{4.42}$$

$$x^n_{k-m} = x^n_{k-m+1} - \frac{1}{\sigma}(i^n_{k-m} - \pi^n_{k-m+1} - r_H), \tag{4.43}$$

$$i^n_{k-m} \geq 0. \tag{4.44}$$

The first-order condition of Equation (4.41) with respect to i^n_{k-m} yields the optimal interest rate

$$i^n_{k-m} = \frac{1}{\lambda+\kappa^2+b\sigma^2}[(\lambda+\kappa^2)r_H + (\lambda+\kappa^2+\kappa\beta\sigma)\pi^n_{k-m+1} + \sigma(\lambda+\kappa^2)x^n_{k-m+1}]. \tag{4.45}$$

The corresponding inflation and output gap are

$$\pi^n_{k-m} = \frac{1}{\lambda+\kappa^2+b\sigma^2}[b\kappa\sigma r_H + (\lambda\beta + b\kappa\sigma + b\beta\sigma^2)\pi^n_{k-m+1} + b\kappa\sigma^2 x^n_{k-m+1}], \tag{4.46}$$

and

$$x^n_{k-m} = \frac{1}{\lambda+\kappa^2+b\sigma^2}[b\sigma r_H + (b\sigma - \kappa\beta)\pi^n_{k-m+1} + b\sigma^2 x^n_{k-m+1}]. \tag{4.47}$$

Note that the interest rates, inflations and output gaps in normal times are all non-negative.

The dynamics can be described by

$$\boldsymbol{S_{k-m}} = \boldsymbol{T} \cdot \boldsymbol{R_{k-m+1}}, \tag{4.48}$$

where $\boldsymbol{S_t} = \begin{pmatrix} i^n_t \\ \pi^n_t \\ x^n_t \end{pmatrix}$, $\boldsymbol{T} = \frac{1}{\lambda+\kappa^2+b\sigma^2}\begin{pmatrix} \lambda+\kappa^2 & \lambda+\kappa^2+\kappa\beta\sigma & \sigma(\lambda+\kappa^2) \\ b\kappa\sigma & \lambda\beta+b\kappa\sigma+b\beta\sigma^2 & b\kappa\sigma^2 \\ b\sigma & b\sigma-\kappa\beta & b\sigma^2 \end{pmatrix}$, $\boldsymbol{R_t} = \begin{pmatrix} r_H \\ \pi^n_t \\ x^n_t \end{pmatrix}$ and $\boldsymbol{R_k} = \begin{pmatrix} r_H \\ 0 \\ 0 \end{pmatrix}$.

Thus, the inflation and output gap in period $k-l$ are determined by

$$\boldsymbol{Q^n_{k-l}} = \boldsymbol{\tilde{T}} \cdot \boldsymbol{R_{k-l+1}}, \tag{4.49}$$

where $\boldsymbol{\tilde{T}} = \frac{1}{\lambda+\kappa^2+b\sigma^2}\begin{pmatrix} b\kappa\sigma & \lambda\beta+b\kappa\sigma+b\beta\sigma^2 & b\kappa\sigma^2 \\ b\sigma & b\sigma-\kappa\beta & b\sigma^2 \end{pmatrix}$.

If the economy is in the downturn in period $k-l$, the inflation and output gap are determined by[11]

$$\boldsymbol{Q^d_{k-l}} = \mathbb{E}_{k-l}\boldsymbol{O^{-1}}\boldsymbol{Q_{k-l+1}} + \frac{r_L}{\sigma}\begin{pmatrix} \kappa \\ 1 \end{pmatrix}. \tag{4.50}$$

Thus, in the downturn, the expectations are

$$\mathbb{E}_{k-l-1}\boldsymbol{Q_{k-l}} = (1-\delta)\boldsymbol{Q^n_{k-l}} + \delta\boldsymbol{Q^d_{k-l}}. \tag{4.51}$$

In period k, if the economy is out of the downturn, no FGC is re-signed and $\boldsymbol{Q^n_k} = \boldsymbol{0}$; if the economy is still in the downturn, the same FGC (b, k) is re-signed and $\boldsymbol{Q^d_k} = \boldsymbol{Q^d_0}$ due to the fact that the situation in period k is the same as the one in period 0. Thus,

$$\mathbb{E}_{k-1}\boldsymbol{Q_k} = \delta\boldsymbol{Q^d_0}. \tag{4.52}$$

[11] We assume that the central banker sets zero nominal interest rate in downturns.

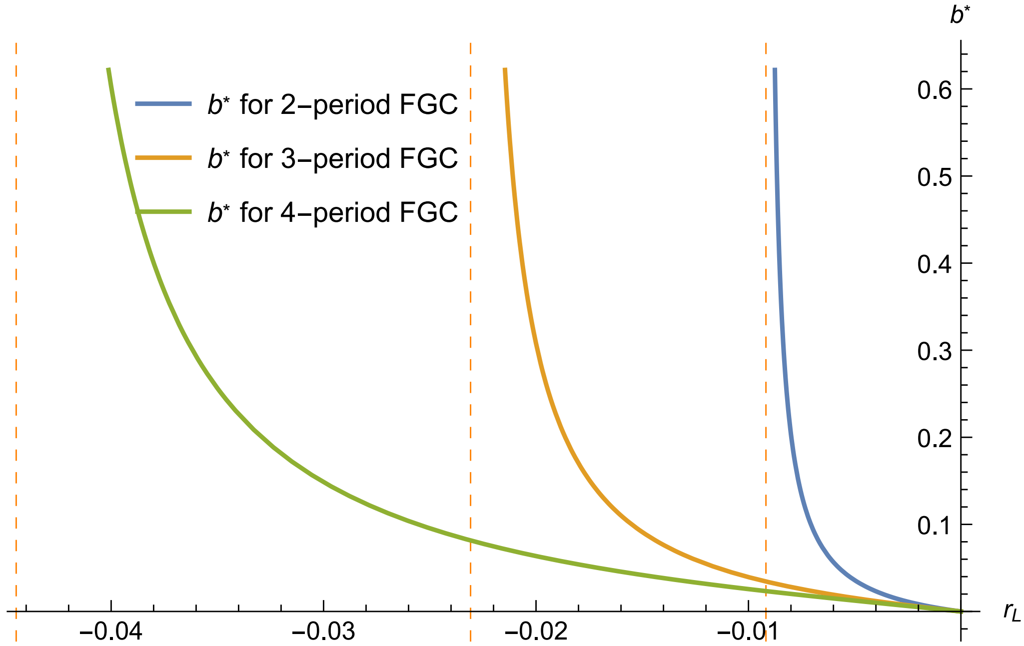

Figure 4.23: The optimal value of b^* for FGCs with different durations.

The expected intertemporal social loss in period 0 is

$$\begin{aligned}\mathbb{E}_0 L^k_{[0,\infty]} &= -0.5\frac{(\boldsymbol{\Lambda Q_0^d})^{\boldsymbol{T}}\boldsymbol{\Lambda Q_0^d} + \beta[(1-\delta)(\boldsymbol{\Lambda Q_1^n})^{\boldsymbol{T}}\boldsymbol{\Lambda Q_1^n} + \delta(\boldsymbol{\Lambda Q_1^d})^{\boldsymbol{T}}\boldsymbol{\Lambda Q_1^d}]}{1-\beta^k\delta^k}\\
&\quad -0.5\frac{\beta^2[(1-\delta)(1+\delta)(\boldsymbol{\Lambda Q_2^n})^{\boldsymbol{T}}\boldsymbol{\Lambda Q_2^n} + \delta^2(\boldsymbol{\Lambda Q_2^d})^{\boldsymbol{T}}\boldsymbol{\Lambda Q_2^d}]}{1-\beta^k\delta^k}\\
&\quad -\cdots\\
&\quad -0.5\frac{\beta^{k-1}[(1-\delta)(1+\delta+\cdots+\delta^{k-2})(\boldsymbol{\Lambda Q_{k-1}^n})^{\boldsymbol{T}}\boldsymbol{\Lambda Q_{k-1}^n} + \delta^{k-1}(\boldsymbol{\Lambda Q_{k-1}^d})^{\boldsymbol{T}}\boldsymbol{\Lambda Q_{k-1}^d}]}{1-\beta^k\delta^k}\\
&= -0.5\frac{(\boldsymbol{\Lambda Q_0^d})^{\boldsymbol{T}}\boldsymbol{\Lambda Q_0^d} + \sum_{j=1}^{k-1}\beta^j[(1-\delta^j)(\boldsymbol{\Lambda Q_j^n})^{\boldsymbol{T}}\boldsymbol{\Lambda Q_j^n} + \delta^j(\boldsymbol{\Lambda Q_j^d})^{\boldsymbol{T}}\boldsymbol{\Lambda Q_j^d}]}{1-\beta^k\delta^k}.
\end{aligned} \tag{4.53}$$

In Figure 4.23, we show the optimal value of b^*, depending on the size of shock, for FGCs of different lengths. Dashed vertical curves represent the critical values below which $b^* = \infty$ for 2-, 3-, 4-period FGCs (from the right to the left, as well as in Figure 4.25). The optimal value of b^* increases with the size of shock, which indicates that the intensity of incentives should go hand in hand with the severity of the recession. A large deflation and output collapse, caused by a large natural real interest-rate shock, can be mitigated by a large expected inflation and output boom created either by a contract of longer-term or by a short-term contract with larger values of b. Thus, compared to long-term FGCs,

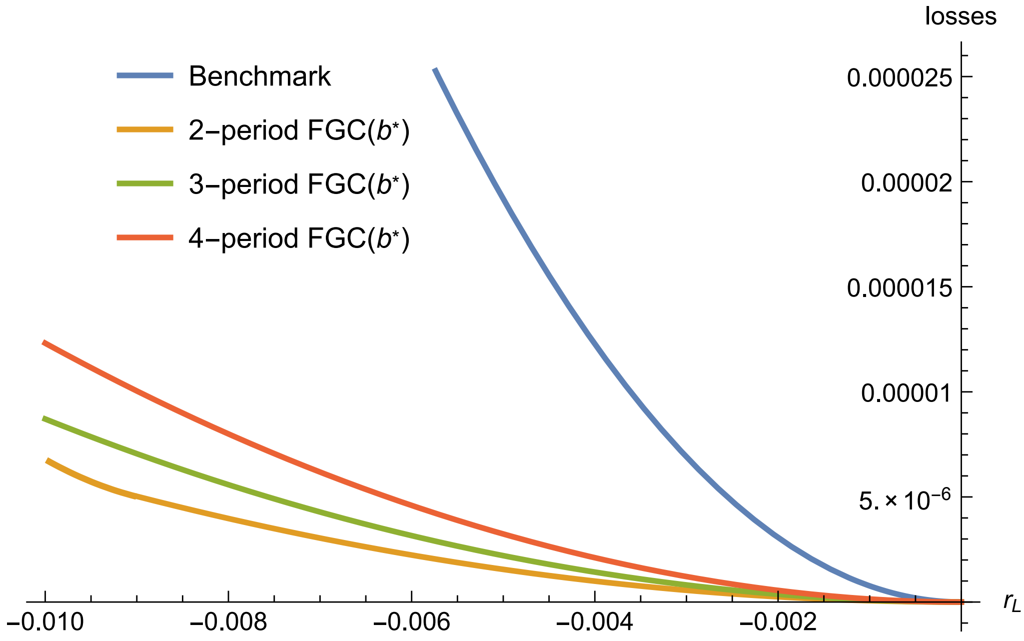

Figure 4.24: The expected social losses for FGCs with different durations, as functions of r_L ($r_L \in [-0.01, 0]$).

short-term FGCs require larger value of b^*.

Figure 4.24 shows that for small and moderate shock sizes, a 2-period FGC is more desirable, as the inflationary expectation raised by a 2-period FGC with a finite value of b^* suffices to compensate the deflation and output collapse caused by the natural real interest-rate shock r_L, while 3- and 4-period FGCs constrain the central bank for a long period—which is excessive. However, as displayed in Figure 4.25, when the shock is severe, longer-term FGCs could lower social losses further compared to short-term contracts. Severe natural real interest-rate shocks require great incentives and long terms to mitigate the deflation and output collapse in downturns. While such contracts can yield even lower social losses in such circumstances, they also constrain the central bank for a long time and may thus be problematic, as unforeseen events requiring greater flexibility may occur in the meantime.

4.5 Discussion and Conclusion

In this chapter we explore longer-term contracts in an alternative contractual environments. We show that when the size of shock is severe, longer-term contracts could lower social losses further compared to short-term contracts. Severe natural real interest rate shocks require large incentive intensities and long contract periods to mitigate the deflation and output collapse in downturns. While such contracts can yield even lower social

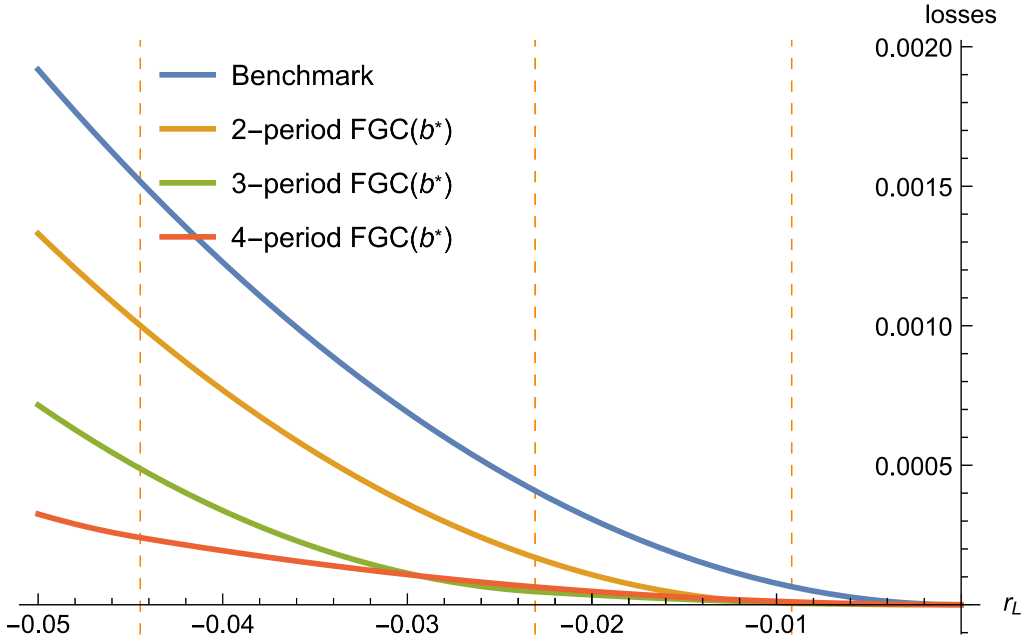

Figure 4.25: The expected social losses for FGCs with different durations, as functions of r_L ($r_L \in [-0.05, 0]$).

losses than short-term contracts in such circumstances, they also constrain the central bank for a long time and may thus be problematic, as unforeseen events requiring greater flexibility may occur in the interim.

5 State-contingent Forward Guidance Contracts

5.1 Introduction

Walsh (1995) examines incentive contracts for central bankers in a static model, and shows that contracts that reward central bankers according to the realized inflation can induce socially desirable outcomes. Lockwood (1997) generalizes Walsh's argument to the case where the unemployment rate follows a first-order autoregressive process. He shows that in such circumstances, incentive contracts for central bankers are socially beneficial if the contract is contingent on the lagged unemployment rate, i.e. the higher the unemployment rate in previous period, the lower central banker's wage. Both Walsh (1995) and Lockwood (1997) propose incentive contracts in which the penalty to central bankers is contingent on the macroeconomic variables that have been realized, while in Chapters 3 and 4, we studied incentive contracts in which the penalty to central bankers is contingent on the discrepancy between the interest rate forecast and the actual interest rate choice.

In this chapter, we introduce State-contingent Forward Guidance Contracts (SFGC). We consider FGCs under which the effective zero-interest-rate policy is contingent on economic developments. In such contracts, the central bankers' forecast would itself depend on macroeconomic variables such as the natural real interest rate or inflation expectation. In downturns, for instance, the central bank announces that it will keep the interest rate at zero until one period after the natural real interest rate has reverted to 2%. In other words, the central bank's zero-interest-rate policy is not calendar-based as studied in Chapters 3 and 4, but state-contingent (until certain criteria are fulfilled).

We investigate the optimal design of SFGCs and the optimal forecast i^f under such a contract. SFGCs have advantage compared to the calendar-based FGCs studied in the previous chapters: The government does not have to re-sign the same contract repeatedly in downturns, while the same effects can be achieved as shown in Section 5.4.

5.2 Evolution of Economics

In the presence of a large negative shock on the natural real interest rate, deflation accompanied by output collapse occurs even after reducing the nominal interest rate to the zero lower bound. In such circumstances, creating inflationary expectation is vital for central banks to lower the real interest rate and thus to stimulate the economy. Since the central bank has limited ability to commit inflation credibly, the government steps in and signs a state-contingent FGC which states that the central banker shall keep the nominal interest rate at zero for certain periods until certain criteria are fulfilled, and the central banker's wage scheme is made dependent on the execution of the zero-interest-rate policy.

To explore the SFGCs, we repeat the model briefly as follows.

The IS Equation and Phillips Curve can be written in matrix form as follows:

$$\mathbb{E}_t \boldsymbol{Q_{t+1}} = \boldsymbol{O} \boldsymbol{Q_t} - \frac{1}{\sigma} \boldsymbol{r_t}, \tag{5.1}$$

where $\boldsymbol{Q_t} = \begin{pmatrix} \pi_t \\ x_t \end{pmatrix}$, $\boldsymbol{O} = \begin{pmatrix} \frac{1}{\beta} & -\frac{\kappa}{\beta} \\ -\frac{1}{\sigma\beta} & 1 + \frac{\kappa}{\sigma\beta} \end{pmatrix}$ and $\boldsymbol{r_t} = \begin{pmatrix} 0 \\ r_t - i_t \end{pmatrix}$.

The central banker's intertemporal loss function in period 0 is

$$L = -\frac{1}{2} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (\pi_t^2 + \lambda x_t^2). \tag{5.2}$$

As in Eggertsson (2006), we assume that a large negative shock on the natural real interest rate occurs in period 0. Thus, in period 0, the natural real interest rate is

$$r_0 = r_L < 0. \tag{5.3}$$

In each period, the economy recovers from the downturn $r_t = r_L$ to the normal time $r_t = r_H > 0$, with a fixed probability $1 - \delta$. The period at which the economy recovers is denoted by τ, where τ is an integer and $\tau > 0$.

The central banker's objective function (5.2) suggests that the optimal interest rate would be the one that leads to zero inflation and output gap in each period. Hence, in normal times, the central banker sets $i_t = r_H$. This leads to $\pi_t^n = x_t^n = 0$, i.e. $\boldsymbol{Q^n} = \boldsymbol{0}$, where n denotes normal times.

Thus, the expectation in downturns is

$$\mathbb{E}_t \boldsymbol{Q_{t+1}} = \delta \boldsymbol{Q^d} + (1 - \delta) \boldsymbol{Q^n} = \delta \boldsymbol{Q^d}, \tag{5.4}$$

where d represents downturn.

Combining Equations (5.1) and (5.4) yields inflation and output gap in downturns

$$\boldsymbol{Q_D^d} = \frac{1}{h(\delta)} \begin{pmatrix} \kappa & 1 - \delta\beta \end{pmatrix}^T r_L, \tag{5.5}$$

where D denotes discretionary policy, $h(\delta) = \sigma\beta\delta^2 - (\sigma + \kappa + \sigma\beta)\delta + \sigma$ and interest rate i_D^d is set at zero in downturns.

Due to the shock on the natural real interest rate and deflationary expectation, deflation and output collapse are excessive in downturns.

Note that $h(\delta^c) = 0$ where $\delta^c = \frac{\sigma+\kappa+\sigma\beta-\sqrt{(\sigma+\kappa+\sigma\beta)^2-4\sigma^2\beta}}{2\sigma\beta}$. As in Eggertsson (2006), Carlstrom et al. (2012) and FGC (2014), the IS Equation and Phillips Curve derived around the steady state are valid when δ is not too large. Hence, throughout the paper, we assume $\delta \in [0, \delta^c)$.

As in previous chapters, after the natural real interest rate shock, the government signs an SFGC with the central banker: The central banker announces the interest rate i_τ^f he will set, where τ represents the period when the economy returns to the normal time. The central banker's wage decreases with the deviation of his actual interest rate choice from his forecast in period τ. The more he deviates, the less he will be paid. Thus, the central banker's objective function in period τ can be written as follows:

$$L(b,k) = -\frac{1}{2}\mathbb{E}_\tau \sum_{t=\tau}^{\tau+k-1} \beta^{t-\tau}[\pi_t^2 + \lambda x_t^2 + b(i_t - i_t^f)^2], \tag{5.6}$$

where b measures the intensity of incentives and k denotes the number of periods the central banker forecasts.

We first study SFGC $(b, 1)$: The central banker forecasts interest rates in normal times for one period.

In periods $[0, \tau - 1]$, the economy is in downturns. The central banker sets the nominal interest rate i^d. In period τ, the natural real interest rate returns to r_H and the central banker sets the nominal interest rate i_τ^n. In period $\tau + 1$, the contract ends and the central banker sets $i_t = r_H$ discretionarily since period $\tau + 1$. Thus, $\pi_t = x_t = 0, \forall t \geq \tau + 1$.

In period τ with $\mathbb{E}_\tau \boldsymbol{Q_{\tau+1}} = \boldsymbol{0}$, the central banker chooses i_τ^n to maximize

$$\max_{i_\tau^n} -\frac{1}{2}[\pi_\tau^2 + \lambda x_\tau^2 + b(i_\tau^n - i_\tau^f)^2], \tag{5.7}$$

$$s.t.$$

$$\pi_\tau = \kappa x_\tau,$$

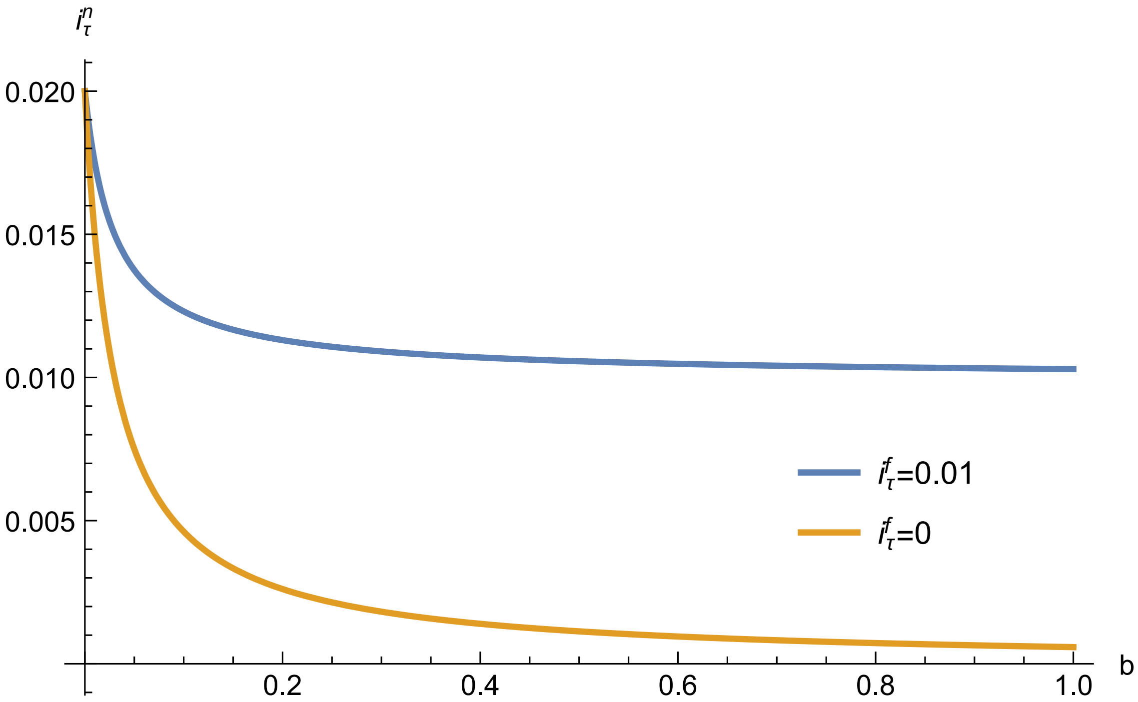

Figure 5.1: The nominal interest rate in period τ in function of the incentive intensity b for different forecasts.

$$x_\tau = -\frac{1}{\sigma}(i_\tau^n - r_H),$$
$$i_\tau^n \geq 0.$$

The objective can be rewritten as

$$\max_{i_\tau^n} -\frac{1}{2}[\frac{\lambda + \kappa^2}{\sigma^2}(i_\tau^n - r_H)^2 + b(i_\tau^n - i_\tau^f)^2], \tag{5.8}$$

$$s.t.$$

$$i_\tau^n \geq 0.$$

Calculating the first-order condition and solving for i_τ^n yields

$$\begin{aligned} i_\tau^n &= \frac{(\lambda + \kappa^2)r_H + b\sigma^2 i_\tau^f}{\lambda + \kappa^2 + b\sigma^2} \\ &= r_H + \frac{b\sigma^2(i_\tau^f - r_H)}{\lambda + \kappa^2 + b\sigma^2} \in [0, r_H]. \end{aligned} \tag{5.9}$$

We note how different values of b impact the chosen interest rate. If costs of deviation are very small (b is small), i_τ^n is close to r_H and thus to the discretionary solution. If b is very large, the central banker sets the interest rate close to his forecast.
Figure (5.1) depicts that the nominal interest rate in period τ deviates from r_H towards

the forecast i^f_τ as b increases.

Note that i^f_τ can be negative[1]. A forecast of negative interest rate shows the central banker's determination to conduct low-interest-rate policy in period τ. For example, when the central banker sets $i_\tau = 1\%$, if his forecast was $i^f_\tau = 0\%$, the deviation cost is $0.0001b$, while if his forecast was $i^f_\tau = -1\%$, the deviation cost is $0.0004b$, four times larger than the first. Thus, in the latter case, the central banker has a stronger incentive to set lower i_τ. *Ceteris paribus*, a lower interest forecast leads to lower interest rate i_τ. Equation (5.9) shows that when $i^f_\tau = \hat{i}^f_\tau$, where $\hat{i}^f_\tau = -\frac{\lambda+\kappa^2}{b\sigma^2} r_H{}^2$, the central banker sets $i^n_\tau = 0$. Thus, the central banker would not forecast an interest rate below $\hat{i}^f_\tau$. Intuitively, forecasting $\hat{i}^f_\tau$ would already lead to the central banker's zero-rate policy in period τ. A lower forecast would result in the same zero interest rate but yield a further unnecessary wage reduction for the central banker. We make the following assumption that there exists a zero lower bound of interest rate forecast.

Assumption 5.1
The central bank makes the interest rate forecast $i^f_\tau \geq 0$.

Figure (5.2) depicts that the nominal interest rate in period τ converges to r_H when i^f_τ approaches the natural real interest rate in normal times. The interest rate is set lower when the value of b is larger.

We combine Equations (5.1), (5.9) and $\boldsymbol{Q}^n_{\tau+1} = \mathbf{0}$, and obtain inflation and output gap in period τ

$$\boldsymbol{Q}^n_\tau \equiv \begin{pmatrix} \pi^n_\tau \\ x^n_\tau \end{pmatrix} = \frac{b\sigma(r_H - i^f_\tau)}{\lambda + \kappa^2 + b\sigma^2} \begin{pmatrix} \kappa \\ 1 \end{pmatrix}, \tag{5.10}$$

where $\boldsymbol{Q}^n_\tau$ represents the inflation and output gap levels once the economy recovers.

Figure (5.3) shows that the inflation and output gap in period τ increase with the value of b. Figure (5.4) shows that as the forecast i^f_τ increases, the inflation and output gap decline. When the forecast $i^f_\tau = r_H = 2\%$, both inflation and output gap are zero, which corresponds to the discretionary case.

In downturns, from the perspective of period t, two situations can occur in period $t+1$. First, in period $t+1$, the natural real interest rate is still r_L with probability δ. As the central bank faces the same economic situation in period $t+1$ as in period t, the central banker sets the same interest rate, i.e. $i^d_{t+1} = i^d_t = i^d$.

[1] An alternative approach to avoiding negative forecasts would be adding an interest rate inclination term i^*_t in central banker's objective function, i.e. $b(i_t - i^f_t - i^*_t)^2$. When the government signs the SFGC with the central banker, apart from the incentive intensity b, the inclination i^*_t is specified as well. This term represents the government's complementary tool, an addition to the incentive intensity. This supplement allows to achieve social optimal results even when the forecasts i^f_t can only have non-negative values.

[2] In our calibration, $\hat{i}^f_\tau = -0.06\%$ when $b = 1$.

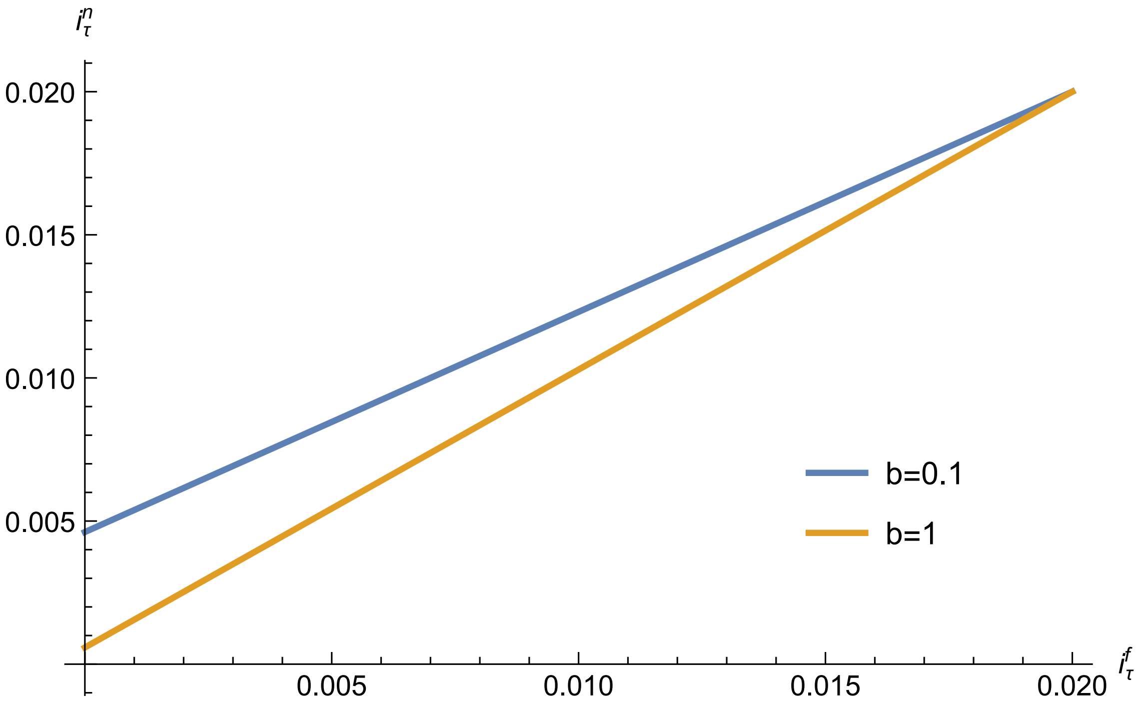

Figure 5.2: The nominal interest rate in period τ in function of the interest rate forecast i^f_τ for different b.

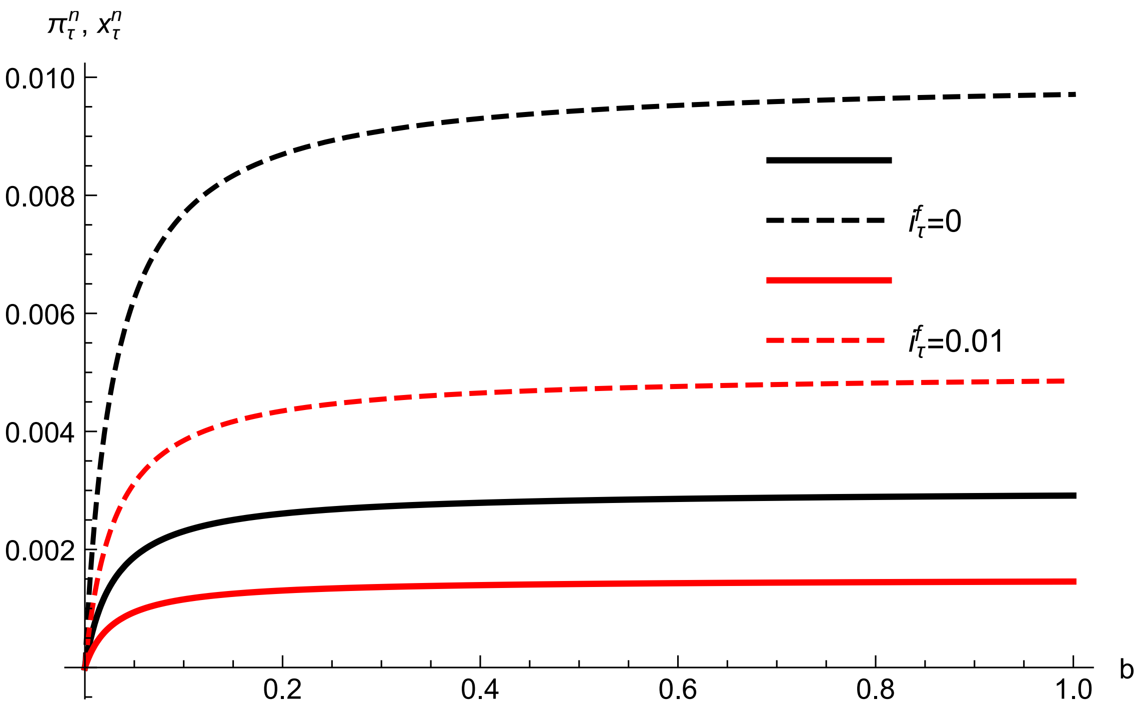

Figure 5.3: Inflation and output gap in period τ in function of the incentive intensity b for different forecasts. Solid and dashed lines represent inflation and output gap, respectively.

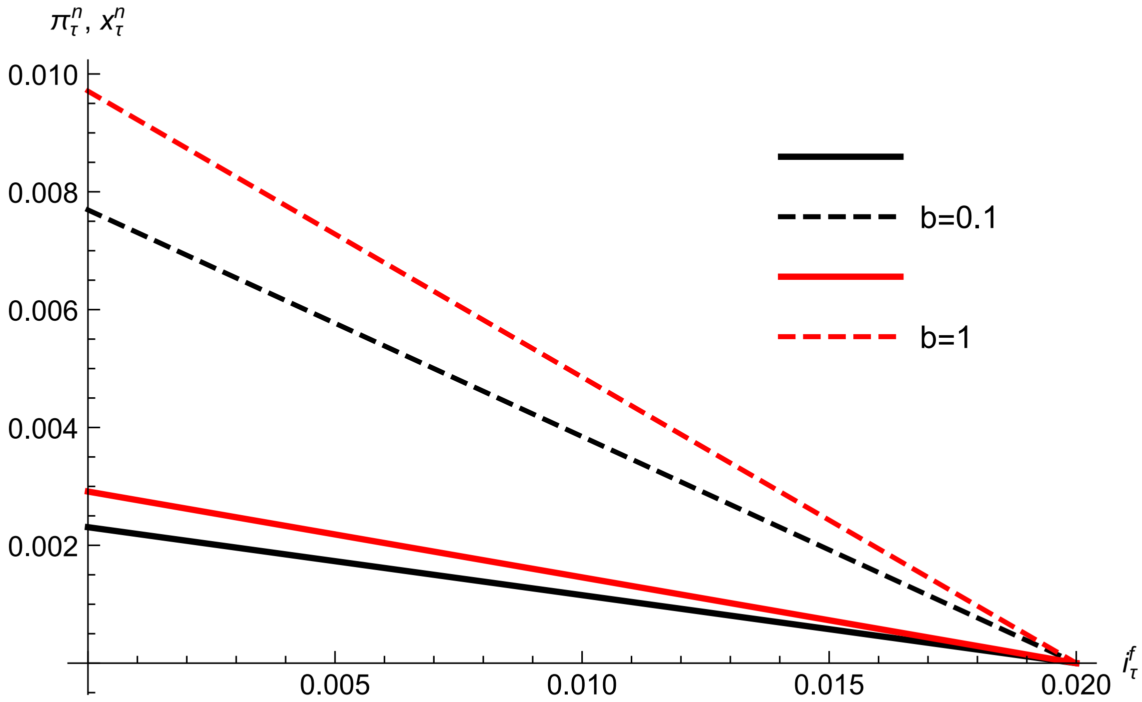

Figure 5.4: Inflation and output gap in period τ in function of the interest rate forecast i^f_τ for different b. Solid and dashed lines represent inflation and output gap, respectively.

Thus, the inflation and output gap in period $t+1$ are the same as the ones in period t

$$\boldsymbol{Q}^d_{t+1} = \boldsymbol{Q}^d_t = \boldsymbol{Q}^d. \tag{5.11}$$

Second, in period $t+1$, the natural real interest rate returns to r_H with probability $1-\delta$. The inflation and output gap are then given by Equation (5.10).
In period t, the expectation for inflation and output gap in period $t+1$ is

$$\mathbb{E}_t \boldsymbol{Q}_{t+1} = \delta \boldsymbol{Q}^d + (1-\delta)\boldsymbol{Q}^n_\tau. \tag{5.12}$$

Lemma 5.1
With the optimal design of SFGC $(b,1)$, the central banker sets the nominal interest rate at zero in downturns.

Proof:
Assume that with the optimal design of SFGC $(b,1)$, the central banker sets nominal interest rate at strictly positive levels in downturns. Setting positive interest rates implies that, in downturns the inflation and output gap induced by SFGC $(b,1)$ are higher than the optimal levels. Equations (5.1) and (5.12) suggest that the inflation and output gap in normal time $\boldsymbol{Q}^n_\tau$ impact the ones in downturns $\boldsymbol{Q}^d$ through the expectation channel.

Higher $\boldsymbol{Q}_\tau^n$ induces higher $\boldsymbol{Q}^d$ at the cost of higher inflation pressure in period τ. Thus, the government can sign an SFGC $(b, 1)$ with a lower value of b, which induces lower inflation and output boom in period τ and optimal levels of inflation and output in downturns. This contradicts the assumption of optimal design of SFGC $(b, 1)$.

□

Thus, Equation (5.1) becomes

$$\boldsymbol{Q}_t^d = \mathbb{E}_t \boldsymbol{O}^{-1} \boldsymbol{Q}_{t+1} + \frac{r_L}{\sigma} \begin{pmatrix} \kappa \\ 1 \end{pmatrix}. \tag{5.13}$$

Equations (5.10), (5.11), (5.12) and (5.13) allow to express the economic outcomes in downturns explicitly as a function of parameters.

Lemma 5.2

With the optimal design of SFGC $(b, 1)$, the inflation and output gap in downturns are

$$\boldsymbol{Q}^d = \frac{1}{h(\delta)} \tilde{\boldsymbol{O}} \begin{pmatrix} \frac{b\sigma(1-\delta)}{\lambda+\kappa^2+b\sigma^2}(r_H - i_\tau^f) \\ r_L \end{pmatrix}. \tag{5.14}$$

The proof and the $2x2$ matrix $\tilde{\boldsymbol{O}}$ are provided in Appendix C.1. Note that the discretionary outcome in Equation (5.5) is recovered when $b = 0$.

Figures (5.5) and (5.6) show that inflation and output gap increase in downturns with the value of b. Figure (5.5) depicts that inflation and output gap decline in downturns with the size of the shock. Figure (5.6) depicts that inflation and output gap decline in downturns with the forecast.

Figure (5.7) depicts the inflation and output gap in downturns in function of the size of the shock, the value of b and the forecast. The values of inflation and output gap are represented by colors. The smaller the values, the colder the color and vice-versa. The upper row shows the inflation and output gap in function of the incentive intensity and the size of the shock, with $i_\tau^f = 0$. The middle row shows the inflation and output gap in function of the incentive intensity and the forecast, with $r_L = -0.02$. The bottom row shows the inflation and output gap in function of the forecast and the size of the shock with $b = 0.5$. All the plots imply that the inflation and output gap decrease in downturns with the size of the shock and the forecast, and increase with the incentive intensity.

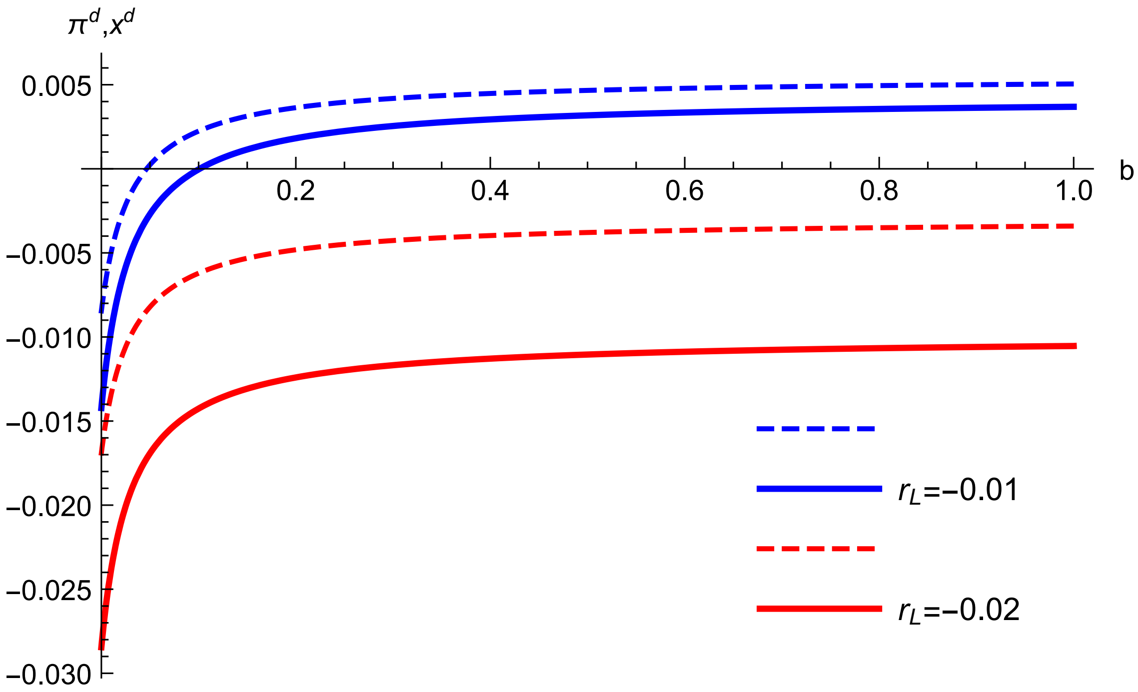

Figure 5.5: Inflation and output gap in downturns in function of the incentive intensity b for different shock sizes, with $i_\tau^f = 0$. Solid lines and dashed lines represent the inflation and output gap, respectively.

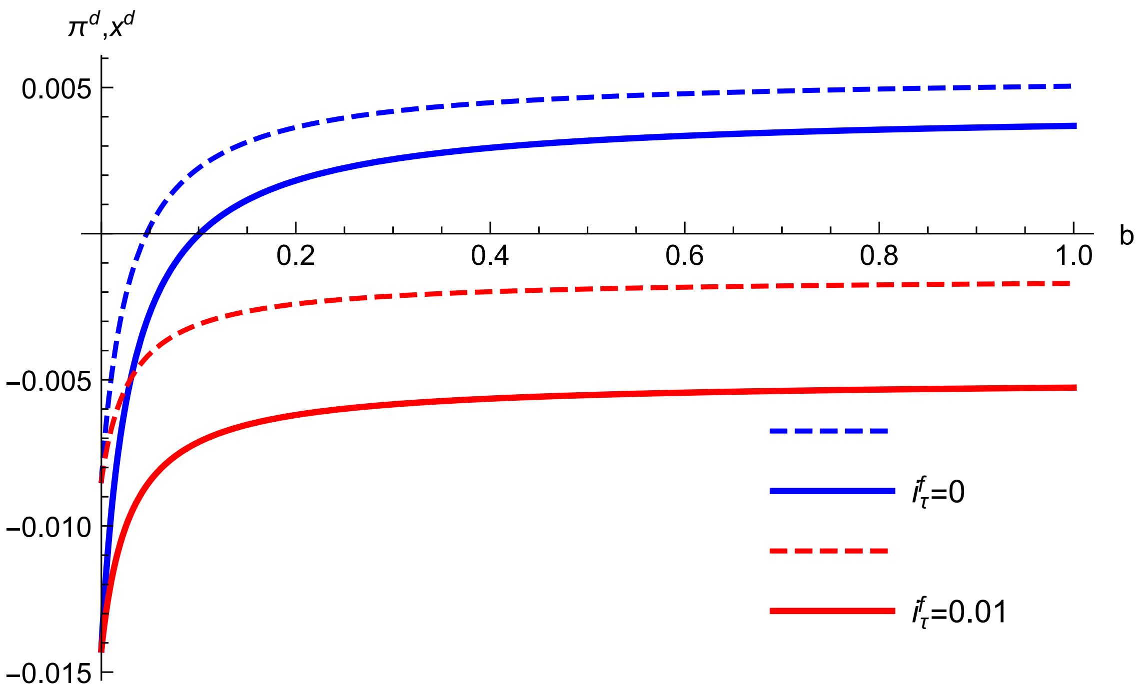

Figure 5.6: Inflation and output gap in downturns in function of the incentive intensity b for different forecast values, with $r_L = -0.01$. Solid lines and dashed lines represent the inflation and output gap, respectively.

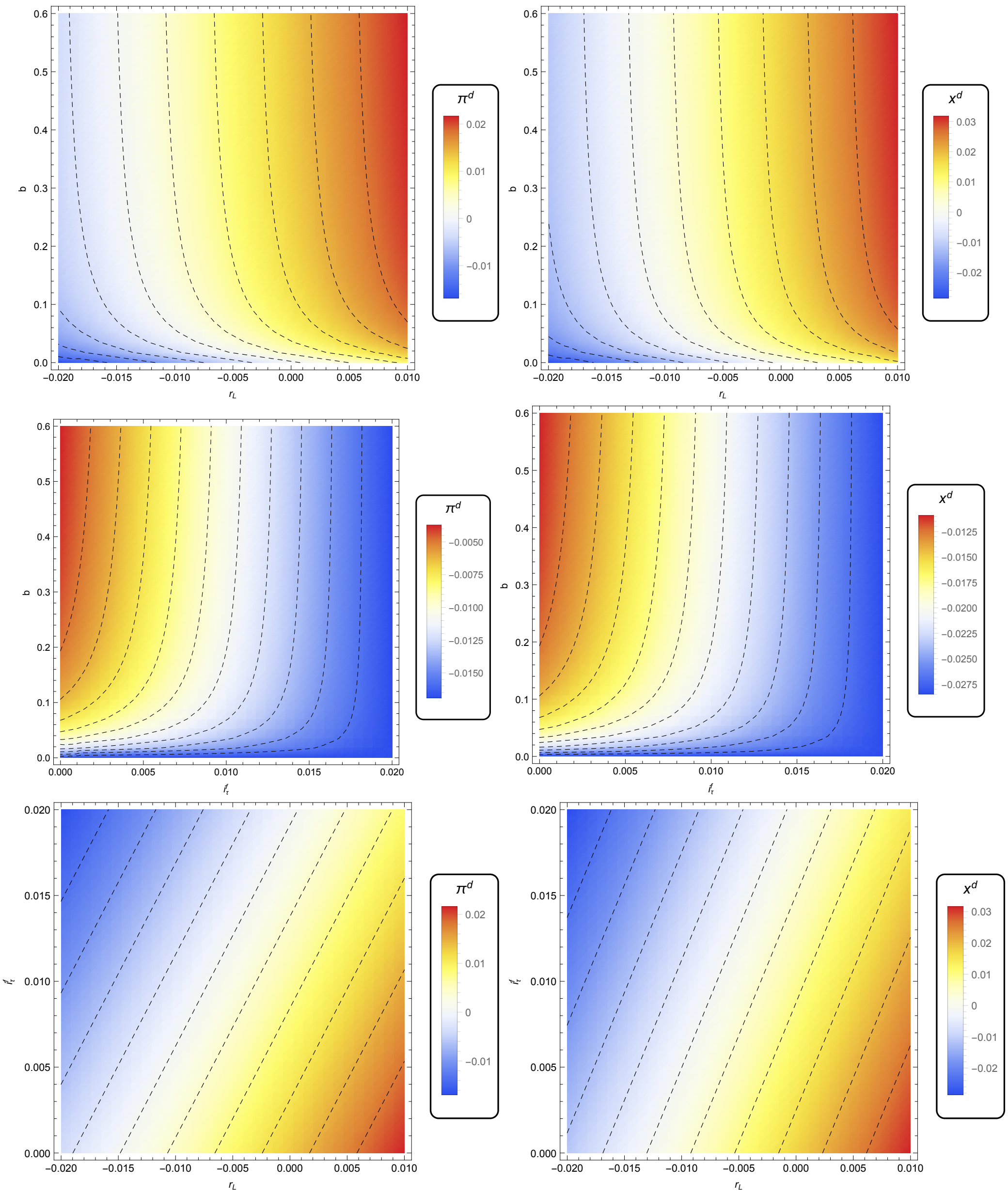

Figure 5.7: The left column and right column represent the inflation and output gap in downturns, respectively. The dashed lines are the contour lines.

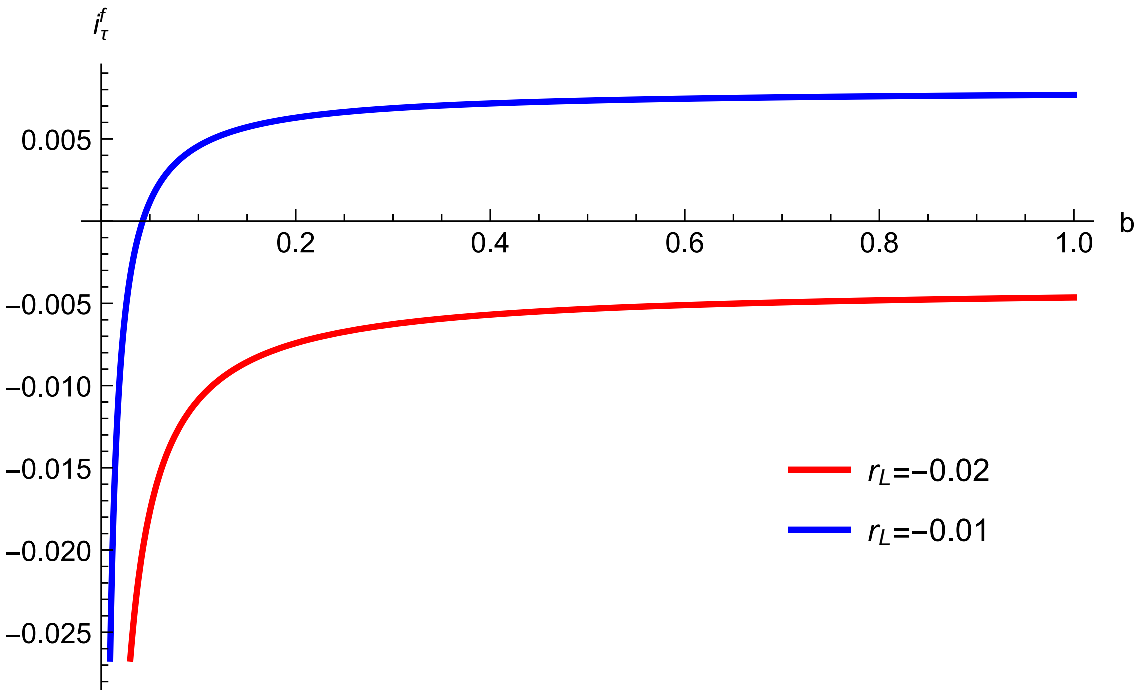

Figure 5.8: The interest rate forecast in function of b for different shock sizes.

5.3 Optimal Design

We derive that with the optimal design of SFGC $(b, 1)$, the central banker forecasts

Proposition 5.1

$$i_\tau^f = \max\{r_H + \frac{(\lambda + \kappa^2 + b\sigma^2)\phi_4}{\phi_5 + \phi_6 b} r_L, 0\}. \tag{5.15}$$

ϕ_4, ϕ_5, ϕ_6 and the proof are given in Appendix C.2.

Figure (5.8) shows that the optimal forecast rises with the value of b. Intuitively, for a given shock size, the larger the value of b, the more credible the forecast and a less the deviation of the forecast from r_H is needed. When the shock size is small or moderate, the central bank forecasts an interest rate that is higher than 0 to create the right amount of inflationary expectation. Figure (5.8) depicts that when $r_L = -0.01$ the interest rate forecast $i_\tau^f > 0$ when $b > 0.04$. However, when $r_L = -0.02$, the central bank forecasts $i_\tau^f = 0$ irrespective of the value of b, as the size of the shock is so severe that a zero forecast is always the optimal one.

Proposition 5.1 implies that $i_\tau^f > 0$ if and only if the following inequality applies:

$$r_L > r_L^c, \tag{5.16}$$

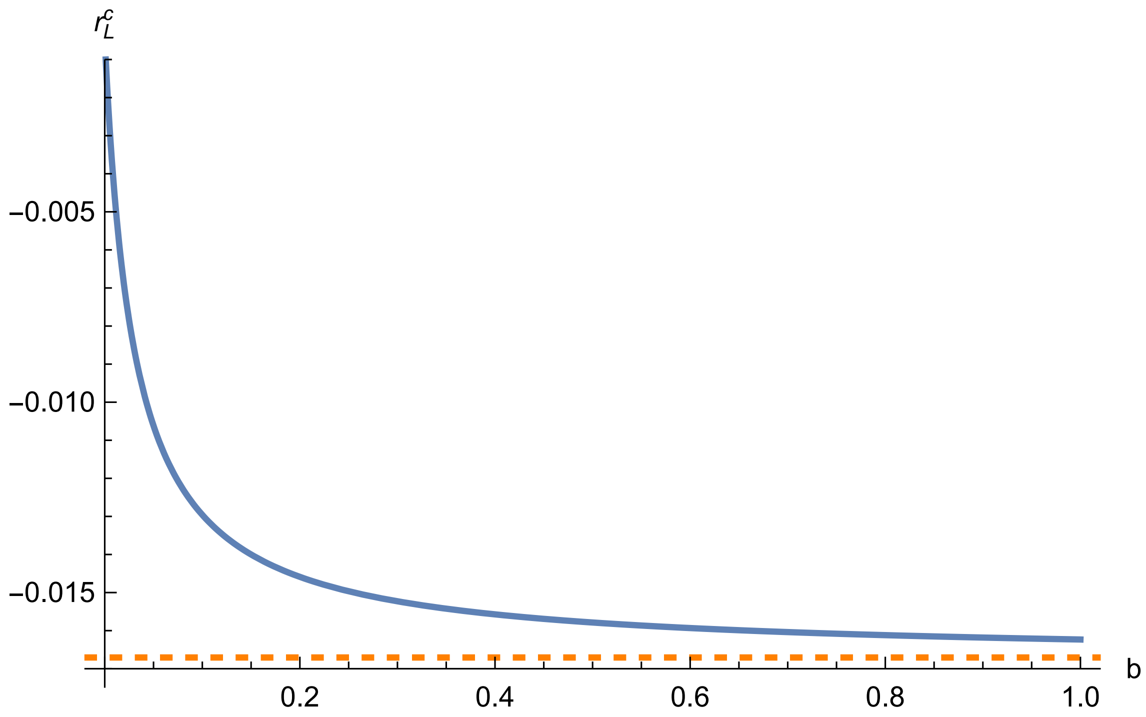

Figure 5.9: The critical value r_L^c.

where $r_L^c = -\frac{\phi_5+\phi_6 b}{(\lambda+\kappa^2+b\sigma^2)\phi_4} r_H$.

Note that the critical value r_L^c decreases with b. Thus, we have the following corollary:

Corollary 5.1
$i_\tau^f = 0$ *for any* $b \geq 0$ *if* $r_L \leq -\frac{\phi_6}{\sigma^2\phi_4} r_H$.

Figure (5.9) depicts that the critical value of the natural real interest rate shocks decreases with the incentive intensity b. The dashed orange curve represents r_L^c when $b = \infty$.

In the case $r_L \in [r_L^c, 0]$, inserting the optimal forecast into Equations (5.10) and (5.14) yields

$$\boldsymbol{Q}_\tau^n = -\frac{\phi_4 b\sigma}{\phi_5+\phi_6 b}\begin{pmatrix}\kappa\\1\end{pmatrix} r_L, \tag{5.17}$$

and

$$\boldsymbol{Q}^d = \frac{1}{h(\delta)}\tilde{\boldsymbol{O}}\begin{pmatrix}-\frac{(1-\delta)\phi_4 b\sigma}{\phi_5+\phi_6 b}\\1\end{pmatrix} r_L. \tag{5.18}$$

In the case $r_L < r_L^c$, the optimal forecast is 0. Equations (5.10) and (5.14) become

$$\boldsymbol{Q}_\tau^n = -\frac{b\sigma r_H}{\lambda+\kappa+b\sigma^2}\begin{pmatrix}\kappa\\1\end{pmatrix}, \tag{5.19}$$

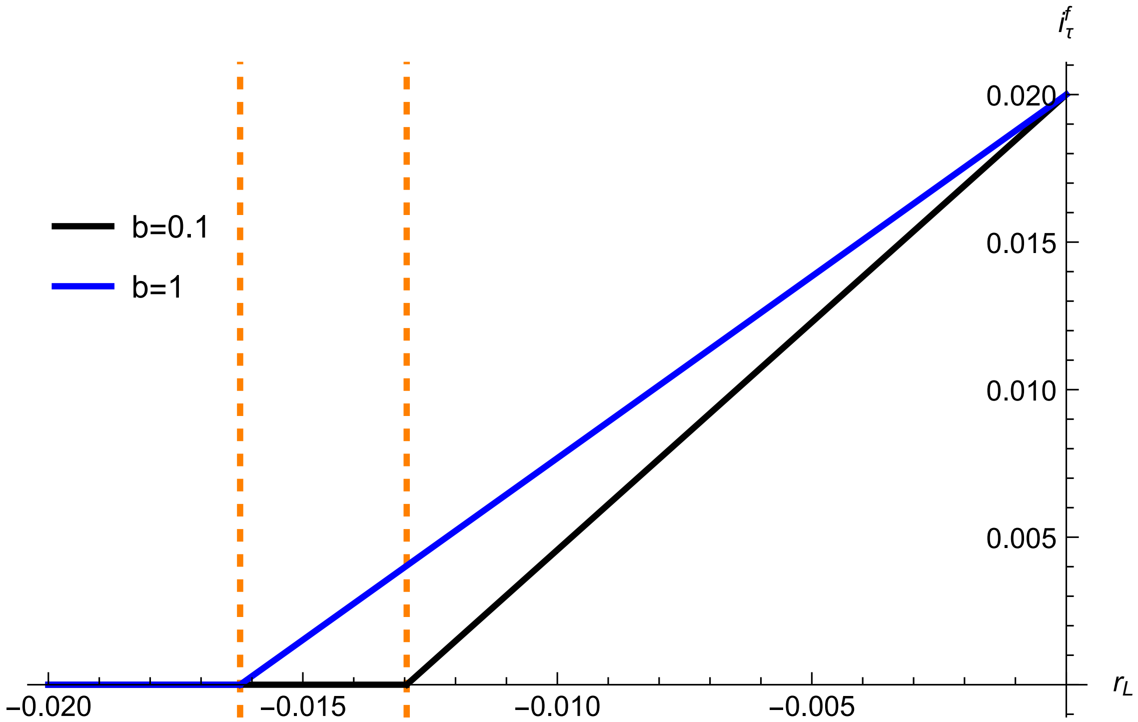

Figure 5.10: Interest rate forecast i_τ^f in function of the shock size for different incentive intensities b.

and

$$\boldsymbol{Q}^d = \frac{1}{h(\delta)}\tilde{\boldsymbol{O}}\begin{pmatrix} \frac{b\sigma(1-\delta)r_H}{\lambda+\kappa+b\sigma^2} \\ 1 \end{pmatrix} r_L. \tag{5.20}$$

Figure (5.10) depicts that the interest rate forecast is zero when $r_L \leq r_L^c$ and increases with r_L when $r_L > r_L^c$.

Figure (5.11) depicts that the interest rate in period τ is kept at a constant level when $r_L \leq r_L^c$. When the value of b is smaller, the central bank sets a higher interest rate due to the smaller weight on the loss of deviation from the zero-interest-rate forecast. The interest rate increases with r_L when $r_L > r_L^c$ as the central bank can achieve enough inflationary expectation in downturns with a large interest-rate forecast in the presence of small size shocks.

Figures (5.12) and (5.13) show the inflation and output gap in period τ and in downturns, respectively.

Since a higher value of b pushes up the inflation and output gap in downturns at the cost of higher inflation and output gap in period τ, in period 0, the government chooses b to maximize

$$\max_b -\frac{1}{2(1-\beta\delta)}\{[(\pi^d)^2 + \lambda(x^d)^2] + \beta(1-\delta)[(\pi_\tau^n)^2 + \lambda(x_\tau^n)^2]\}, \tag{5.21}$$

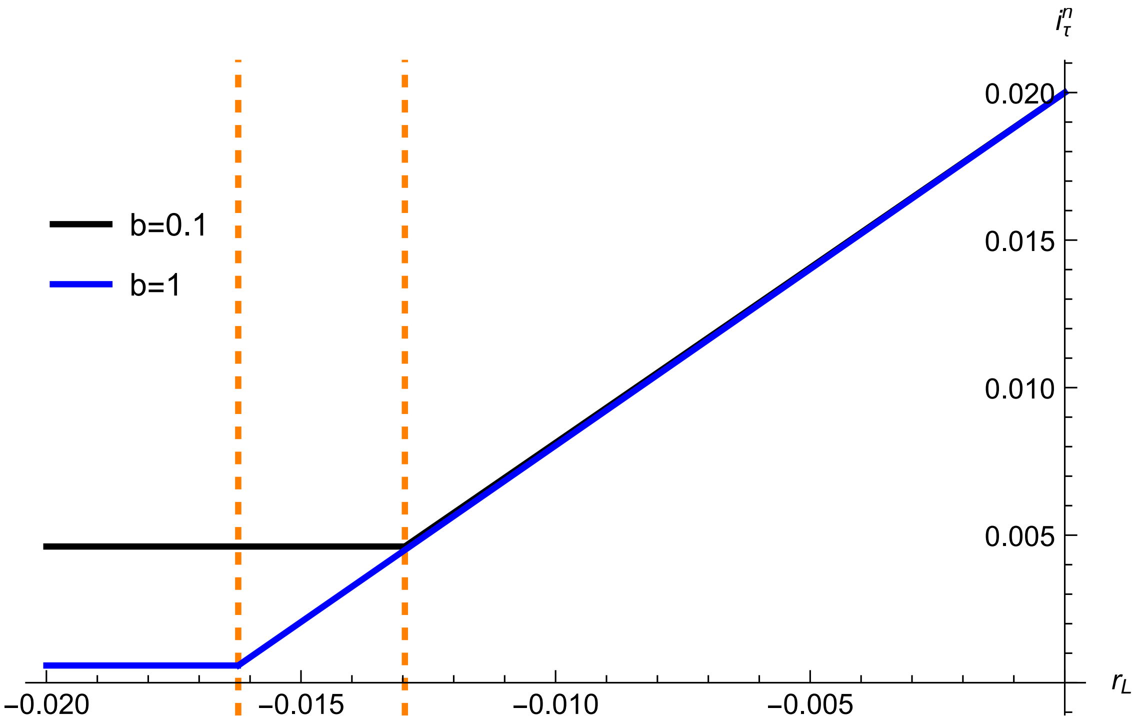

Figure 5.11: Interest rate i_τ^n in function of the shock size for different incentive intensities b.

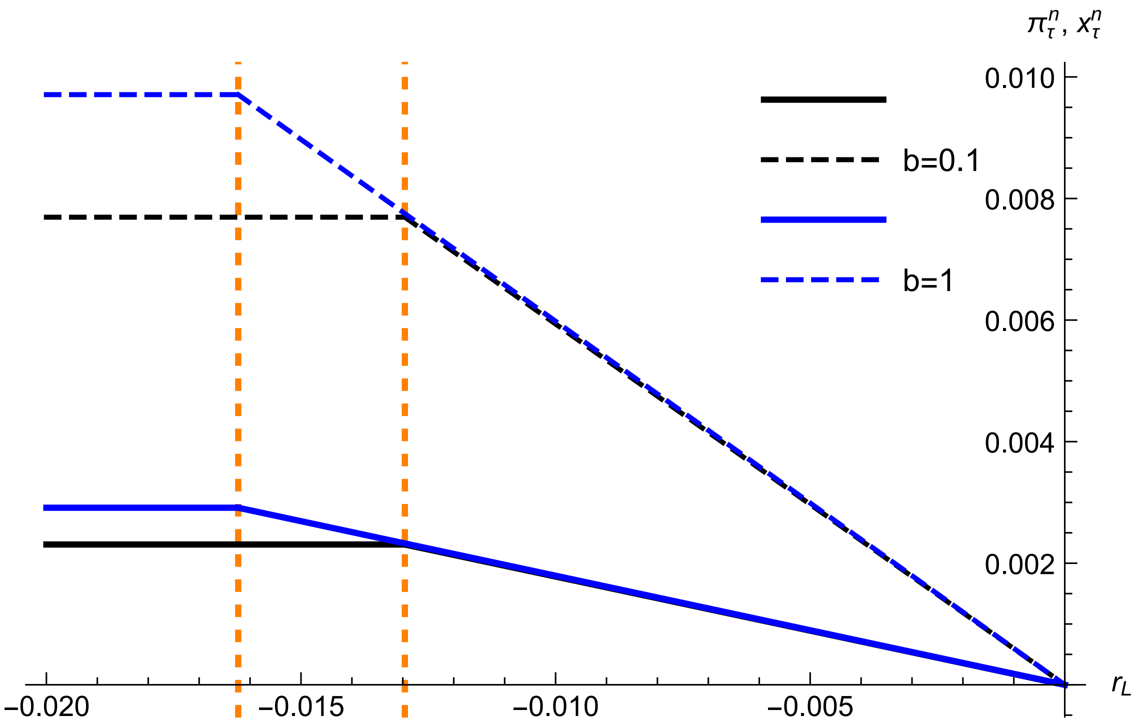

Figure 5.12: Inflation and output gap in period τ in function of the shock size for different incentive intensities b. Solid lines and dashed lines represent the inflation and output gap, respectively.

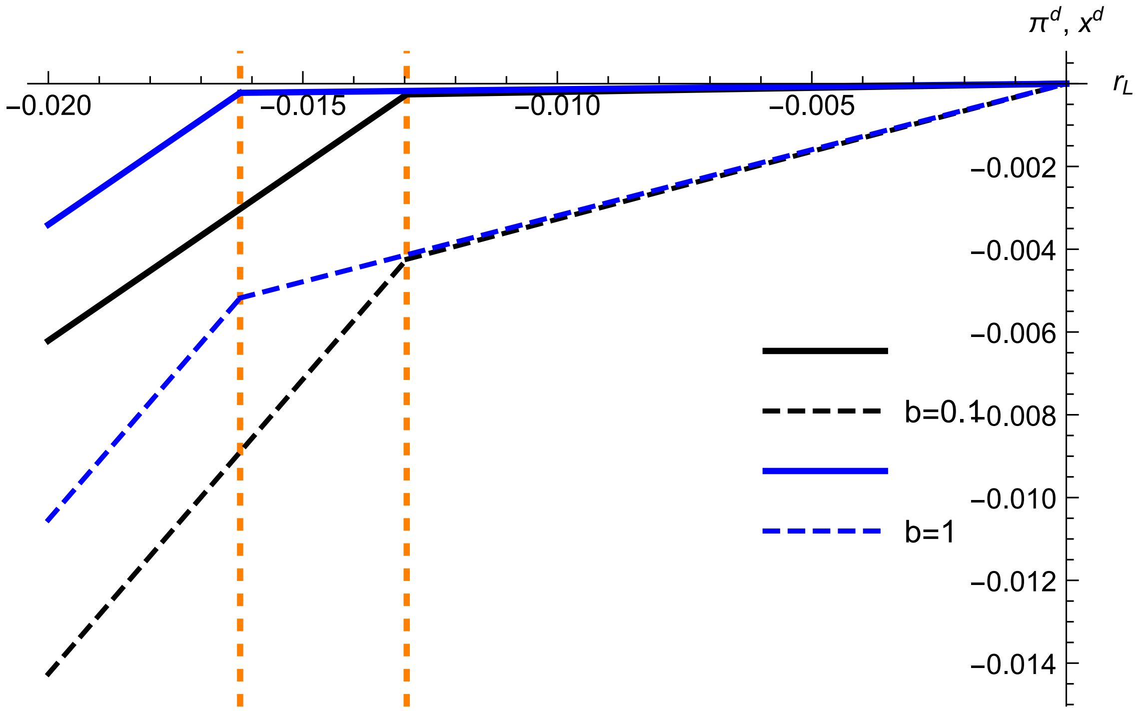

Figure 5.13: Inflation and output gap in period τ in function of the incentive intensity b with the optimal forecast. Solid lines and dashed lines represent inflation and output gap, respectively.

subject to Equations (5.17) and (5.18).

Numerical results show that the expected intertemporal social loss with $b = 1$ attains 99% of the one with infinite value of b. Therefore, in the presence of a shock with $r_L \geq r_L^c$, the government signs an SFGC $(1, 1)$ with the central banker.

Figure (5.15) shows that social loss increases with $|r_L|$. A large value of b is desirable. Here we achieve the main finding of this chapter: If we give central banker the freedom to forecast an interest rate higher than zero, then the optimal value of b is infinity, i.e. a highly scrupulous central banker is social beneficial. However, with $b = 1$ we already achieve a very high welfare gain compared to the benchmark case.

5.4 Equivalence of SFGC $(b, 1)$ and Simple Renewable FGC

In period 0, right after the shock, two types of contracts can be signed due to the stochastic character of the shock: State-contingent Forward Guidance Contracts (SFGC) and simple renewable Forward Guidance Contracts (FGC) as studied in chapter 3. Since the recovery is stochastic (with probability $1 - \delta$ in each period), SFGC $(b, 1)$ states that the contract

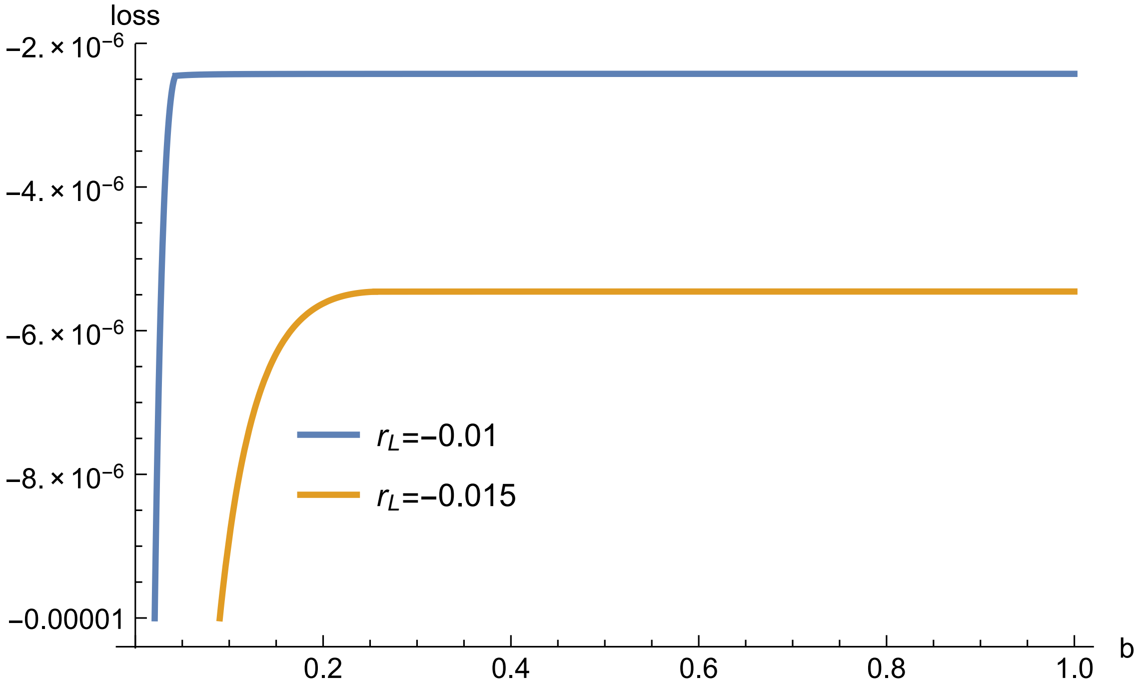

Figure 5.14: The expected intertemporal social loss in function of the incentive intensity b with the optimal forecast.

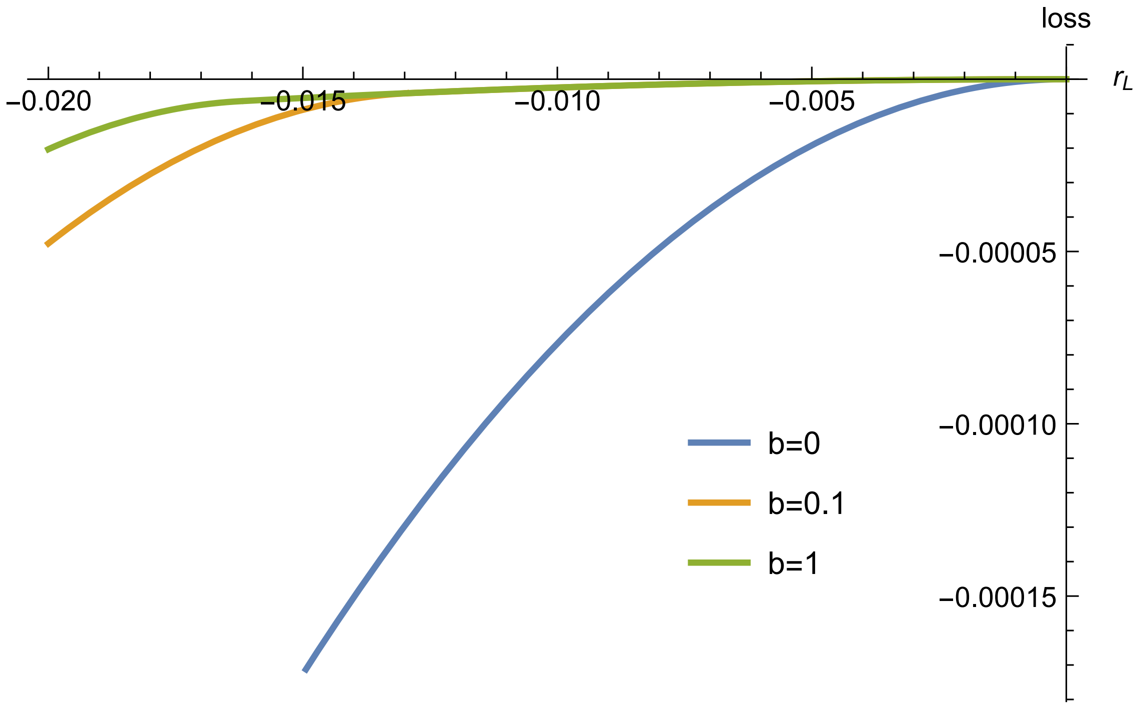

Figure 5.15: The expected intertemporal social loss in function of the size of shock for different values of b.

ends one period after the shock has vanished. In other words, the expiry date of SFGC is contingent on the recovery date. In contrast, the length of FGC is not contingent on the state of the economy being in the downturn or in normal times. FGC signed in period t expires in period $t+1$. Nevertheless, if the economy is still in the downturn in period $t+1$, the same FGC can be re-signed, since the situation is the same as in period t. Otherwise, if the economy is out of downturn in period $t+1$, no FGC is re-signed.

We find that signing the FGC in each period in downturns yields the same effect as signing an SFGC $(b, 1)$ in period 0: The central banker's choice of interest rate once the economy has recovered is constrained by his forecast in downturns, which ultimately creates inflationary expectations in downturns.

However, two slight differences should be pointed out. On the one hand, signing an SFGC $(b, 1)$ in period 0 once and for all avoids the unnecessary administrative issues of re-signing contracts in each period in downturns. On the other hand, signing an FGC gives the government and the central bank flexibility in redesigning the contract in case of a new development.

5.5 Discussion and Conclusion

In this chapter, we have studied a contract that is contingent on the natural real interest rate. However, the natural real interest rate is not observable or measurable directly. One might consider to use, the inflation expectation survey in financial market, for instance, to replace the natural real interest rate as a macroeconomic indicator. Specifically, in downturns, the government signs an SFGC with the central bank. The central bank announces to conduct the zero-interest-rate policy until the market's inflation expectation is above a certain level.

Unlike calendar-based FGCs studied in Chapters 3 and 4, SFGCs allow the central bank to conduct state-contingent zero-interest-rate policy. With SFGCs the government does not have to re-sign the FGCs repeatedly in downturns, while the same effects can be achieved.

6 Conclusions and Outlook

We have studied in Chapters 3 and 4 two variants of FGCs in the New Keynesian Framework. We started with the simple, renewable contracts that are signed in one period and become effective in the next period. With the optimally-designed contract, i.e. the optimally-chosen incentive intensity, the deflation and output gap collapse in downturns are alleviated due to the inflation expectation and output boom expectation. These beneficial effects arise because the FGCs enable central banks to commit to loose monetary policy after the vanishing of the downturn. When the size of the natural real interest rate shock is severe, longer-term FGCs can further improve social welfare by fostering even larger inflationary expectation. We addressed the longer-term FGCs in an alternative environment where the contract is signed and become effective in the same period and the length of the contract can be adjusted as needed. We showed that while such contracts could further lower the social loss compared to the short-term contracts, they also constrain the central bank for a longer period. This might hamper the central bank's flexibility in reacting to unforeseen economic developments in the interim.

Aside from these calendar-based FGCs, we have studied in Chapter 5 a contract that is contingent on the natural real interest rate. The contract, signed in downturns, is in effect as long as certain criteria are fulfilled. For example, the contract expires one period after the natural real interest rate reaches 2%. State-contingent contracts exempt the government from re-signing the simple, renewable FGCs repeatedly, while the same effect can be achieved.

We also discussed which type of FGCs can be used when there is uncertainty about natural real interest rate shocks, a situation which typically calls for moderate incentive intensity. Future work could also consider other complications in the use of FGCs that were not addressed in this thesis, such as the uncertainty about the transmission channel of monetary policy, for instance. In the following, we list several complications that we are currently addressing in our on-going projects.

Complication 1: Mismeasurement of parameters

In the thesis, we have assumed that the discretionary central banker has the same objective function as the households: $l_t = 0.5(\pi_t^2 + \lambda x_t^2)$. One might argue that central bankers put different weight on the output gap target relative to the inflation target, i.e. that central bankers have a λ that is different from the society's. Consider, for instance, a conservative central banker who puts more weight on the inflation mandate, i.e. the central banker has a λ smaller than the society's. As suggested in Equations (3.9)-(3.11), (3.15) and (3.16), the central banker would set a lower nominal interest rate than the socially-optimal value once the economy recovers to the normal times, leading to higher inflation and output gap in both downturns and normal times. However, the value of λ is relatively small compared to κ and σ in our calibration and in most other literature. Thus, a different weight on output gap would not change our results significantly.

Complication 2: Mismeasurement of variables

The IS Curve (3.1) suggests that the current output gap (differential between the actual output gap and its natural counterpart) depends on the expected output gap in the next period and the differential between the real interest rate and the natural real interest rate. Thus, the precise measurement of the natural output and the natural real interest rate is vital for the assessment of the economy being contractionary or expansionary and of the monetary policy stance being tight or loose.

The natural output—the desired level of output in the long-run—is the aggregate production of the economy in the absence of nominal rigidities. If the actual output is above the natural output, the economy is overusing its productive resources, leading to a boom and vice versa. Thus, the output gap is an important indicator to assess whether the economy is underperforming or overheating, and also an important guide for monetary policy. The natural real interest rate is used to represent a reference level of the real interest rate: If the real interest rate is above its natural counterpart, the monetary policy is assessed as tight and economic activities contract, and vice versa. Thus, the natural real interest rate provides a metric to gauge the tightness of the monetary policy stance being contractionary or stimulative.

If both the output gap and the natural real interest rate are measured correctly, the central bank's job is simple: setting the real interest rate higher than the natural counterpart in the presence of positive output gap and vice versa. Thus, it is ultimately important to measure the level of the natural output and the natural real interest rate. However, these two variables are not directly observable or measurable. They can only be estimated using theoretical derivations. For example, the natural output can be calculated using the estimated potential labor force, capital and overall productivity. The natural real interest rate can be computed using the consumers' preference and the technology level, for instance. However, preferences and technology level may vary with structural shifts, which are hard to measure. Future research could consider the use of FGCs under a veil of uncertainty about the natural output and natural real interest rate.

Complication 3: Multiple and time-varying shocks

In this thesis, we have abstracted from the supply shock and assumed that there exists only one type of shock: the shock on the natural real interest rate. In reality, however, different types of shock coexist and occur simultaneously or successively. For example, we could consider a circumstance where two types of shock occur consecutively: after the vanishing of natural real interest rate shock, a supply shock with unknown size arises—and unlike the stochastic natural real interest shock with two *ad hoc* realizations, the supply shock follows the first-order autoregressive process. After the natural real interest rate shock has vanished, the supply shock arises and dies out over time. In the presence of an inflationary supply shock, sticking to the zero-interest-rate forecast made in downturns would result in excessively high inflation in normal times. Thus, the uncertainty about the size of the supply shock prevents the central bank from announcing an unconditional zero-interest-rate forecast in downturns, leading to excessive deflation and output collapse in downturns. In one of our on-going projects we introduce two types of FGCs to resolve this problem:

- Forward Guidance Contracts with Escape Clause: Central bank is subject to the contract if and only if certain criteria are fulfilled, for instance if the inflation is below certain level. Otherwise, the central bank can act discretionarily. This kind of contracts allows the central bank to abandon the contract in certain circumstances. When the size of inflationary supply shock is too large, for instance, the central bank can discard the promise of zero-interest-rate policy made in downturns and fight the inflation with full force.

- Switching Forward Guidance Contracts: The central bank issues a zero-interest-rate forecast in downturns. After the vanishing of the natural real interest rate shock, if the size of the supply shock is too large, the central bank could switch to the inflation forecast to lower the unduly-large inflation caused by the zero-interest-rate forecast made in downturns. With such a contract, the central bank does not need to give up the forecast made in downturns, and can switch to an alternative forecast regime to suppress the overshooting inflation.

How FGCs should be designed and applied in various circumstances is left for future research. Nevertheless, this thesis provides a benchmark for future research on the use of forward guidance and Forward Guidance Contracts in the presence of the zero lower bound. FGCs are a flexible tool that is worth exploring further.

A Proofs for Chapter 3

A.1 Proof of Lemma 3.1

In this appendix we examine the circumstances under which our previous assumption holds that the central bank will select an interest rate of zero in a downturn. For this purpose, we use (3.1) and (3.2) to replace π_t and x_t in the central bank's instantaneous loss function in period t in the presence of a contract. The derivative of the resulting expression with respect to i_t has to be weakly positive at $i_L^C = 0$. Otherwise, it would be profitable to raise interest rates. Formally, this condition can be stated as

$$\kappa \pi_L^C + \lambda x_L^C \leq 0. \tag{A.1}$$

As a next step, we evaluate (A.1) at π_L^C and x_L^C, where the latter two variables are specified in (3.15) and (3.16):

$$\kappa \left[Af(b) + \pi_L^D \right] + \lambda \left[Bf(b) + x_L^D \right] \leq 0.$$

Solving for $f(b)$ yields

$$f(b) \leq -\frac{\kappa \pi_L^D + \lambda x_L^D}{\kappa A + \lambda B} = \frac{\kappa^2 + \lambda(1 - \beta\delta)}{\kappa(\kappa A + \lambda B)} |\pi_L^D|.$$

A.2 Proof of Lemma 3.2

To prove the lemma, we proceed in several steps. First, we determine the value of b that minimizes the social losses represented by (3.24), assuming that the resulting value of b satisfies (3.19), i.e. is small enough to ensure that the zero lower bound is binding in the downturn. Second, we show that, for this value of b, (3.19) is actually satisfied. Third, we examine optimal central bank policy if condition (3.19) fails to hold, i.e. in the case where b is such that the zero lower bound is not binding. Fourth, we show that the government would never find it optimal to select such a value for b.

Step #1 Inserting (3.20) and (3.21) into (3.24) and computing the derivative with respect to b reveals that this derivative is proportional to $f(b) - f^*$, where f^* has been defined in (3.25). We note that f^* is positive because $\pi_L^D < 0$, $x_L^D < 0$, $A > 0$, and $B > 0$.
Recall that $f(b)$ is a strictly monotonically increasing function with $f(0) = 0$. Hence for $\lim_{b\to\infty} f(b) = r_H/\sigma > f^*$, there is a unique value of b, b^*, that satisfies $f(b) = f^*$. This value minimizes expected social losses. By contrast, for $\lim_{b\to\infty} f(b) = r_H/\sigma \leq f^*$, social losses are a strictly monotonically decreasing function of b $\forall b \geq 0$. Loosely speaking, the optimal value of b is infinitely high in this case. For the remainder of the proof we focus on the case where values of b exist for which the zero lower bound would not bind in equilibrium under optimal central bank policy, i.e. we focus on $\lim_{b\to\infty} f(b) = r_H/\sigma > f^*$.

Step #2 It is unclear as yet whether for the value of b, b^*, identified in the previous step, the zero lower bound is actually binding in equilibrium. To show this, we prove that with respect to i_t, the derivative of the central bank's loss function, with the Phillips Curve and the IS Curve used to substitute for inflation and output, is weakly positive at $i_t = 0$ in a downturn. Formally, this condition can be stated as

$$\left.\frac{\partial l_t^{CB}}{\partial i_t}\right|_{i_L^C=0} \geq 0. \tag{A.2}$$

Since

$$l_t^{CB} = l_L^{CB} = \frac{1}{2}\left[\left(\pi_L^C\right)^2 + \lambda\left(x_L^C\right)^2\right] + \frac{1}{2}bi_t^2$$

with $i_t = i_L^C$ and

$$\frac{\partial l_t^{CB}}{\partial i_t} = \pi_t\frac{\partial \pi_t}{\partial i_t} + \lambda x_t\frac{\partial x_t}{\partial i_t} + bi_t = \pi_L^C\left(-\frac{\kappa}{\sigma}\right) + \lambda x_L^C\left(-\frac{1}{\sigma}\right) + bi_L^C,$$

(A.2) can be rewritten as

$$\kappa\pi_L^C + \lambda x_L^C \leq 0. \tag{A.3}$$

Using $\pi_L^C = Af(b^*) + \pi_L^D$ and $x_L^C = Bf(b^*) + x_L^D$, replacing $f(b^*)$ by f^*, and using the definition of f^* in (3.25), it is straightforward, though tedious, to show that (A.3) is satisfied for $b = b^*$.

Step #3 So far, we have determined the optimal choice of b from all values for which the zero lower bound binds in equilibrium. However, it is conceivable that the government would select a value of b such that this would not be the case, i.e. a value for which (3.19) is violated. Hence, we consider the equilibrium for this range of b in this step. In the fourth step, we demonstrate that the government would never select such a value for b^*.

If the zero lower bound does not bind, the following first-order condition holds in state L under an FGC, as can easily be shown:

$$\kappa\hat{\pi}_L^C + \lambda\hat{x}_L^C - b\sigma i_L^C = 0. \tag{A.4}$$

The Phillips Curve (3.2), $\hat{\pi}_L^C = \beta\left(\delta\hat{\pi}_L^C + (1-\delta)\pi_H^C\right) + \kappa\hat{x}_L^C$, the IS Curve (3.1), $\hat{x}_L^C = -\sigma^{-1}\left[i_L^C - \left(\delta\hat{\pi}_L^C + (1-\delta)\pi_H^C\right) - r_L\right] + \left(\delta\hat{x}_L^C + (1-\delta)x_H^C\right)$, (3.9), (3.10), and (A.4) can be used to compute

$$\hat{\pi}_L^C = z(b)\frac{\kappa(1-\delta)\left(\beta\lambda + b\sigma\left(\kappa + \sigma\left(1+\beta(1-\delta)\right)\right)\right)}{\kappa^2+\lambda+b\sigma^2} r_H + z(b)\kappa r_L, \tag{A.5}$$

$$\hat{x}_L^C = z(b)\frac{(1-\delta)\left(-\beta\kappa^2 + b\sigma\left(\kappa + \sigma\left(1-\beta\delta\right)\right)\right)}{\kappa^2+\lambda+b\sigma^2} r_H + z(b)(1-\beta\delta) r_L, \tag{A.6}$$

where

$$z(b) := \frac{b\sigma}{\kappa^2 + \lambda(1-\beta\delta) + b\sigma\left(\sigma(1-\delta)(1-\beta\delta)\right) - \delta\kappa)}. \tag{A.7}$$

We observe that $z(b)$ is a monotonically increasing function of b (recall our previous assumption $\sigma(1-\delta)(1-\beta\delta) - \delta\kappa > 0$).

Step #4 Suppose that b is sufficiently high for the zero lower bound not to be binding ($b \geq \hat{b}$). Then (A.5) and (A.6) can be used to write per-period losses in a downturn as

$$\begin{aligned}\hat{l}_L^C &= \left(\hat{\pi}_L^C\right)^2 + \lambda\left(\hat{x}_L^C\right)^2 \\ &= z(b)^2\left[\left(\frac{\kappa(1-\delta)\left(\beta\lambda + b\sigma\left(\kappa+\sigma\left(1+\beta(1-\delta)\right)\right)\right)}{\kappa^2+\lambda+b\sigma^2} r_H + \kappa r_L\right)^2\right. \\ &\qquad \left. + \lambda\left(\frac{(1-\delta)\left(-\beta\kappa^2 + b\sigma\left(\kappa+\sigma\left(1-\beta\delta\right)\right)\right)}{\kappa^2+\lambda+b\sigma^2} r_H + (1-\beta\delta) r_L\right)^2\right].\end{aligned} \tag{A.8}$$

Using $f(b) = \frac{b\sigma}{\lambda+\kappa^2+b\sigma^2} r_H$ and $\frac{1}{\lambda+\kappa^2+b\sigma^2} r_H = \frac{r_H - \sigma f(b)}{\kappa^2+\lambda}$ (see (3.12)), we can restate (A.8) as follows:

$$\begin{aligned}\hat{l}_L^C = z(b)^2 &\left[\left(\kappa(1-\delta)\left(\frac{r_H-\sigma f(b)}{\kappa^2+\lambda}\beta\lambda + f(b)\left(\kappa+\sigma\left(1+\beta(1-\delta)\right)\right)\right) + \kappa r_L\right)^2\right. \\ &\left. + \lambda\left((1-\delta)\left(-\frac{r_H-\sigma f(b)}{\kappa^2+\lambda}\beta\kappa^2 + f(b)\left(\kappa+\sigma\left(1-\beta\delta\right)\right)\right) + (1-\beta\delta) r_L\right)^2\right].\end{aligned} \tag{A.9}$$

We will now explain that (A.9) is a monotonically increasing function of b for $b \geq \hat{b}$, where $\hat{b}$ is implicitly defined by (3.19). This follows from two observations. First, we have already noted that $z(b)$ monotonically increases with $b \; \forall b \geq 0$. Second, the term in brackets in (A.9) is a quadratic function of $f(b)$. It is straightforward, though tedious, to show that the minimum of this term, interpreted as a function of $f(b)$, is at $f(b) = f(\hat{b})$, where $f(\hat{b})$ is given by the right-hand side of (3.19). Hence (A.9) monotonically increases with b for $b \geq \hat{b}$.

Because at $b = \hat{b}$, $l_L^C = \hat{l}_L^C$ holds[1] and l_L^C, evaluated at $\hat{b}$, has to be larger than at $b = b^*$ as b^* is the value of b minimizing l_L^C, we can conclude that the government would not choose a value of b with $b \geq \hat{b}$.

A.3 Proof of Lemma 3.3

A.3.1 Preliminary steps

We need to define the strategy of the government in the candidate equilibrium precisely. We assume that the government will always sign a new contract for the next period in state L, independently of whether a contract has been signed for the current period. Moreover, we consider the case where the government never signs a contract for the next period if the economy is in state H, irrespective of whether a contract exists for the current period.

Next we examine whether, for the government, profitable deviations exist in a particular period, when the government takes its own future behavior, the behaviors of the central bank and of the private sector as given. There are four potential deviations. First, in a situation where a contract is present in the current period and where the current economic state is L, the government chooses not to sign a contract for the next period. Second, the government refuses to offer a contract in state L, given that no contract is present in the current period. Third, in state H the government offers a contract for the next period if a contract is active in the current period. Fourth, in state H without a contract in the current period, the government introduces a contract for the next period.

It is comparably straightforward to show that the third and fourth deviation cannot be profitable. Showing that the other deviations are undesirable for the government is more cumbersome and requires a few preliminary steps and some additional notation.

Let $V_s(C)$ be the discounted future social losses for optimal central-bank and private-

[1] It is not difficult to verify that, for $b = \hat{b}$, $\pi_L^C = \hat{\pi}_L^C = -\frac{\lambda\beta\kappa r_L}{\kappa^3+\sigma(1+\beta(1-\delta))\kappa^2+\lambda\kappa+\sigma\lambda(1-\beta\delta)}$ and $x_L^C = \hat{x}_L^C = \frac{\beta\kappa^2 r_L}{\kappa^3+\sigma(1+\beta(1-\delta))\kappa^2+\lambda\kappa+\sigma\lambda(1-\beta\delta)}$, which implies the continuity of social losses, interpreted as a function of b, at $b = \hat{b}$.

sector behaviors, given the current state $s \in \{L, H\}$, the fact that the government pursues the strategy described above, and that a contract has been signed for the current period. $V_s(N)$ is the analogous expression for the case where no contract is present in the current period. Moreover, let l_s^{XY} with $s \in \{H, L\}$ and $X, Y \in \{C, N\}$ be the per-period losses in state s if currently there is a contract ($X = C$) or no contract ($X = N$) and if in the current period a contract is signed for the next period ($Y = C$) or not ($Y = N$).

We obtain the following equations:

$$V_L(C) = l_L^{CC} + \beta\left(\delta V_L(C) + (1-\delta)V_H(C)\right), \quad (A.10)$$

$$V_H(C) = l_H^{CN} + \beta V_H(N), \quad (A.11)$$

$$V_L(N) = l_L^{NC} + \beta\left(\delta V_L(C) + (1-\delta)V_H(C)\right), \quad (A.12)$$

$$V_H(N) = l_H^{NN} + \beta V_H(N). \quad (A.13)$$

We note that $l_H^{NN} = 0$, $l_L^{CN} = l_L^{NN}$, and $l_L^{CC} = l_L^{NC}$, where the latter two conditions follow from the observation that the zero lower bound always binds in state $s = L$, irrespective of whether a contract was signed in the previous period. This observation will be shown formally later.

As a result, we obtain

$$V_L(C) = \frac{1}{1-\beta\delta} l_L^{CC} + \frac{\beta(1-\delta)}{1-\beta\delta} l_H^{CN}, \quad (A.14)$$

$$V_H(C) = l_H^{CN}, \quad (A.15)$$

$$V_L(N) = \frac{1}{1-\beta\delta} l_L^{CC} + \frac{\beta(1-\delta)}{1-\beta\delta} l_H^{CN}, \quad (A.16)$$

$$V_H(N) = 0. \quad (A.17)$$

A.3.2 Deviation in state L when a contract was signed in the previous period

We are now in a position to specify the condition that ensures that the government does not find it optimal to refuse to offer a contract for the next period, given that the current state is L and that a contract was signed in the previous period:

$$l_L^{CN} + \beta\left(\delta V_L(N) + (1-\delta)V_H(N)\right) \geq V_L(C). \quad (A.18)$$

The right-hand side of the inequality states the losses incurred if the government does not deviate. The expression on the left-hand side represents social losses if the government does not offer a contract in the period under consideration but pursues its equilibrium strategy in all future periods. With the help of (A.14)-(A.17), (A.18) can be simplified to

$$l_L^{CN} \geq l_L^{CC} + \beta(1-\delta)l_H^{CN}. \tag{A.19}$$

This condition will be analyzed in more detail later.

A.3.3 Deviation in state L when a contract was not signed in the previous period

In state L, the government will find it optimal to offer a contract for the next period, provided that no contract was signed in the previous period, if

$$l_L^{NN} + \beta\left(\delta V_L(N) + (1-\delta)V_H(N)\right) \geq V_L(N). \tag{A.20}$$

Because $l_L^{NN} = l_L^{CN}$ and $V_L(N) = V_L(C)$, this condition is equivalent to (A.18) and thus to (A.19).

A.3.4 Evaluating condition (A.19)

To evaluate condition (A.19), we have to determine l_L^{CN}, l_L^{CC}, and l_H^{CN}. For this purpose, we observe that l_L^{CC} and l_H^{CN} are per-period losses that also occur in the candidate equilibrium. Hence, we obtain

$$l_L^{CC} = 0.5[(\pi_L^C)^2 + \lambda(x_L^C)^2], \tag{A.21}$$

$$l_H^{CN} = 0.5[(\pi_H^C)^2 + \lambda(x_H^C)^2]. \tag{A.22}$$

To determine l_L^{CN}, we have to compute inflation and the output gap, π_L^{CN} and x_L^{CN}, for the case where the government does not offer a contract in state L for the next period but reverts to its putative equilibrium strategy in all future periods, i.e. it will offer a contract in state L and no contract in state H. In such a situation, expectations of inflation and the output gap are

$$\mathbb{E}_t[\pi_{t+1}] = \delta\pi_L^{NC} + (1-\delta)\pi_H^{NN} = \delta\pi_L^C, \tag{A.23}$$

$$\mathbb{E}_t[x_{t+1}] = \delta x_L^{NC} + (1-\delta)x_H^{NN} = \delta x_L^C. \tag{A.24}$$

It is tedious but straightforward to show that inserting these two expressions into (3.1) and (3.2), evaluated at $i_t = 0$, yields

$$\pi_L^{CN} = Pf(b) + \pi_L^D, \quad \text{(A.25)}$$
$$x_L^{CN} = Qf(b) + x_L^D, \quad \text{(A.26)}$$

where π_L^D, x_L^D, and $f(b)$ have been introduced in (3.5), (3.12), and (3.4) respectively, and P and Q are given by

$$P := \delta\left[\left(\frac{\kappa}{\sigma} + \beta\right) A + \kappa B\right], \quad \text{(A.27)}$$
$$Q := \delta\left(\frac{A}{\sigma} + B\right). \quad \text{(A.28)}$$

Recall that A and B have been defined in (3.17) and (3.18).

A.3.5 Verifying that the zero lower bound binds for the deviations

It remains to be verified that the zero lower bound is also binding for the deviations analyzed above if (3.19) is satisfied, which ensures that it is binding in state L in equilibrium when a contract is present in the current period. We note that this is the case for

$$\kappa\pi_L^{CN} + \lambda x_L^{CN} \leq 0. \quad \text{(A.29)}$$

Using (3.19), (A.25), (A.26), and $x_L^D = \frac{1-\beta\delta}{\kappa}\pi_L^D$, which follows from (3.4) and (3.5), and re-arranging results in the condition yields

$$\frac{\kappa P + \lambda Q}{\kappa A + \lambda B} \leq 1. \quad \text{(A.30)}$$

It is straightforward to show $A - P = \frac{\kappa(1-\delta)(\kappa+\sigma(1+\beta))}{\sigma} > 0$ and $B - Q = \frac{(1-\delta)(\kappa+\sigma)}{\sigma} > 0$, which, together with $P > 0$ and $Q > 0$, implies (A.30).

A.3.6 Simplifying condition (A.19)

Finally, we simplify condition (A.19) to identify the set of parameter values for which no profitable deviation for the government exists. The condition can be written as

$$\begin{aligned} 0 \leq & \left(\pi_L^{NN}\right)^2 + \lambda\left(x_L^{NN}\right)^2 - \left(\pi_L^C\right)^2 - \lambda\left(x_L^C\right)^2 - \beta(1-\delta)\left(\pi_H^C\right)^2 - \beta(1-\delta)\lambda\left(x_H^C\right)^2 \\ = & -\left(A^2 - P^2 + \lambda B^2 - \lambda Q^2 + \beta(1-\delta)(\kappa^2+\lambda)\right)(f(b))^2 \\ & -2\left(A - P + \lambda(B-Q)\frac{1-\beta\delta}{\kappa}\right) f(b)\pi_L^D. \end{aligned}$$

As $\pi_L^D < 0$, $A^2 - P^2 + \lambda B^2 - \lambda Q^2 + \beta(1-\delta)(\kappa^2+\lambda) > 0$, and $\left(A - P + \lambda(B-Q)\frac{1-\beta\delta}{\kappa}\right) > 0$ (due to $A > P$ and $B > Q$), we can conclude that this expression is weakly positive for all values of $f(b)$ with $f(b) \geq 0$ that are smaller than or equal to $2\tilde{f}$, where

$$\tilde{f} := \frac{A - P + \lambda(B-Q)\frac{1-\beta\delta}{\kappa}}{A^2 - P^2 + \lambda B^2 - \lambda Q^2 + \beta(1-\delta)(\kappa^2+\lambda)}|\pi_L^D|. \tag{A.31}$$

Hence no profitable deviation exists for the government if $f^* \leq 2\tilde{f}$, where

$$f^* = \frac{A + \lambda B\frac{1-\beta\delta}{\kappa}}{A^2 + \lambda B^2 + \beta(1-\delta)(\lambda+\kappa^2)}|\pi_L^D|. \tag{A.32}$$

A.4 Proof of Proposition 3.2

Suppose that the value of b corresponded to the optimal value b^{**} for the realization $r_L = \overline{r}_L$. Clearly, for this particular realization of r_L, social welfare would be higher than in the benchmark case. In the following we show that this value of b also leads to welfare improvements for all other realizations of r_L. For this purpose, we note that b^{**} is a monotonically increasing function of $|r_L|$, as both $\tilde{f}$ and f^* are increasing linear functions of $|\pi_L^D|$, which, in turn, monotonically increases with $|r_L|$. As social losses interpreted as a function of $f(b)$ are monotonically decreasing for all $f(b) \leq f^*$, we can conclude that the value of b optimal for $\overline{r}_L$ would also increase welfare for all other realizations of r_L.

A.5 Proof of Lemma 3.4

With a given $\tilde{b}$, inserting (A.5) and (A.6) into (A.4) yields

$$i_L^C(\tilde{b}) = \frac{(\kappa A + \lambda B)[\sigma(1-\delta)(1-\beta\delta) - \delta\kappa]f(\tilde{b}) + [\kappa^2 + \lambda(1-\beta\delta)]r_L}{b\sigma[\sigma(1-\delta)(1-\beta\delta) - \delta\kappa] + \kappa^2 + \lambda(1-\beta\delta)}, \tag{A.33}$$

where $f(\tilde{b})$ is given[2] in (3.25) in Lemma 3.2.

Equation (A.33) implies that $i_L^C > 0$ if and only if $r_L > a\tilde{r}_L$, where

$$a := \frac{(\kappa A + \lambda B)[\kappa A + \lambda B(1-\beta\delta)]}{[\kappa^2 + \lambda(1-\beta\delta)][A^2 + \lambda B^2 + \beta(1-\delta)(\lambda+\kappa^2)]}. \tag{A.34}$$

Therefore, the zero lower bound is binding when $r_L \leq a\tilde{r}_L$.

We next prove that it is socially desirable to offer the FGC($\tilde{b}$) when $r_L \leq a\tilde{r}_L$.

We can write the discounted social loss under discretion as in (3.24):

$$V_L(D) = \frac{1}{1-\beta\delta} l_L^D, \tag{A.35}$$

where $l_L^D = 0.5[(\pi_L^D)^2 + \lambda(x_L^D)^2]$.

The government would offer the contract when $r_L \leq a\tilde{r}_L$ if and only if the discounted social loss with FGC($\tilde{b}$) were lower than the one in the benchmark case:

$$V_L(C) < V_L(D). \tag{A.36}$$

Solving (A.36) yields

$$r_L < 0.5\tilde{r}_L.$$

In our calibration[3], $a = 0.95 > 0.5$. Therefore, for all $r_L \leq a\tilde{r}_L$, (A.36) is satisfied and it is socially desirable to offer the FGC in these circumstances.

A.6 Proof of Proposition 3.3

We have derived (3.15) and (3.16), assuming the zero lower bound is binding. In a similar vein, we now derive the inflation and output gap in downturn with a given FGC($\tilde{b}$), without assuming that the zero lower bound is binding. We obtain

$$\pi_L^C = Af(\tilde{b}) + \frac{\kappa}{\sigma(1-\delta)(1-\beta\delta) - \delta\kappa}(r_L - i_L^C) \tag{A.37}$$

and

$$x_L^C = Bf(\tilde{b}) + \frac{1-\beta\delta}{\sigma(1-\delta)(1-\beta\delta) - \delta\kappa}(r_L - i_L^C). \tag{A.38}$$

The government would offer the FGC if and only if (A.36) applied.

[2] Recall that $\tilde{b}$ is the optimal value of b when $r_L = \tilde{r}_L$.

[3] Numerical result shows that for all δ that satisfy Assumption 3.1, $a > 0.5$ is fulfilled.

Inserting (A.33), (A.37), and (A.38) into (A.36) yields

$$r_L < \tilde{r}_L^c,$$

where

$$\tilde{r}_L^c = \frac{a_1 a_2 + \lambda a_3 a_4 - \sqrt{(a_1 a_2 + \lambda a_3 a_4)^2 + (a_5 - a_1^2 - \lambda a_3^2)[a_2^2 + \lambda a_4^2 + \beta(1-\delta)(\lambda + \kappa^2)]}}{a_5 - a_1^2 - \lambda a_3^2} f(\tilde{b}),$$

$$a_1 := \frac{\kappa \sigma \tilde{b}}{\sigma[\sigma(1-\delta)(1-\beta\delta) - \delta\kappa]\tilde{b} + \kappa^2 + \lambda(1-\beta\delta)},$$

$$a_2 := \frac{\sigma A[\sigma(1-\delta)(1-\beta\delta) - \delta\kappa]\tilde{b} + \lambda(1-\beta\delta)A - \lambda\kappa B}{\sigma[\sigma(1-\delta)(1-\beta\delta) - \delta\kappa]\tilde{b} + \kappa^2 + \lambda(1-\beta\delta)},$$

$$a_3 := \frac{(1-\beta\delta)\sigma \tilde{b}}{\sigma[\sigma(1-\delta)(1-\beta\delta) - \delta\kappa]\tilde{b} + \kappa^2 + \lambda(1-\beta\delta)},$$

$$a_4 := \frac{\sigma B[\sigma(1-\delta)(1-\beta\delta) - \delta\kappa]\tilde{b} + \kappa^2 B - \kappa(1-\beta\delta)A}{\sigma[\sigma(1-\delta)(1-\beta\delta) - \delta\kappa]\tilde{b} + \kappa^2 + \lambda(1-\beta\delta)},$$

$$a_5 := \frac{\kappa^2 + \lambda(1-\beta\delta)^2}{[\sigma(1-\delta)(1-\beta\delta) - \delta\kappa]^2}.$$

B Proofs for Chapter 4

B.1 Benchmarks

In this section, we provide the formal detail of the two benchmark solutions–discretion and commitment. In the last subsection, we consider the commitment scenario for the special case when the natural real interest rate bounces back to the steady-state value r_H one period after the shock.

B.1.1 Discretionary Policy

In this subsection, we consider a discretionary central bank and assume that it is common knowledge that discretionary policy is conducted in every period.
The expected intertemporal social loss in period 0 is

$$\mathbb{E}_0 l_{[0,\infty]} = -\frac{1}{2}\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t(\pi_t^2 + \lambda x_t^2). \tag{B.1}$$

When the central bank chooses its policy in a particular period t, the Lagrangian is

$$\begin{aligned}\mathcal{L} =&\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t[\frac{1}{2}(\pi_t^2 + \lambda x_t^2) \\ &+ \psi_{1,t}(x_t - \mathbb{E}_t x_{t+1} + \frac{1}{\sigma}(i_t - \mathbb{E}_t \pi_{t+1} - r_t)) \\ &+ \psi_{2,t}(\pi_t - \kappa x_t - \beta \mathbb{E}_t \pi_{t+1})].\end{aligned} \tag{B.2}$$

In a discretionary solution, the expectations $\mathbb{E}_t x_{t+1}$ and $\mathbb{E}_t \pi_{t+1}$ are taken as given when the government selects its policy. Therefore, the first-order conditions amount to

$$\frac{\partial \mathcal{L}}{\partial \pi_t} = \pi_t + \psi_{2,t} = 0, \tag{B.3}$$

and

$$\frac{\partial \mathcal{L}}{\partial x_t} = \lambda x_t + \psi_{1,t} - \kappa \psi_{2,t} = 0, \tag{B.4}$$

with the constraints

$$i_t \geq 0, \psi_{1,t} \geq 0, i_t\psi_{1,t} = 0. \tag{B.5}$$

The initial conditions are

$$\psi_{1,-1} = 0, \psi_{2,-1} = 0. \tag{B.6}$$

The steady-state solution when the shock has subsided is characterized by[1]

$$\pi_{ss} = 0, x_{ss} = 0, i_{ss} = r_H, \psi_{1,ss} = 0, \psi_{2,ss} = 0. \tag{B.9}$$

We denote the period when the natural real interest rate reverts r_H by τ, where $\tau \geq 1$. In the periods $t \geq \tau$, $\psi_{1,t} = 0$.

Combining Equations (B.3) and (B.4) yields

$$\lambda x_t + \kappa\pi_t = 0. \tag{B.10}$$

Inserting the Phillips Curve (3.2) in Equation (B.10) leads to

$$\pi_{t+1} = \frac{1}{\beta}(1 + \frac{\kappa^2}{\lambda})\pi_t. \tag{B.11}$$

By taking the limit on both sides of Equation (B.11), we obtain

$$\lim_{t\to\infty} \pi_t = \lim_{t\to\infty}[\frac{1}{\beta}(1 + \frac{\kappa^2}{\lambda})]^{t-\tau}\pi_\tau = 0. \tag{B.12}$$

Since $\frac{1}{\beta}(1 + \frac{\kappa^2}{\lambda}) > 1$, the unique solution is $\pi_t = 0$. With Equation (B.10), $x_t = 0$. With the IS Equation (3.1), we conclude that $i_t = r_H$.

In periods $t \in [0, \tau - 1]$, $i_t = 0$ and $\psi_{1,t} > 0$, as this minimizes the per-period social loss.

The IS Equation for $t \in [0, \tau - 1]$ becomes

$$\mathbb{E}_t x_{t+1} = x_t - \frac{1}{\sigma}(\mathbb{E}_t\pi_{t+1} + r_t), \tag{B.13}$$

Rearranging the IS Equation (B.13) and the Phillips Curve (3.2) for the downturn $t \in$

[1] Deflationary spirals characterized by

$$\pi = -r_H < 0, x = -\frac{1-\beta}{\kappa}r_H < 0, i = 0, \tag{B.7}$$

with Lagrange Multipliers

$$\psi_1 = \frac{\kappa^2 + \lambda(1-\beta)}{\kappa}r_H, \psi_2 = r_H, \tag{B.8}$$

can be excluded by setting the interest rate above zero once the economy has returned to $r_H > 0$.

$[0, \tau - 1]$ yields

$$\mathbb{E}_t \boldsymbol{Q_{t+1}} \equiv \mathbb{E}_t \begin{pmatrix} \pi_{t+1} \\ x_{t+1} \end{pmatrix} = \boldsymbol{O}\boldsymbol{Q_t} - \frac{1}{\sigma}\boldsymbol{r_t}, \tag{B.14}$$

where $\boldsymbol{O} := \begin{pmatrix} \frac{1}{\beta} & -\frac{\kappa}{\beta} \\ -\frac{1}{\sigma\beta} & 1 + \frac{\kappa}{\sigma\beta} \end{pmatrix}$ and $\boldsymbol{r_t} := \begin{pmatrix} 0 \\ r_t \end{pmatrix}$.

$\boldsymbol{O}$ is invertible due to $|\boldsymbol{O}| = \frac{1}{\beta} \neq 0$.

As a preliminary step, we address the solution when $r_t = r_L$ for a specific number of periods τ[2]. In period τ, r_t returns to r_H and stays there forever.

Step 1: Inserting the IS Equation (3.1) in the Phillips Curve (3.2) yields

$$\begin{aligned} \pi_t &= \beta\pi_{t+1} + \kappa(x_{t+1} - \frac{1}{\sigma}(i_t - \pi_{t+1} - r_t)) \\ &= \beta\pi_{t+1} - \frac{\kappa}{\sigma}(i_t - \pi_{t+1} - r_t) + \pi_{t+1} - \beta\pi_{t+2} \\ &= -\frac{\kappa}{\sigma}(i_t - r_t) + (1 + \beta + \frac{\kappa}{\sigma})\pi_{t+1} - \beta\pi_{t+2}. \end{aligned} \tag{B.15}$$

In the periods $t \in [0, \tau - 1]$, $i_t = 0$ and $r_t = r_L$.

The evolution of inflation in the downturn is

$$\pi_t = \frac{\kappa}{\sigma}r_L + (1 + \beta + \frac{\kappa}{\sigma})\pi_{t+1} - \beta\pi_{t+2}. \tag{B.16}$$

The terminal conditions are

$$\pi_\tau = 0, \tag{B.17}$$

$$\pi_{\tau-1} = \frac{\kappa}{\sigma}r_L, \tag{B.18}$$

and

$$\pi_{\tau-2} = (2 + \beta + \frac{\kappa}{\sigma})\frac{\kappa}{\sigma}r_L. \tag{B.19}$$

In order to solve Equation (B.16), we first solve the particular equation

$$\chi = \frac{\kappa}{\sigma}r_L + (1 + \beta + \frac{\kappa}{\sigma})\chi - \beta\chi, \tag{B.20}$$

with the auxiliary variable χ.

We obtain

$$\chi = -r_L. \tag{B.21}$$

[2] See Carlstrom et al. (2012).

Solving the characteristic equation of Equation (B.16)

$$\iota^2 - (1 + \beta + \frac{\kappa}{\sigma})\iota + \beta = 0, \tag{B.22}$$

yields

$$\iota_{1/2} = \frac{1 + \beta + \frac{\kappa}{\sigma} \pm \sqrt{(1 + \beta + \frac{\kappa}{\sigma})^2 - 4\beta}}{2}, \tag{B.23}$$

where ι is the auxiliary variable.
The two solutions satisfy $\iota_1 < \beta < 1 < \iota_2$.
Using Equations (B.21) and (B.23), the inflation dynamics in the downturn can be written as

$$\pi_t = -r_L + \upsilon_1 \iota_1^{\tau - t} + \upsilon_2 \iota_2^{\tau - t}, \tag{B.24}$$

where υ_1 and υ_2 are two real valued coefficients. υ_1 and υ_2 are determined by the terminal conditions

$$\pi_{\tau-1} = -r_L + \upsilon_1 \iota_1 + \upsilon_2 \iota_2 = \frac{\kappa}{\sigma} r_L, \tag{B.25}$$

and

$$\pi_{\tau-2} = -r_L + \upsilon_1 \iota_1^2 + \upsilon_2 \iota_2^2 = (2 + \beta + \frac{\kappa}{\sigma})\frac{\kappa}{\sigma} r_L. \tag{B.26}$$

If υ_2 is non-zero, a larger value of τ implies a larger value of $|\pi_0|$. This is shown in Figures B.1 and B.2.

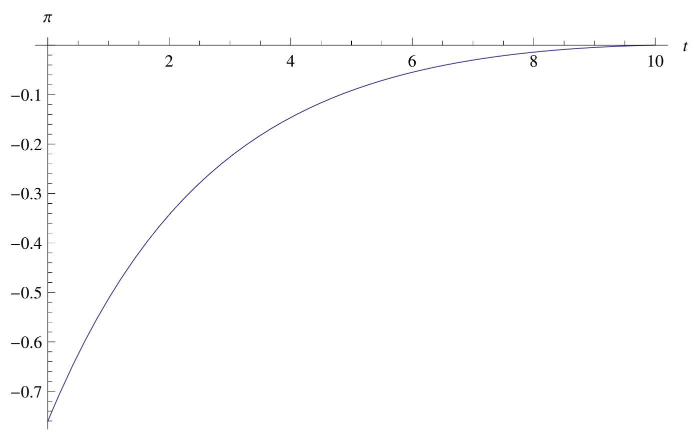

Figure B.1: The evolution of inflation after a negative natural real interest rate shock $r_L = -0.03$ when the economy returns to the steady state at $\tau = 10$.

Step 2: We next consider the case when the recovery date is unknown. We look for solutions when the inflation level and output gap are constant before the natural real interest rate bounces back. The justification is that households and central banks face the same

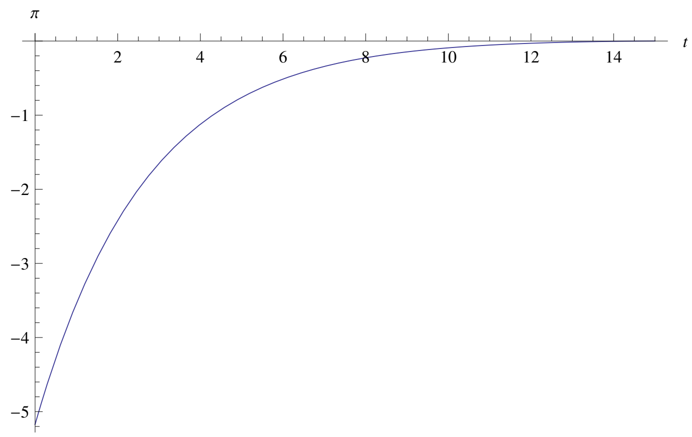

Figure B.2: The evolution of inflation after a negative natural real interest rate shock of $r_L = -0.03$ when the economy returns to the steady state at $\tau = 15$.

infinite-horizon problem and thus the same IS Equation and Phillips Curve in each period in which the natural real interest rate is at r_L and has not returned to r_H.

$$\mathbb{E}_t \boldsymbol{Q_{t+1}} = \delta \boldsymbol{Q_D^d} + (1-\delta)\mathbf{0} = \delta \boldsymbol{Q_D^d}, \tag{B.27}$$

where D denotes discretion and d represents the downturn.

Inserting Equation (B.27) into Equation (B.14) yields

$$\boldsymbol{Q_D^d} = \frac{1}{h(\delta)} \begin{pmatrix} \kappa & 1-\delta\beta \end{pmatrix}^T r_L, \tag{B.28}$$

where $h(\delta) := \sigma(1-\delta)(1-\delta\beta) - \kappa\delta$.

The signs of the inflation and output gap are determined by the denominator $h(\delta)$.

Figure B.3 implies that π_t and x_t decrease when the probability δ approaches the critical point δ^c from the left. The opposite occurs when δ approaches the threshold δ^c from the right.

We note that for $\delta = 0$, i.e. the natural real interest rate reverts to r_H one period after the shock, $\pi_D^d = \frac{\kappa}{\sigma} r_L < 0$.

For $\delta > \delta^c$, the solution displays positive inflation levels during the downturn, which is implausible[3], as the approximation method to obtain the IS Equation and Phillips Curve does not work well in such circumstances. Hence, we will assume that $\delta < \delta^c$ throughout

[3] In Carlstrom et al. (2012), when the interest rate is lowered unconditionally for extended periods, some peculiar behavior of inflation and output gap (e.g. reversals) occurs. The reason is that the IS Equation and Phillips Curve are derived around the steady state and the inflation rate and output gap deviate too far from their steady-state values if the interest rate is unconditionally pegged at a lower value for extended periods.

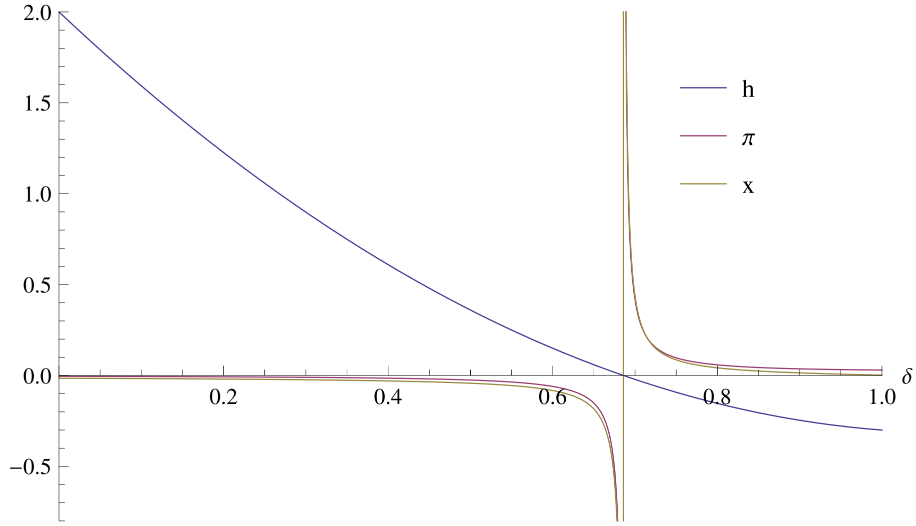

Figure B.3: The blue line represents the denominator $h(\delta)$. It crosses the x-axis at $\delta^c = 0.69$. The red and yellow lines represent the inflation and output gap levels in the downturn, respectively.

the paper[4].

A final remark on the discretionary case is in order. One can also construct equilibria with varying inflation during the downturn[5].

B.1.2 Commitment Policy

In the commitment case, the central bank commits to an entire path of policy choices, depending on whether the economy is still in the downturn or has returned to normal time.

The first-order conditions of Equation (B.2) become

$$\pi_t + \psi_{2,t} - \psi_{2,t-1} - \frac{1}{\sigma\beta}\psi_{1,t-1} = 0, \tag{B.29}$$

$$\lambda x_t + \psi_{1,t} - \frac{1}{\beta}\psi_{1,t-1} - \kappa\psi_{2,t} = 0, \tag{B.30}$$

and

$$i_t \geq 0, \psi_{1,t} \geq 0, i_t\psi_{1,t} = 0. \tag{B.31}$$

To derive the solution, we assume that the central bank's policy choice is characterized by two different points in time, i.e. τ and $\hat{\tau}$ ($\tau \leq \hat{\tau}$), where τ is the date at which the natural

[4] See Footnote 27 in Eggertsson (2006).
[5] Refer to Appendix C in Carlstrom et al. (2012).

real interest rate returns to r_H and $\hat{\tau}$ is the date[6] at which interest rate is set at positive level. Then, the central bank is assumed to set interest rates as follows:
In the periods $t \in [0, \tau - 1]$, $r_t = r_L < 0$, $i_t = 0$ and $\psi_{1,t} > 0$.
In the periods $t \in [\tau, \hat{\tau} - 1]$, $r_t = r_H > 0$, $i_t = 0$ and $\psi_{1,t} > 0$.
In the periods $t \in [\hat{\tau}, \infty)$, $r_t = r_H > 0$, $i_t > 0$ and $\psi_{1,t} = 0$.
Hence, the central bank chooses zero nominal interest rates during the downturn and for an additional time $\hat{\tau} - \tau$, and finally moves to positive nominal interest rates since period $\hat{\tau}$.
The idea of committing to zero interest rate in the periods $[\tau, \hat{\tau} - 1]$ is to create inflationary expectations during the downturn $[0, \tau - 1]$.
In the periods $t \in [0, \tau - 1]$, $i_t = 0$, $r_t = r_L$. In the downturn, we have

$$\mathbb{E}_t \boldsymbol{Q_{t+1}} = \delta \boldsymbol{Q^d_C} + (1 - \delta) \boldsymbol{Q_\tau}, \tag{B.32}$$

where $\boldsymbol{Q_\tau} = \begin{pmatrix} \pi_\tau \\ x_\tau \end{pmatrix} \neq \mathbf{0}$ is the value of inflation and output gap once the natural real interest rate has returned to r_H and $\boldsymbol{Q^d_C} = \begin{pmatrix} \pi^d_C \\ x^d_C \end{pmatrix}$ is the value of inflation and output gap during the downturn.
Inserting Equation (B.32) in Equation (B.14), we calculate

$$\boldsymbol{Q^d_C} = \boldsymbol{O_C} \boldsymbol{Q_\tau} + \boldsymbol{Q^d_D}, \tag{B.33}$$

where $\boldsymbol{O_C} := \frac{1-\delta}{h(\delta)} \begin{pmatrix} \kappa + \beta\sigma(1-\delta) & \kappa\sigma \\ 1 & \sigma(1 - \beta\delta) \end{pmatrix}$.
In the periods $t \in [\tau, \hat{\tau} - 1]$, $i_t = 0$, $r_t = r_H$ and $\psi_{1,t} > 0$. Equation (B.15) becomes

$$\pi_t = \frac{\kappa}{\sigma} r_H + (1 + \beta + \frac{\kappa}{\sigma}) \pi_{t+1} - \beta \pi_{t+2}. \tag{B.34}$$

In the periods $[\hat{\tau}, \infty)$, $i_t > 0$, $r_t = r_H$ and $\psi_{1,t} = 0$.
In period $\hat{\tau}$, the first-order conditions are

$$\pi_{\hat{\tau}} + \psi_{2,\hat{\tau}} - \psi_{2,\hat{\tau}-1} - \frac{1}{\sigma\beta} \psi_{1,\hat{\tau}-1} = 0, \tag{B.35}$$

and

$$\lambda x_{\hat{\tau}} - \frac{1}{\beta} \psi_{1,\hat{\tau}-1} - \kappa \psi_{2,\hat{\tau}} = 0. \tag{B.36}$$

In the periods $t \in [\hat{\tau} + 1, \infty)$, $\psi_{1,t} = 0$ and $\psi_{1,t-1} = 0$.

[6] Note that $\hat{\tau}$ is dependent on τ.

Equations (B.29) and (B.30) become

$$\pi_t + \psi_{2,t} - \psi_{2,t-1} = 0, \tag{B.37}$$

and

$$\lambda x_t - \kappa\psi_{2,t} = 0. \tag{B.38}$$

Combining Equations (B.37) and (B.38) yields

$$\pi_{t+1} = \frac{\lambda}{\kappa}(x_t - x_{t+1}). \tag{B.39}$$

Inserting the Phillips Curve (3.2) in Equation (B.39), yields the dynamics of inflation and output gap

$$\boldsymbol{Q_{t+2}} = (1 + \frac{\lambda + \kappa^2}{\beta\lambda})\boldsymbol{Q_{t+1}} - \frac{1}{\beta}\boldsymbol{Q_t}. \tag{B.40}$$

Its characteristic equation is

$$q^2 - \frac{\beta\lambda + \lambda + \kappa^2}{\beta\lambda}q + \frac{1}{\beta} = 0, \tag{B.41}$$

with the auxiliary variable $q \in \mathbb{R}$.

The solutions are

$$q_1 = \frac{\beta\lambda + \lambda + \kappa^2 - \sqrt{(\beta\lambda + \lambda + \kappa^2)^2 - 4\beta\lambda^2}}{2\beta\lambda}, \tag{B.42}$$

and

$$q_2 = \frac{\beta\lambda + \lambda + \kappa^2 + \sqrt{(\beta\lambda + \lambda + \kappa^2)^2 - 4\beta\lambda^2}}{2\beta\lambda}. \tag{B.43}$$

We note that $q_1 < 1 < q_2$.

Therefore, we can write the dynamics for all $t \geq \hat{\tau} + 1$ as

$$\boldsymbol{Q_t} = \boldsymbol{O'} \begin{pmatrix} q_1^{t-\hat{\tau}} \\ q_2^{t-\hat{\tau}} \end{pmatrix}, \tag{B.44}$$

where $\boldsymbol{O'} := \begin{pmatrix} o_{\pi 1} & o_{\pi 2} \\ o_{x1} & o_{x2} \end{pmatrix}$.

Since $q_2 > 1$, the terminal conditions $\pi_{ss} = \lim_{t\to\infty} \pi_t = 0$ and $x_{ss} = \lim_{t\to\infty} x_t = 0$ imply that the coefficients $o_{\pi 2}$ and o_{x2} have to be zero. Otherwise, the dynamics would display explosive behavior.

Hence, the dynamics for all $t \geq \hat{\tau} + 1$ are

$$\pi_t = o_{\pi 1} q_1^{t-\hat{\tau}}, \tag{B.45}$$

and

$$x_t = o_{x_1} q_1^{t-\hat{\tau}}. \tag{B.46}$$

We provide a simple example in Subsection B.1.3.

B.1.3 One-period Shock

In this subsection, we consider the commitment scenario for the special case when the natural real interest rate bounces back to the steady-state value r_H one period after the shock, i.e. $\tau = 1$, and the recovery date is publicly known. This corresponds to the stochastic recovery mode with $\delta = 0$. This case can be fully solved analytically. As it is particularly instructive, we explore in full detail.

We start with the assumption that the optimal interest rate is non-zero as soon as the natural real interest rate is non-negative, i.e. $\tau = \hat{\tau} = 1$.

In period 0, $r_0 = r_L$, $i_0^d = 0$ and $\psi_{1,0} > 0$.

Equation (B.33) implies

$$x_1 = \frac{\sigma\beta + \kappa}{\sigma\beta} x_0 - \frac{1}{\sigma\beta}\pi_0 - \frac{1}{\sigma} r_L, \tag{B.47}$$

and

$$\pi_1 = \frac{1}{\beta}\pi_0 - \frac{\kappa}{\beta} x_0. \tag{B.48}$$

Equations (B.29) and (B.30) imply

$$\pi_0 = -\psi_{2,0}, \tag{B.49}$$

and

$$x_0 = \frac{\kappa}{\lambda}\psi_{2,0} - \frac{1}{\lambda}\psi_{1,0}. \tag{B.50}$$

In period 1, $r_1 = r_H$, $i_1 > 0$ and $\psi_{1,1} = 0$.

The Phillips Curve (3.2) and IS Equation (3.1) are

$$\begin{aligned} \pi_2 &= \frac{1}{\beta}\pi_1 - \frac{\kappa}{\beta} x_1 \\ &= \frac{\kappa + \sigma}{\sigma\beta^2}\pi_0 - \frac{\kappa\sigma + \kappa^2 + \kappa\beta\sigma}{\sigma\beta^2} x_0 + \frac{\kappa}{\sigma\beta} r_L, \end{aligned} \tag{B.51}$$

and

$$\begin{aligned} x_2 &= x_1 + \frac{1}{\sigma}(i_1 - \pi_2 - r_H) \\ &= \frac{\sigma^2\beta^2 + \kappa\sigma + \kappa^2 + 2\kappa\sigma\beta}{\sigma^2\beta^2}x_0 - \frac{\sigma\beta + \kappa + \sigma}{\sigma^2\beta^2}\pi_0 - \frac{1}{\sigma}r_H - \frac{\sigma\beta + \kappa}{\sigma^2\beta}r_L + \frac{1}{\sigma}i_1. \end{aligned} \tag{B.52}$$

The First-order Conditions (B.35) and (B.36) deliver

$$\pi_1 = \frac{1}{\beta\sigma}\psi_{1,0} + \psi_{2,0} - \psi_{2,1}, \tag{B.53}$$

and

$$x_1 = \frac{1}{\lambda\beta}\psi_{1,0} + \frac{\kappa}{\lambda}\psi_{2,1}. \tag{B.54}$$

Eliminating $\psi_{1,0}$ and $\psi_{2,0}$ by Equations (B.49) and (B.50), we obtain

$$\pi_1 = -\frac{\beta\sigma + \kappa}{\beta\sigma}\pi_0 - \frac{\lambda}{\beta\sigma}x_0 - \psi_{2,1}, \tag{B.55}$$

and

$$x_1 = -\frac{1}{\beta}x_0 - \frac{\kappa}{\lambda\beta}\pi_0 + \frac{\kappa}{\lambda}\psi_{2,1}. \tag{B.56}$$

Combining Equations (B.48) and (B.55) yields

$$\psi_{2,1} = -\frac{\beta\sigma + \kappa + \sigma}{\beta\sigma}\pi_0 - \frac{\lambda - \kappa\sigma}{\beta\sigma}x_0. \tag{B.57}$$

With Equations (B.56) and (B.57), we obtain

$$x_1 = -\frac{\kappa\lambda + \sigma\lambda - \sigma\kappa^2}{\beta\sigma\lambda}x_0 - \frac{2\kappa\sigma + \sigma\beta\kappa + \kappa^2}{\beta\sigma\lambda}\pi_0. \tag{B.58}$$

Combining Equations (B.47) and (B.58) yields a constraint on x_0 and π_0,

$$\frac{\sigma\beta + \kappa}{\sigma\beta}x_0 - \frac{1}{\sigma\beta}\pi_0 - \frac{1}{\sigma}r_L = -\frac{\kappa\lambda + \sigma\lambda - \sigma\kappa^2}{\beta\sigma\lambda}x_0 - \frac{2\kappa\sigma + \sigma\beta\kappa + \kappa^2}{\beta\sigma\lambda}\pi_0. \tag{B.59}$$

In period 2, $r_2 = r_H$, $i_2 > 0$ and $\psi_{1,2} = 0$.
The First-order Conditions (B.37) and (B.38) imply

$$\pi_2 = \psi_{2,1} - \psi_{2,2}, \tag{B.60}$$

and

$$x_2 = \frac{\kappa}{\lambda}\psi_{2,2}. \tag{B.61}$$

According to Equations (B.51) and (B.57), we rearrange Equation (B.60)

$$\begin{aligned}\psi_{2,2} &= \psi_{2,1} - \pi_2 \\ &= \frac{\kappa\sigma + \kappa^2 + 2\kappa\sigma\beta - \lambda\beta}{\sigma\beta^2} x_0 - \frac{\beta(\sigma\beta + \kappa + \sigma) + \kappa + \sigma}{\sigma\beta^2} \pi_0 - \frac{\kappa}{\sigma\beta} r_L.\end{aligned} \tag{B.62}$$

Therefore, Equation (B.61) becomes

$$x_2 = \frac{\kappa}{\lambda}[\frac{\kappa\sigma + \kappa^2 + 2\kappa\sigma\beta - \lambda\beta}{\sigma\beta^2} x_0 - \frac{\beta(\sigma\beta + \kappa + \sigma) + \kappa + \sigma}{\sigma\beta^2} \pi_0 - \frac{\kappa}{\sigma\beta} r_L]. \tag{B.63}$$

Equations (B.52) and (B.63) determine the value of i_1. If i_1 turns out to be negative, the zero lower bound is still binding, i.e. $\psi_{1,1} = 0$. In such circumstances, our initial assumption that $\hat{\tau} = \tau$ is incorrect and need to be adjusted. Hence, we next assume $\hat{\tau} = \tau + 1$, which means that the interest rate is kept at zero for one more period. We then repeat the procedure above until we reach the case when the zero lower bound is not violated[7]. In our calibration, $i_1 > 0$. Therefore, the assumption $\hat{\tau} = \tau$ applies here.
The Phillips Curve (3.2) in period 2 delivers π_3

$$\pi_3 = \frac{1}{\beta}\pi_2 - \frac{\kappa}{\beta} x_2. \tag{B.64}$$

In period 3, $r_3 = r_H$, $i_3 > 0$ and $\psi_{1,3} = 0$.
The First-order Conditions (B.37) and (B.38) imply

$$\pi_3 = \psi_{2,2} - \psi_{2,3}, \tag{B.65}$$

and

$$x_3 = \frac{\kappa}{\lambda}\psi_{2,3}. \tag{B.66}$$

The IS Equation (3.1) in period 2 is

$$x_3 = x_2 + \frac{1}{\sigma}(i_2 - \pi_3 - r_H). \tag{B.67}$$

Equations (B.65), (B.66) and (B.67) determine the values of x_3 and i_2.
In the same manner, we obtain the values of x_4 and i_3. We can write the initial conditions of Equation (B.44) as

$$x_{\hat{\tau}+2} = x_3 = o_{x1} q_1 + o_{x2} q_2, \tag{B.68}$$

and

$$x_{\hat{\tau}+3} = x_4 = o_{x1} q_1^2 + o_{x2} q_2^2. \tag{B.69}$$

[7] This is the same procedure as the one in the Technical Appendix of Eggertsson (2006).

The terminal conditions $\lim_{t\to\infty}\pi_t = 0$ and $\lim_{t\to\infty} x_t = 0$ require $o_{x2} = 0$ to rule out the explosive root. Then, we have

$$q_1 = \frac{x_{\hat{\tau}+3}}{x_{\hat{\tau}+2}} = \frac{x_4}{x_3}. \tag{B.70}$$

This is the second constraint on the initial inflation and output gap.

Together with the first Constraint (B.59), Equation (B.70) delivers the values of x_0 and π_0 and hence the path of the inflation and output gap.

The coefficient o_{x1} is obtained by

$$o_{x1} = \frac{x_3}{q_1}. \tag{B.71}$$

The evolution of the output gap ($t \geq 3$) is

$$x_t = o_{x1} q_1^{t-2}. \tag{B.72}$$

The evolution of inflation ($t \geq 3$) is given by

$$\pi_t = \frac{\lambda}{\kappa}(x_{t-1} - x_t) = \frac{\lambda o_{x1}}{\kappa}(1 - q_1) q_1^{t-3}. \tag{B.73}$$

The evolution of inflation ($t \geq 3$) can also be written as

$$\pi_t = o_{\pi 1} q_1^{t-2}, \tag{B.74}$$

where $o_{\pi 1} = \frac{\lambda o_{x1}}{\kappa q_1}(1 - q_1)$.

The evolution of interest rate stemming from the IS Equation (3.1) is

$$\begin{aligned} i_t &= \sigma(x_{t+1} - x_t) + \pi_{t+1} + r_t \\ &= \sigma(x_{t+1} - x_t) + \pi_{t+1} + r_H \\ &= -\frac{\kappa\sigma}{\lambda}\pi_{t+1} + \pi_{t+1} + r_H \\ &= r_H + o_{x1}(\sigma - \frac{\lambda}{\kappa})(q_1 - 1) q_1^{t-2}. \end{aligned} \tag{B.75}$$

Figures B.4, B.5 and B.6 display the evolution of the inflation rate, output gap and interest rate, respectively. Figures B.7 and B.8 display the evolution of the inflation rate, output gap and interest rate at a refined scale.

The intertemporal social losses for the commitment denoted by $l^C_{[0,\infty]}$ are

$$l^C_{[0,\infty]} = -\sum_{t=0}^{\infty} \beta^t \frac{1}{2}(\pi_t^2 + \lambda x_t^2) = -3.1 \times 10^{-6}. \tag{B.76}$$

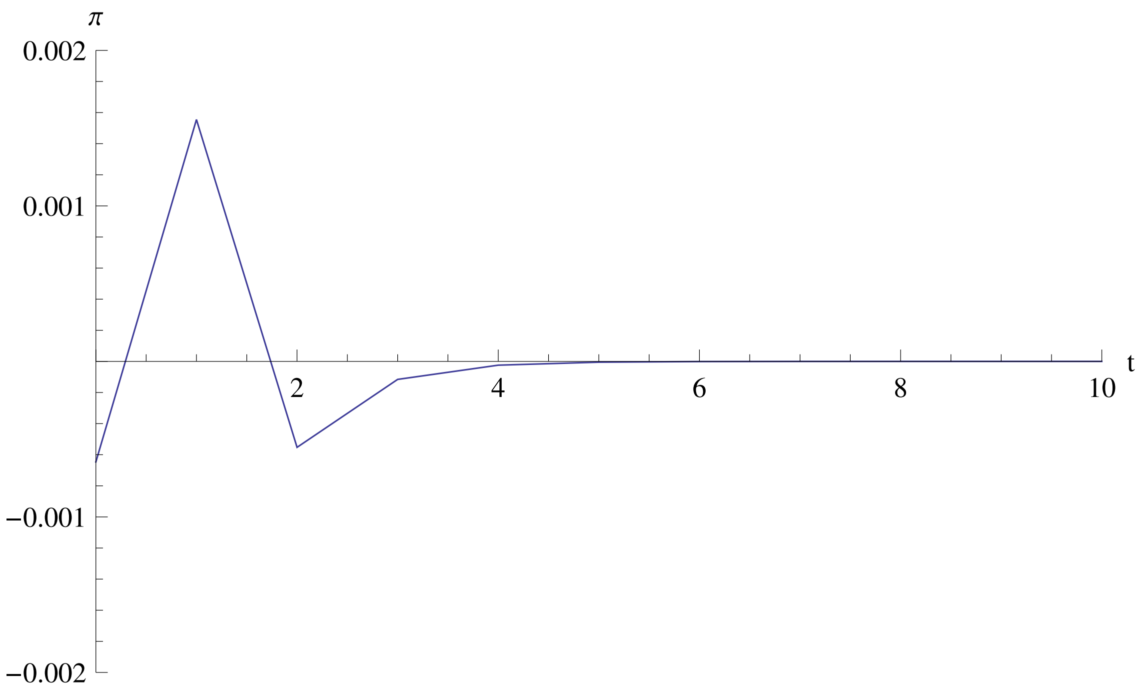

Figure B.4: The evolution of the inflation from period 0 to period 10.

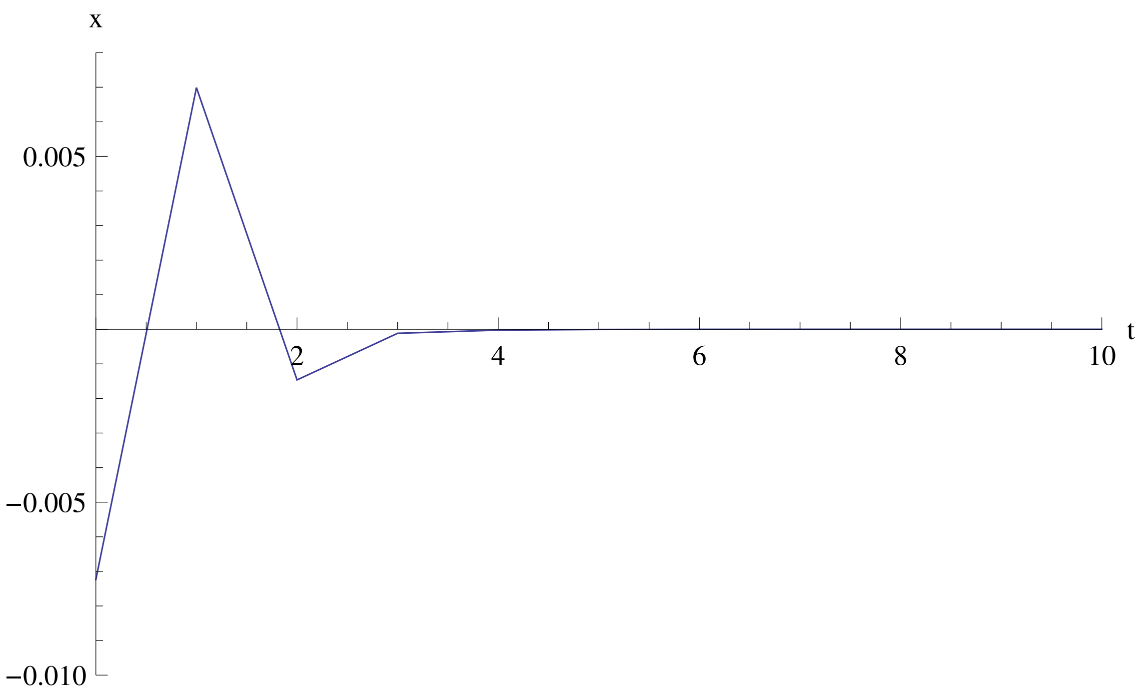

Figure B.5: The evolution of the output gap from period 0 to period 10.

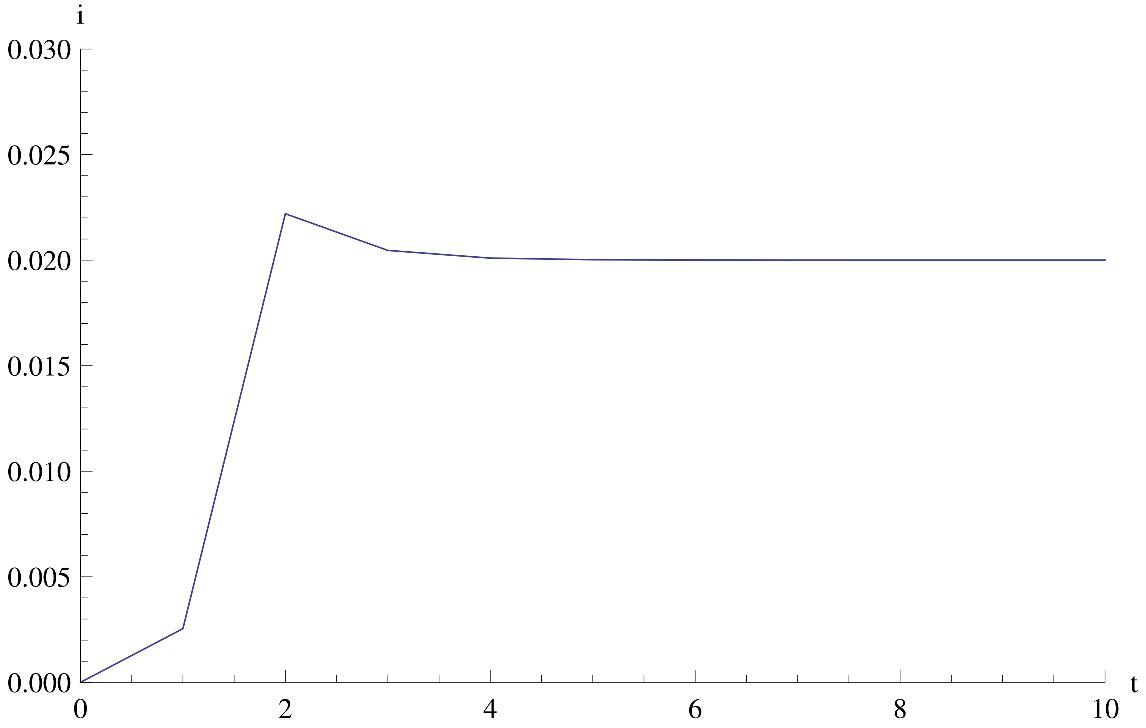

Figure B.6: The evolution of the interest rate from period 0 to period 10.

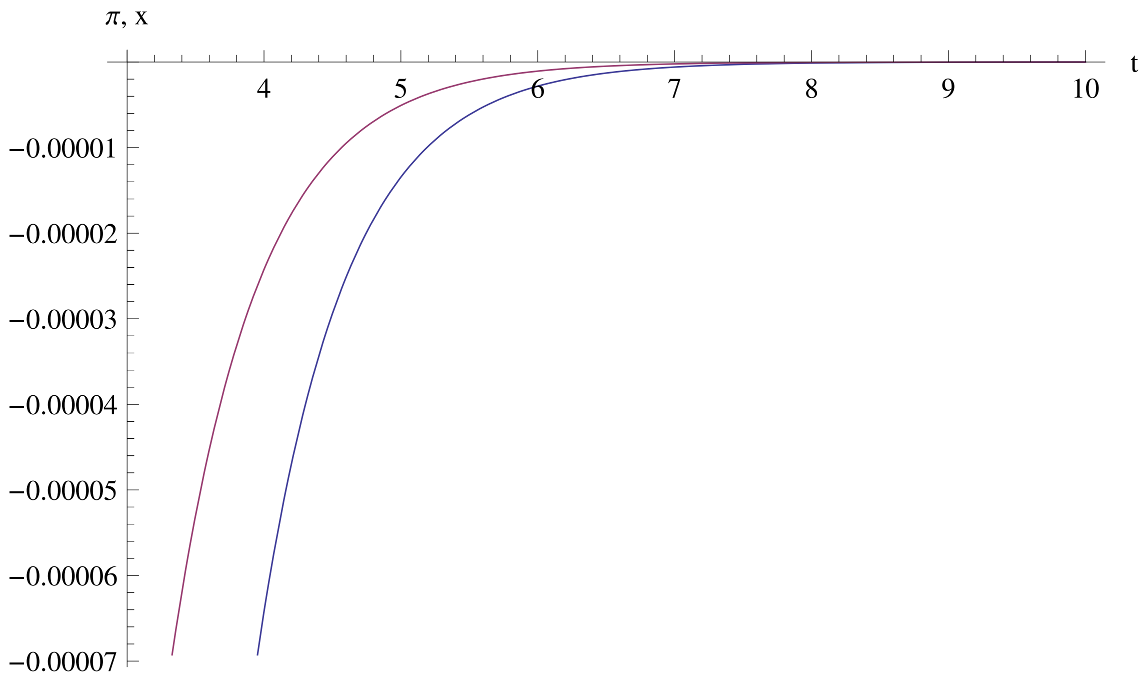

Figure B.7: The evolution of inflation and of output gap from period 3 to period 10.

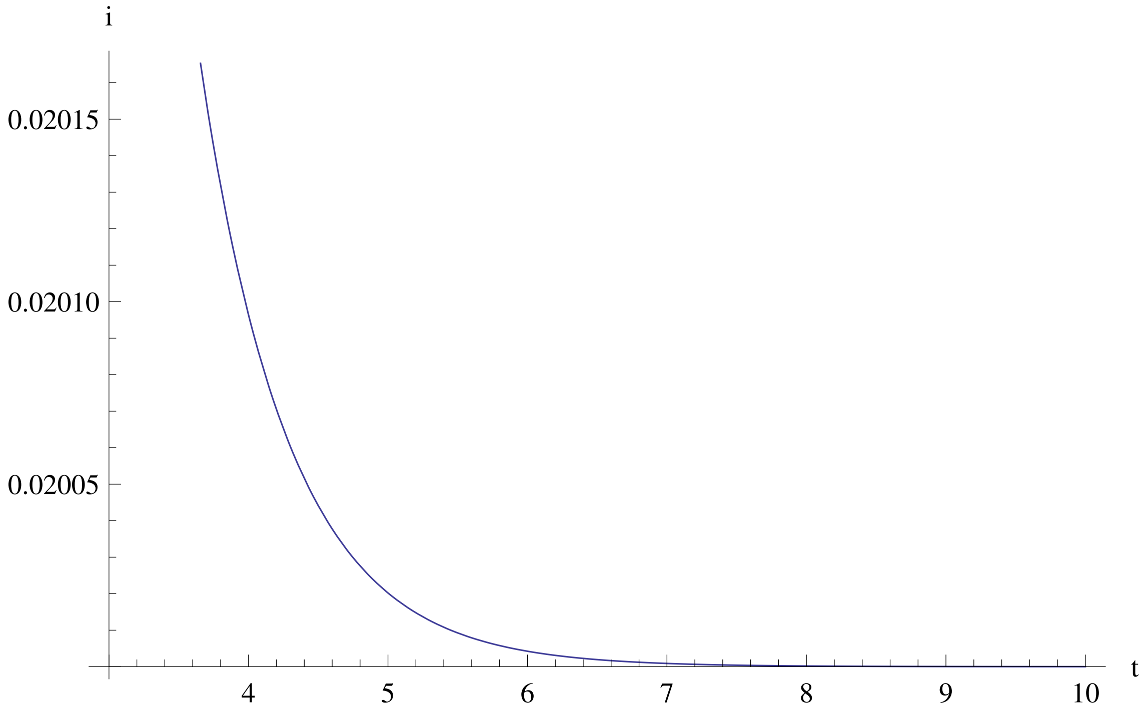

Figure B.8: The evolution of the interest rate from period 3 to period 10.

We compare it to the discretionary case:

$$x_0 = \frac{1}{\sigma} r_L.$$

$$\pi_0 = \frac{\kappa}{\sigma} r_L.$$

$$x_t = \pi_t = 0, \forall t \geq 1.$$

The intertemporal social losses for the discretionary denoted by $l^D_{[0,\infty]}$ are

$$l^D_{[0,\infty]} = -\sum_{t=0}^{\infty} \beta^t \frac{1}{2}(\pi_t^2 + \lambda x_t^2) = -\frac{1}{2}(\pi_0^2 + \lambda x_0^2) = -\frac{(\kappa^2 + \lambda)(r_L)^2}{2\sigma^2} = -1.35 \times 10^{-5}.$$

Thus, the intetemporal social loss with commitment policy is more than four times smaller than the one with discretionary policy.

B.2 Proof of Lemma 4.2

Inserting Equation (4.11) into Equation (5.11) yields

$$\boldsymbol{Q_1^d} = \delta \boldsymbol{O^{-1}} \boldsymbol{Q_0^d} + \frac{r_L - i_1^d}{\sigma} \begin{pmatrix} \kappa \\ 1 \end{pmatrix}. \tag{B.77}$$

In period 0, the expected inflation and output gap in period 1 are

$$\mathbb{E}_0 \boldsymbol{Q_1} = (1-\delta)\boldsymbol{Q_1^n} + \delta \boldsymbol{Q_1^d}. \tag{B.78}$$

Combining Equations (4.5) and (B.78) yields

$$\boldsymbol{Q_0^d} = (1-\delta)\boldsymbol{O^{-1}} \boldsymbol{Q_1^n} + \delta \boldsymbol{O^{-1}} \boldsymbol{Q_1^d} + \frac{r_L - i_0^d}{\sigma} \begin{pmatrix} \kappa \\ 1 \end{pmatrix}. \tag{B.79}$$

Inserting Equation (B.77) in Equation (B.79) delivers

$$\boldsymbol{Q_0^d} = (1-\delta)\boldsymbol{O^{-1}} \boldsymbol{Q_1^n} + \delta^2 \boldsymbol{O^{-2}} \boldsymbol{Q_0^d} + \delta \frac{r_L - i_1^d}{\sigma} \boldsymbol{O^{-1}} \begin{pmatrix} \kappa \\ 1 \end{pmatrix} + \frac{r_L - i_0^d}{\sigma} \begin{pmatrix} \kappa \\ 1 \end{pmatrix}. \tag{B.80}$$

Rearranging Equation (B.80) yields

$$\begin{aligned} \boldsymbol{Q_0^d} =& (1-\delta)(\boldsymbol{I} - \delta^2 \boldsymbol{O^{-2}})^{-1} \boldsymbol{O^{-1}} \boldsymbol{Q_1^n} \\ &+ \delta \frac{r_L - i_1^d}{\sigma} (\boldsymbol{I} - \delta^2 \boldsymbol{O^{-2}})^{-1} \boldsymbol{O^{-1}} \begin{pmatrix} \kappa \\ 1 \end{pmatrix} + \frac{r_L - i_0^d}{\sigma} (\boldsymbol{I} - \delta^2 \boldsymbol{O^{-2}})^{-1} \begin{pmatrix} \kappa \\ 1 \end{pmatrix}, \end{aligned} \tag{B.81}$$

where $\boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Inserting Equation (4.9) in Equation (B.81) yields, after some algebraic manipulations

$$\boldsymbol{Q_0^d} = \frac{1}{f(\delta)} \left(\boldsymbol{O_0} \begin{pmatrix} \frac{b\sigma^2}{\lambda + \kappa^2 + b\sigma^2}(1-\delta) r_H \\ r_L \end{pmatrix} - \boldsymbol{\tilde{O}_0} \begin{pmatrix} i_0^d \\ i_1^d \end{pmatrix} \right), \tag{B.82}$$

where

$$\boldsymbol{O_0} := \begin{pmatrix} \kappa o_1 & \kappa g(\delta) \\ o_2 & (1-\beta\delta)g(\delta) \end{pmatrix}, \tag{B.83}$$

$$\tilde{\boldsymbol{O}}_{\mathbf{0}} := \begin{pmatrix} \kappa o_3 & \delta\kappa o_1 \\ o_4 & \delta o_2 \end{pmatrix}, \tag{B.84}$$

where the functions $f(\delta)$ and $g(\delta)$, as well as the elements o_1, o_2, o_3 and o_4, are given by

$$f(\delta) := g(\delta)h(\delta), \tag{B.85}$$

$$g(\delta) := \sigma\beta\delta^2 + o_1\delta + \sigma, \tag{B.86}$$

$$h(\delta) := \sigma\beta\delta^2 - o_1\delta + \sigma, \tag{B.87}$$

$$o_1 := \sigma\beta + \kappa + \sigma, \tag{B.88}$$

$$o_2 := \kappa + \sigma - \sigma\beta^2\delta^2, \tag{B.89}$$

$$o_3 := \sigma(1 + \beta\delta^2), \tag{B.90}$$

$$o_4 := \sigma - \beta(\beta\sigma + \kappa)\delta^2. \tag{B.91}$$

Note that

$$o_3 + o_1\delta = g(\delta), \tag{B.92}$$

$$o_3 - o_1\delta = h(\delta), \tag{B.93}$$

$$o_4 + o_2\delta = (1-\beta\delta)g(\delta), \tag{B.94}$$

$$o_4 - o_2\delta = (1+\beta\delta)h(\delta). \tag{B.95}$$

Inserting Equation (B.82) in Equation (B.77), we obtain

$$\boldsymbol{Q_1^d} = \frac{1}{f(\delta)} \left(\boldsymbol{O_1} \begin{pmatrix} \frac{b\sigma}{\lambda+\kappa^2+b\sigma^2}\delta(1-\delta)r_H \\ r_L \end{pmatrix} - \tilde{\boldsymbol{O}}_{\mathbf{1}} \begin{pmatrix} i_0^d \\ i_1^d \end{pmatrix} \right), \tag{B.96}$$

where

$$\boldsymbol{O_1} := \begin{pmatrix} \kappa(o_1^2 - \beta\sigma o_3) & \kappa g(\delta) \\ \kappa o_1 + \sigma o_2 & (1-\beta\delta)g(\delta) \end{pmatrix}, \tag{B.97}$$

$$\tilde{\boldsymbol{O}}_{\mathbf{1}} := \begin{pmatrix} \delta\kappa o_1 & \kappa o_3 \\ \delta o_2 & o_4 \end{pmatrix}. \tag{B.98}$$

Formally, each element in $\tilde{\boldsymbol{O}}_{\mathbf{0}}$ and $\tilde{\boldsymbol{O}}_{\mathbf{1}}$ is positive, irrespective of the parameter values. The only exception is o_4. However, for reasonable parameter constraints ($\beta \leq 1$, $\kappa \leq 1$

and $\sigma \geq 1$), o_4 is also positive. This suggests that positive levels of i_0^d and i_1^d lower the values of $\boldsymbol{Q_0^d}$ and $\boldsymbol{Q_1^d}$.

B.3 Proof of Proposition 4.1

We establish Proposition 4.1 through several steps.

Step 1: We consider the central banker's problem in periods 0 and 1 and assume momentarily that the zero bound is not binding. If the derived solutions produce non-positive interest rates, we can conclude that the zero bound is indeed binding. In period 0, the central banker chooses i_0^d to maximize

$$\max_{i_0^d}\{-0.5[(\pi_0^d)^2 + \lambda(x_0^d)^2 + b(i_0^d)^2], \} \tag{B.99}$$

subject to Equation (4.12), a given value of b and in anticipation of interest rates i_1^d, i_1^n that will be chosen in future periods, depending on the realization of the natural real interest rate.

If we neglect the zero bound, the first-order condition with respect to i_0^d yields

$$\pi_0^d \frac{\partial \pi_0^d}{\partial i_0^d} + \lambda x_0^d \frac{\partial x_0^d}{\partial i_0^d} + b i_0^d = 0. \tag{B.100}$$

In period 1, when the economy is still in the downturn, the central banker chooses i_1^d to maximize its objective function

$$\max_{i_1^d}\{-0.5[(\pi_1^d)^2 + \lambda(x_1^d)^2 + b(i_1^d)^2], \} \tag{B.101}$$

subject to Equation (4.13), a given value of b and in anticipation of interest rates i_2^d, i_2^n that will be chosen in future periods, depending on the realization of the natural real interest rate.

The first-order condition with respect to i_1^d yields

$$\pi_1^d \frac{\partial \pi_1^d}{\partial i_1^d} + \lambda x_1^d \frac{\partial x_1^d}{\partial i_1^d} + b i_1^d = 0. \tag{B.102}$$

Equations (4.5) and (5.11) imply

$$\frac{\partial \boldsymbol{Q_0^d}}{\partial i_0^d} = \frac{\partial \boldsymbol{Q_1^d}}{\partial i_1^d} = -\frac{1}{\sigma}\begin{pmatrix} \kappa \\ 1 \end{pmatrix}. \tag{B.103}$$

Combining Equations (B.100) and (B.103) yields

$$\kappa\pi_0^d + \lambda x_0^d = b\sigma i_0^d. \tag{B.104}$$

Combining Equations (B.102) and (B.103) yields

$$\kappa\pi_1^d + \lambda x_1^d = b\sigma i_1^d. \tag{B.105}$$

Inserting Equations (4.12) and (4.13) into Equations (B.104) and (B.105) yields

$$(o_7 + \sigma f(\delta)b)i_0^d + \delta o_5 i_1^d = \sigma o_5 \frac{b\sigma}{\lambda + \kappa^2 + b\sigma^2}(1-\delta)r_H + o_6 g(\delta) r_L \tag{B.106}$$

and

$$(o_7 + \sigma f(\delta)b)i_1^d + \delta o_5 i_0^d = \delta o_8 \frac{b\sigma}{\lambda + \kappa^2 + b\sigma^2}(1-\delta)r_H + o_6 g(\delta) r_L, \tag{B.107}$$

where

$$o_5 := \kappa^2 o_1 + \lambda o_2, \tag{B.108}$$
$$o_6 := \kappa^2 + \lambda(1 - \beta\delta), \tag{B.109}$$
$$o_7 := \kappa^2 o_3 + \lambda o_4, \tag{B.110}$$
$$o_8 := \kappa^2(o_1^2 - \beta\sigma o_3) + \lambda(\kappa o_1 + \sigma o_2). \tag{B.111}$$

We note that all terms o_5, o_6, o_7 and o_8 are positive.

Equation (B.106) is the first-order condition of Equation (B.99). If

$$\sigma o_5 \frac{b\sigma}{\lambda + \kappa^2 + b\sigma^2}(1-\delta)r_H + o_6 g(\delta) r_L \leq 0, \tag{B.112}$$

in period 0 the central banker sets $i_0^d = 0$.

Equation (B.107) is the first-order condition of Equation (B.101). If

$$\delta o_8 \frac{b\sigma}{\lambda + \kappa^2 + b\sigma^2}(1-\delta)r_H + o_6 g(\delta) r_L \leq 0, \tag{B.113}$$

in period 1, the central banker sets $i_1^d = 0$.

All terms in Equations (B.106) and (B.107)—except r_L—are positive. Moreover, $\frac{b\sigma}{\lambda+\kappa^2+b\sigma^2}$ increases monotonically with b and $\lim_{b\to\infty} \frac{b\sigma}{\lambda+\kappa^2+b\sigma^2} = \frac{1}{\sigma}$. Hence, both Equations (B.106) and (B.107) define critical threshold values for r_L, such that Conditions (B.112) and (B.113) are satisfied for values of r_L smaller than these thresholds.

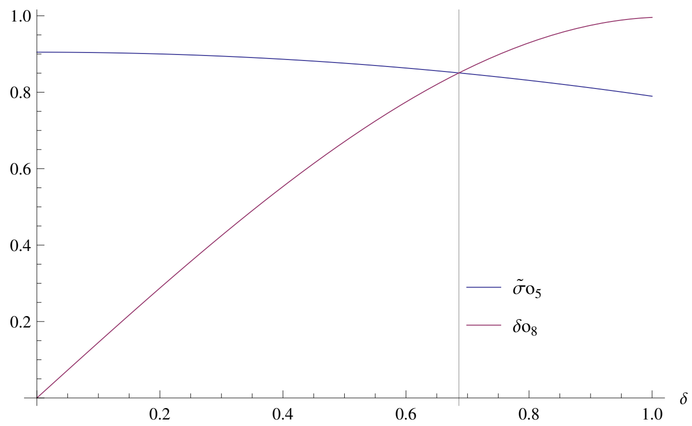

Figure B.9: The evolution of δo_8 and σo_5 with respect to δ. The vertical black line represents the critical value δ^c.

Note that $\sigma o_5 - \delta o_8 = [\kappa^2(\sigma\beta\delta + o_1) + \lambda(\kappa + \sigma + \sigma\beta\delta)]h(\delta) > 0, \forall\delta \in (0, \delta^c)$[8]. Thus, Condition (B.112) is tighter than Condition (B.113), since $\delta o_8 < \sigma o_5$, as is shown in Figure B.9.

Thus, we explore Condition (B.112). It implies that the lower bound is indeed binding irrespective of the value of b if

$$r_L \leq \hat{r}_L, \tag{B.114}$$

where $\hat{r}_L = -\frac{(1-\delta)o_5}{o_6 g(\delta)} r_H$.

This establishes point i in the proposition.

Step 2: In this step we provide two explicit formulae for the interest rate, again neglecting momentarily the zero lower bound on interest rates.

Solving Equations (B.106) and (B.107) yields

$$i_0^d = \frac{[\sigma^2 o_5 f(\delta)b + \sigma o_5 o_7 - \delta^2 o_5 o_8]\frac{b\sigma(1-\delta)}{\lambda+\kappa^2+b\sigma^2} r_H + [\sigma f(\delta)b + o_7 - \delta o_5]o_6 g(\delta) r_L}{(\sigma f(\delta)b + o_7)^2 - \delta^2 o_5^2} \tag{B.115}$$

and

$$i_1^d = \frac{[\sigma\delta o_8 f(\delta)b + \delta o_8 o_7 - \sigma\delta o_5^2]\frac{b\sigma(1-\delta)}{\lambda+\kappa^2+b\sigma^2} r_H + [\sigma f(\delta)b + o_7 - \delta o_5]o_6 g(\delta) r_L}{(\sigma f(\delta)b + o_7)^2 - \delta^2 o_5^2}. \tag{B.116}$$

The denominator is positive for any $\delta \in (0, \delta^c)$, due to the fact that $\sigma f(\delta)b \geq 0$, $o_7 > 0$, and $o_7^2 - \delta^2 o_5^2 = [\lambda^2(1 - \beta^2\delta^2) + \kappa^4 + 2\lambda\kappa^2]f(\delta) > 0$.

Note that $[\sigma^2 o_5 f(\delta)b + \sigma o_5 o_7 - \delta^2 o_5 o_8] - [\sigma\delta o_8 f(\delta)b + \delta o_8 o_7 - \sigma\delta o_5^2] = [\sigma f(\delta)b + o_7 +$

[8] See Appendix B.1.

$\delta o_5](\sigma o_5 - \delta o_8) > 0\ \forall \delta \in (0, \delta^c)$.
Hence, $\sigma^2 o_5 f(\delta)b + \sigma o_5 o_7 - \delta^2 o_5 o_8 > \sigma\delta o_8 f(\delta)b + \delta o_8 o_7 - \sigma\delta o_5^2$ and thus $i_1^d \leq i_0^d$.

Step 3: In this step, we argue that with the optimal value of b, the central banker sets $i_1^d = 0$.
Assume that with the optimal b, the central banker sets $i_0^d > 0$ and $i_1^d > 0$.
From Equation (4.12), we observe that a lower value of i_0^d increases π_0^d and x_0^d. As $i_0^d > 0$ was the assumed optimal choice, π_0^d and x_0^d can not both be negative. The same applies to π_1^d and x_1^d.
In the downturn, the optimal value of i^d is the one that maximizes $-0.5[(\pi^d)^2 + \lambda(x^d)^2 + b(i^d)^2]$. Namely, the central banker chooses i^d optimally to balance the social loss $-0.5[(\pi^d)^2 + \lambda(x^d)^2]$ and his wage loss $-0.5b(i^d)^2$. Due to the incentive term $-0.5b(i^d)^2$, the central banker chooses i^d that is lower than the one that minimizes the social loss $-0.5[(\pi^d)^2 + \lambda(x^d)^2]$. This results in relatively higher π^d and x^d.
From Equation (4.9), we observe that inflation and output in period 1 are positive if the economy returns to normal times in this period. Moreover, lowering b by Δb reduces those values, say by $\Delta\pi_1^n$ and Δx_1^n, and thus social losses in period 1. The same argument applies to cases when the economy recovers in any other period with an odd number, as the FGC $(b, 2)$ is still in effect in those periods.
From Equations (4.12) and (4.13), we observe that a decline of b causes all economic outcomes during the downturn, π_0^d, π_1^d, x_0^d and x_1^d to be lower through two channels. On the one hand, the inflationary expectations are lowered as the boom that is created in normal times is smaller when the FGC $(b, 2)$ is still in effect. On the other hand, the constraint on the central banker's setting high i^d is eased due to the reduced weight the central banker puts on deviation loss $-0.5b(i^d)^2$. The same applies to all economic outcomes in any period if the economy is in the downturn.
Therefore, lowering the value of b reduces the social losses in normal times and downturns. Hence, the original value of b cannot be optimal.
Thus, with an optimal value of b chosen by the government, the nominal interest rates i_0^d and i_1^d cannot both be set at non-zero values. Intuitively, setting $i_0^d > 0$ and $i_1^d > 0$ implies that the value of b is too high. Since $i_1^d \leq i_0^d$ as is proved in Step 2, it can only be $i_1^d = 0$ or $i_0^d = i_1^d = 0$.

Step 4: We finally show[9] that $i_0^d = 0$ when the contract parameter b is chosen optimally and $r_L < 0$.

[9] Note that the proof by contradiction of Step 3 does not apply to proving $i_0^d = 0$. If we assume with the optimal value of b, $i_0^d > 0$ and $i_1^d = 0$. Then, lowering the value of b reduces the social losses in period 0 and in normal time in period 1. But it dampens the economy in the downturn in period 1.

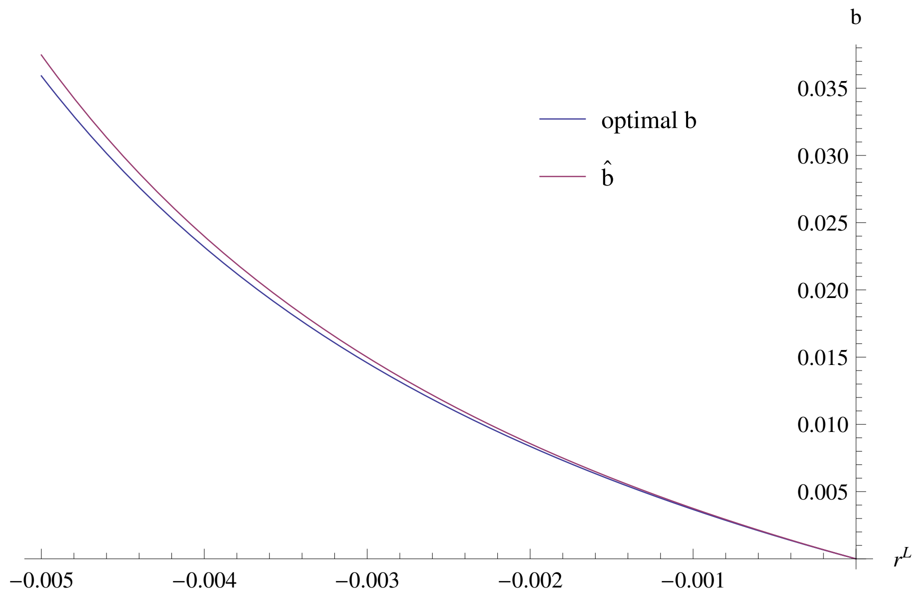

Figure B.10: The evolution of the optimal b and $\hat{b}$ in function of r_L.

Due to $i_1^d = 0$, we rewrite Equation (B.106) as

$$\begin{aligned} i_0^d &= \frac{b\sigma^2 o_5(1-\delta)r_H + (\lambda+\kappa^2+b\sigma^2)o_6 g(\delta)r_L}{(o_7+\sigma f(\delta)b)(\lambda+\kappa^2+b\sigma^2)} \\ &= \frac{[\sigma^2 o_5(1-\delta)r_H + \sigma^2 o_6 g(\delta)r_L]b + (\lambda+\kappa^2)o_6 g(\delta)r_L}{(o_7+\sigma f(\delta)b)(\lambda+\kappa^2+b\sigma^2)}. \end{aligned} \tag{B.117}$$

As the denominator of Equation (B.117) is positive and $(\lambda+\kappa^2)o_6 g(\delta)r_L < 0$, $i_0^d = 0$ irrespective of the value of b if

$$o_5\sigma^2(1-\delta)r_H + \sigma^2 g(\delta)o_6 r_L \leq 0, \tag{B.118}$$

which is exactly the same as Condition (B.114).
Since $i_0^d = i_1^d = 0$ for any value of b if $r_L \leq \hat{r}_L$, we can concentrate on the case $\hat{r}_L < r_L < 0$. Equation (B.117) implies $i_0^d = 0$ if and only if the value of b satisfies

$$b \leq \hat{b}, \tag{B.119}$$

where $\hat{b} = -\frac{\lambda+\kappa^2}{\sigma^2}\frac{o_6 g(\delta)}{o_5(1-\delta)r_H + o_6 g(\delta)r_L} r_L$.
We next prove that the optimal b satisfies this condition.
In Proposition 4.2, we assume $b \leq \hat{b}$, i.e. $i_0^d = i_1^d = 0$ and then derive the optimal value of b.
Figure B.10 displays the evolution of the optimal b and $\hat{b}$ with $\delta = 0.5$. It shows that the optimal b satisfies the assumption. The formal proof is provided at the end of the proof of

Proposition 4.2.

B.4 Proof of Proposition 4.2

We calculate the first-order condition of Equation (4.14) with respect to b,

$$\pi_0^d \frac{\partial \pi_0^d}{\partial b} + \lambda x_0^d \frac{\partial x_0^d}{\partial b} + \beta(1-\delta)(\pi_1^n \frac{\partial \pi_1^n}{\partial b} + \lambda x_1^n \frac{\partial x_1^n}{\partial b}) + \beta\delta(\pi_1^d \frac{\partial \pi_1^d}{\partial b} + \lambda x_1^d \frac{\partial x_1^d}{\partial b}) = 0. \quad \text{(B.120)}$$

As is shown in Proposition 4.1, the central banker sets zero interest rate in downturns for optimal b. Equations (4.12) and (4.13) become

$$\boldsymbol{Q_0^d} = \frac{1}{f(\delta)} \boldsymbol{O_0} \begin{pmatrix} \frac{b\sigma^2}{\lambda+\kappa^2+b\sigma^2}(1-\delta)r_H \\ r_L \end{pmatrix}, \quad \text{(B.121)}$$

and

$$\boldsymbol{Q_1^d} = \frac{1}{f(\delta)} \boldsymbol{O_1} \begin{pmatrix} \frac{b\sigma}{\lambda+\kappa^2+b\sigma^2}\delta(1-\delta)r_H \\ r_L \end{pmatrix}. \quad \text{(B.122)}$$

The first-order conditions of Equations (B.121) and (B.122) with respect to b are

$$\frac{\partial \boldsymbol{Q_0^d}}{\partial b} = \frac{1}{f(\delta)} \begin{pmatrix} \kappa o_1 \\ o_2 \end{pmatrix} \frac{\sigma^2(\lambda+\kappa^2)}{(\lambda+\kappa^2+b\sigma^2)^2}(1-\delta)r_H, \quad \text{(B.123)}$$

and

$$\frac{\partial \boldsymbol{Q_1^d}}{\partial b} = \frac{1}{f(\delta)} \begin{pmatrix} \kappa(o_1^2 - \sigma\beta o_3) \\ \kappa o_1 + \sigma o_2 \end{pmatrix} \frac{\sigma(\lambda+\kappa^2)}{(\lambda+\kappa^2+b\sigma^2)^2}\delta(1-\delta)r_H. \quad \text{(B.124)}$$

The first-order condition of Equation (4.9) with respect to b yields

$$\frac{\partial \boldsymbol{Q_1^n}}{\partial b} = \begin{pmatrix} \kappa \\ 1 \end{pmatrix} \frac{\sigma(\lambda+\kappa^2)}{(\lambda+\kappa^2+b\sigma^2)^2} r_H. \quad \text{(B.125)}$$

Inserting Equations (B.121), (B.122), (4.9), (B.123), (B.124) and (B.125) into Equation (B.120), we achieve

$$b = -\frac{(\lambda+\kappa^2)\Delta_1 g(\delta) r_L}{\sigma(\Delta_2 r_H + \sigma\Delta_1 g(\delta) r_L)},$$

where $\Delta_1 = \sigma\kappa^2 o_1 + \sigma\lambda o_2(1-\beta\delta) + \beta\delta^2\kappa^2(o_1^2 - \beta\sigma o_3) + \beta\lambda\delta^2(1-\beta\delta)(\kappa o_1 + \sigma o_2)$ and $\Delta_2 = \sigma^2(1-\delta)\kappa^2 o_1^2 + \sigma^2\lambda(1-\delta)o_2^2 + \beta\delta^3(1-\delta)\kappa^2(o_1^2 - \beta\sigma o_3)^2 + \beta\lambda\delta^3(1-\delta)(\kappa o_1 + \sigma o_2)^2 + \beta(\kappa^2+\lambda)f(\delta)^2$.

Further simplification yields

$$b = \frac{\lambda+\kappa^2}{\sigma^2} \frac{1}{\frac{r_L^c}{r_L} - 1}.$$

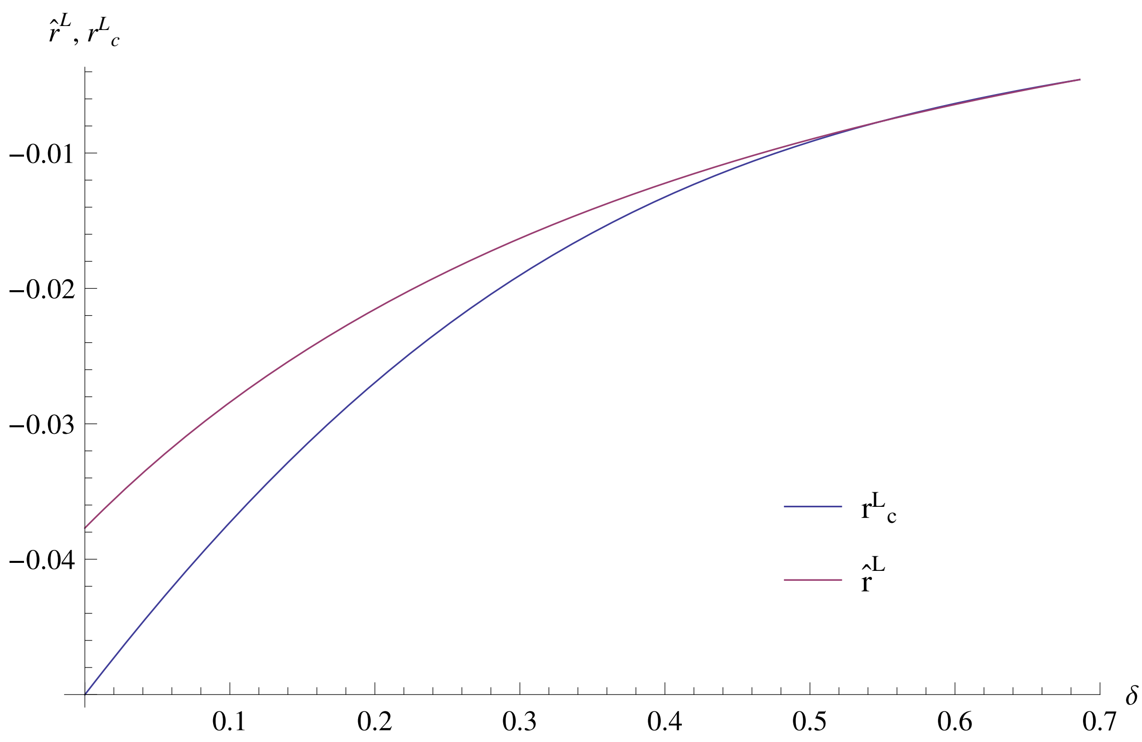

Figure B.11: The values of r_L^c and $\hat{r}_L$ in function of δ. r_L^c and $\hat{r}_L$ cross at $\hat{r}_L = 0.55$ and $r_L^c = 0.69$.

$r_L^c = -\frac{\Delta_2}{\sigma\Delta_1 g(\delta)} r_H.$

Thus, we can write Condition (B.119) as

$$-\frac{(\lambda+\kappa^2)\Delta_1 g(\delta) r_L}{\sigma(\Delta_2 r_H + \sigma\Delta_1 g(\delta) r_L)} \leq -\frac{\lambda+\kappa^2}{\sigma^2}\frac{o_6 g(\delta)}{o_5(1-\delta)r_H + o_6 g(\delta) r_L} r_L, \tag{B.126}$$

which stands if and only if

$$\frac{1}{r_L^c - r_L} \geq \frac{1}{\hat{r}_L - r_L}. \tag{B.127}$$

In Step 4 of the proof of Proposition 4.1, we concentrate on the case $\hat{r}_L < r_L < 0$. As is displayed in Figure B.11, $r_L^c \leq \hat{r}_L$ when $\delta \leq \hat{\delta}$ and $r_L^c > \hat{r}_L$ when $\delta > \hat{\delta}$, where $\hat{\delta} = 0.55$ defines the threshold value. Thus, Condition (B.127) is fulfilled when $\delta \leq \hat{\delta}$ or when $\delta > \hat{\delta}$, $\hat{r}_L < r_L \leq r_L^c$. When $\delta > \hat{\delta}$ and $\hat{r}_L < r_L^c < r_L$, Condition (B.127) is violated. Throughout the paper, we assume that $\delta \leq \hat{\delta} < \delta^c$.

B.5 Proof of Lemma 4.3

Case 1: We consider the central banker's problem in period 1 and assume momentarily that the zero bound is not binding. Thus, we rewrite Equation (4.22),

$$\boldsymbol{Q_1^{d1}} = \frac{\delta}{f(\delta)}\boldsymbol{O^{-1}O_0}\begin{pmatrix}(1-\delta)r_H\\ r^{L1}\end{pmatrix} + \frac{r^{L1}-i_1^{d1}}{\sigma}\begin{pmatrix}\kappa\\ 1\end{pmatrix}. \tag{B.128}$$

The central banker chooses i_1^{d1} to maximize

$$\max_{i_1^{d1}}\{-0.5[(\pi_1^{d1})^2 + \lambda(x_1^{d1})^2 + b(i_1^{d1})^2]\}, \tag{B.129}$$

subject to Equation (B.128), a given value of b and anticipating that FGC ($b=\infty, 2$) will be signed in the next period provided that the economy is still in the downturn.
If we neglect the zero bound, the first-order condition with respect to i_1^{d1} yields

$$\pi_1^{d1}\frac{\partial\pi_1^{d1}}{\partial i_1^{d1}} + \lambda x_1^{d1}\frac{\partial x_1^{d1}}{\partial i_1^{d1}} + b i_1^{d1} = 0. \tag{B.130}$$

Equation (B.128) implies

$$\frac{\partial\boldsymbol{Q_1^{d1}}}{\partial i_1^{d1}} = -\frac{1}{\sigma}\begin{pmatrix}\kappa\\ 1\end{pmatrix}. \tag{B.131}$$

Combining Equations (B.130) and (B.131) yields

$$\kappa\pi_1^{d1} + \lambda x_1^{d1} = b\sigma i_1^{d1}. \tag{B.132}$$

Inserting Equation (B.128) into Equation (B.132) yields

$$i_1^{d1} = \frac{[\kappa^2(o_1^2-\sigma\beta o_3)+\lambda(\kappa o_1+\sigma o_2)]\delta(1-\delta)r_H + [\kappa^2(o_1-\sigma\beta\delta)+\lambda(\kappa+\sigma-\sigma\beta\delta)]g(\delta)\delta r^{L1}}{(\lambda+\kappa^2+b\sigma^2)f(\delta)} + \frac{\lambda+\kappa^2}{\lambda+\kappa^2+b\sigma^2}r^{L1}. \tag{B.133}$$

We note that if $\delta = 0$, Equation (B.133) becomes

$$i_1^{d1} = \frac{\lambda+\kappa^2}{\lambda+\kappa^2+b\sigma^2}r^{L1}. \tag{B.134}$$

Since $\lambda+\kappa^2 > 0$, $\lambda+\kappa^2+b\sigma^2 > 0$, $f(\delta) > 0$ and $[\kappa^2(o_1-\sigma\beta\delta)+\lambda(\kappa+\sigma-\sigma\beta\delta)]g(\delta)\delta > 0$, Equation (B.133) implies that i_1^{d1} increases with r^{L1}. Thus, i_1^{d1} achieves the maximum

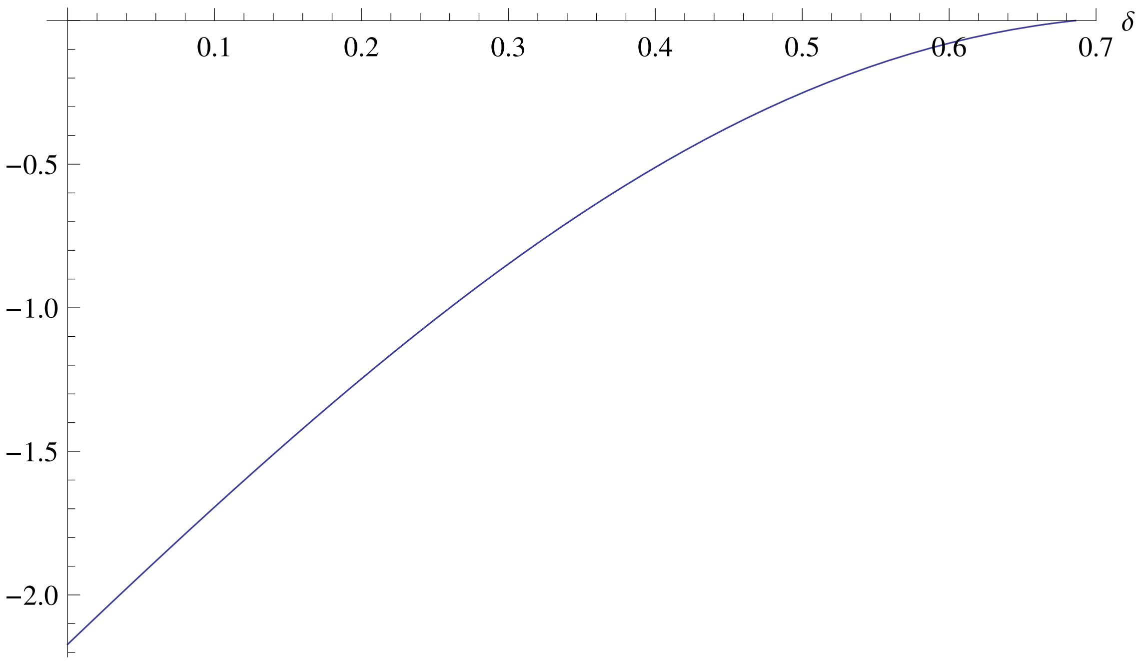

Figure B.12: The numerator of Equation (B.135) in function of δ.

when $r^{L1} = r_L^c$. Inserting $r^{L1} = r_L^c$ into Equation (B.133) yields

$$\begin{aligned} i_1^{d1}(r_L^c) =& \frac{\sigma\Delta_1[\kappa^2(o_1^2 - \sigma\beta o_3) + \lambda(\kappa o_1 + \sigma o_2)]\delta(1-\delta)}{\sigma\Delta_1(\lambda + \kappa^2 + b\sigma^2)f(\delta)} r_H \\ &- \frac{[\kappa^2(o_1 - \sigma\beta\delta) + \lambda(\kappa + \sigma - \sigma\beta\delta)]\delta\Delta_2 + (\lambda + \kappa^2)h(\delta)\Delta_2}{\sigma\Delta_1(\lambda + \kappa^2 + b\sigma^2)f(\delta)} r_H. \end{aligned} \tag{B.135}$$

Since $\sigma\Delta_1[\kappa^2(o_1^2 - \sigma\beta o_3) + \lambda(\kappa o_1 + \sigma o_2)]\delta(1-\delta) - [\kappa^2(o_1 - \sigma\beta\delta) + \lambda(\kappa + \sigma - \sigma\beta\delta)]\delta\Delta_2 + (\lambda + \kappa^2)h(\delta)\Delta_2 < 0$ for any $\delta \in (0, \delta^c)$ as is displayed in Figure B.12, the optimal nominal interest rate is negative. Since the derived solution produces negative interest rates, we can conclude that the central bank would set zero nominal interest rate irrespective of the value of b.

Case 2: In a similar way, we obtain

$$\begin{aligned} i_1^{d2} =& \frac{[\kappa^2(o_1 - \sigma\beta\delta) + \lambda(\kappa + \sigma - \sigma\beta\delta)]\Delta_2 - [\kappa^2(o_1^2 - \sigma\beta o_3) + \lambda(\kappa o_1 + \sigma o_2)](1-\delta)\sigma\Delta_1}{(\lambda + \kappa^2 + b\sigma^2)f(\delta)\Delta_2} \delta g(\delta) r^{L2} \\ &+ \frac{\lambda + \kappa^2}{\lambda + \kappa^2 + b\sigma^2} r^{L2}. \end{aligned} \tag{B.136}$$

Since $[\kappa^2(o_1 - \sigma\beta\delta) + \lambda(\kappa + \sigma - \sigma\beta\delta)]\delta g(\delta)\Delta_2 - [\kappa^2(o_1^2 - \sigma\beta o_3) + \lambda(\kappa o_1 + \sigma o_2)]\delta(1 - \delta)\sigma\Delta_1 g(\delta) + (\lambda + \kappa^2)f(\delta)\Delta_2 > 0$ for any $\delta \in (0, \delta^c)$ as is displayed in Figure B.13, the optimal nominal interest rate is negative for any $r^{L2} < 0$, i.e. the zero lower bound is

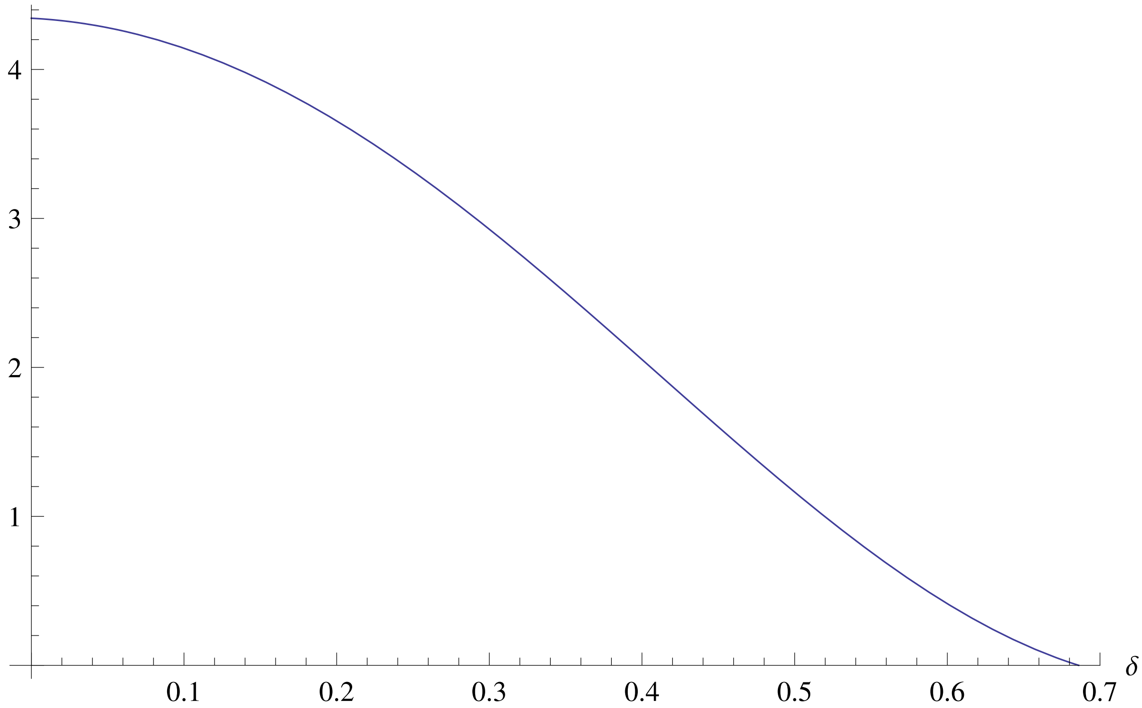

Figure B.13: The numerator of Equation (B.136) in function of δ.

binding.

B.6 Proof of Lemma 4.4

In period 0, if the natural real interest rate is r^{L1}, the dynamics of inflation and output gap is

$$\boldsymbol{Q_0^{d1}} = \mathbb{E}_0 \boldsymbol{O^{-1}} \boldsymbol{Q_1} + \frac{r^{L1} - i_0^{d1}}{\sigma} \begin{pmatrix} \kappa \\ 1 \end{pmatrix}. \tag{B.137}$$

Combining Equations (4.9) and (4.23) yields

$$\mathbb{E}_0 \boldsymbol{Q_1} = (1-\delta) \frac{\sigma b r_H}{\lambda + \kappa^2 + b\sigma^2} \begin{pmatrix} \kappa \\ 1 \end{pmatrix} + \delta \begin{pmatrix} \pi_1^{d1} \\ x_1^{d1} \end{pmatrix}. \tag{B.138}$$

where π_1^{d1} and x_1^{d1} are determined by Equation (4.22).
Inserting Equation (B.138) into Equation (B.137), we obtain

$$\boldsymbol{Q_0^{d1}} = (1-\delta) \frac{b r_H}{\lambda + \kappa^2 + b\sigma^2} \begin{pmatrix} \kappa o_1 \\ \kappa + \sigma \end{pmatrix} + \frac{\delta}{\sigma} \begin{pmatrix} (\sigma\beta + \kappa)\pi_1^{d1} + \kappa\sigma x_1^{d1} \\ \pi_1^{d1} + \sigma x_1^{d1} \end{pmatrix} + \frac{r^{L1} - i_0^{d1}}{\sigma} \begin{pmatrix} \kappa \\ 1 \end{pmatrix}. \tag{B.139}$$

We consider the central banker's social loss minimization problem in period 0 and assume momentarily that the zero bound is not binding. Hence, the central banker chooses i_0^{d1} to maximize

$$\max_{i_0^{d1}} \{-0.5[(\pi_0^{d1})^2 + \lambda(x_0^{d1})^2 + b(i_0^{d1})^2]\}, \tag{B.140}$$

subject to Equation (B.139) and a given value of b.
If we neglect the zero bound, the first-order condition with respect to i_0^{d1} yields

$$\pi_0^{d1}\frac{\partial \pi_0^{d1}}{\partial i_0^{d1}} + \lambda x_0^{d1}\frac{\partial x_0^{d1}}{\partial i_0^{d1}} + b i_0^{d1} = 0. \tag{B.141}$$

Equation (B.139) implies

$$\frac{\partial \boldsymbol{Q_0^{d1}}}{\partial i_0^{d1}} = -\frac{1}{\sigma}\begin{pmatrix} \kappa \\ 1 \end{pmatrix}. \tag{B.142}$$

Combining Equations (B.141) and (B.142) yields

$$\kappa\pi_0^{d1} + \lambda x_0^{d1} = b\sigma i_0^{d1}. \tag{B.143}$$

Inserting Equation (B.139) into Equation (B.143) yields

$$\begin{aligned} i_0^{d1} =& [\kappa^2 o1 + \lambda(\kappa+\sigma)]\frac{\sigma(1-\delta)b}{(\lambda+\kappa^2+b\sigma^2)^2} r_H + \delta\frac{[\lambda + \kappa(\kappa+\sigma\beta)]\pi_1^{d1} + \sigma(\lambda+\kappa^2)x_1^{d1}}{\lambda+\kappa^2+b\sigma^2} \\ &+ \frac{\lambda+\kappa^2}{\lambda+\kappa^2+b\sigma^2} r^{L1}. \end{aligned} \tag{B.144}$$

Figure B.14 demonstrates that the optimal nominal interest rate for different values of b is negative and the larger the value of b is, the closer i_0^{d1} is to zero. Since the derived solution produces negative interest rates, we can conclude that the central bank would set zero nominal interest rate regardless of the value of b for any $r^{L1} \leq r_L^c$. In a similar way, we obtain

$$\begin{aligned} i_0^{d2} =& [\kappa^2 o1 + \lambda(\kappa+\sigma)]\frac{\sigma(1-\delta)b}{(\lambda+\kappa^2+b\sigma^2)^2} r_H + \delta\frac{[\lambda + \kappa(\kappa+\tilde{\sigma\beta})]\pi_1^{d2} + \sigma(\lambda+\kappa^2)x_1^{d2}}{\lambda+\kappa^2+b\sigma^2} \\ &+ \frac{\lambda+\kappa^2}{\lambda+\kappa^2+b\sigma^2} r^{L2}, \end{aligned} \tag{B.145}$$

and

$$i_0^{d3} = [\kappa^2 o1 + \lambda(\kappa+\sigma)]\frac{\sigma b[(1-\delta)r_H + \delta r^{L3}]}{(\lambda+\kappa^2+b\sigma^2)^2} r_H + \frac{\lambda+\kappa^2}{\lambda+\kappa^2+b\sigma^2} r^{L3}. \tag{B.146}$$

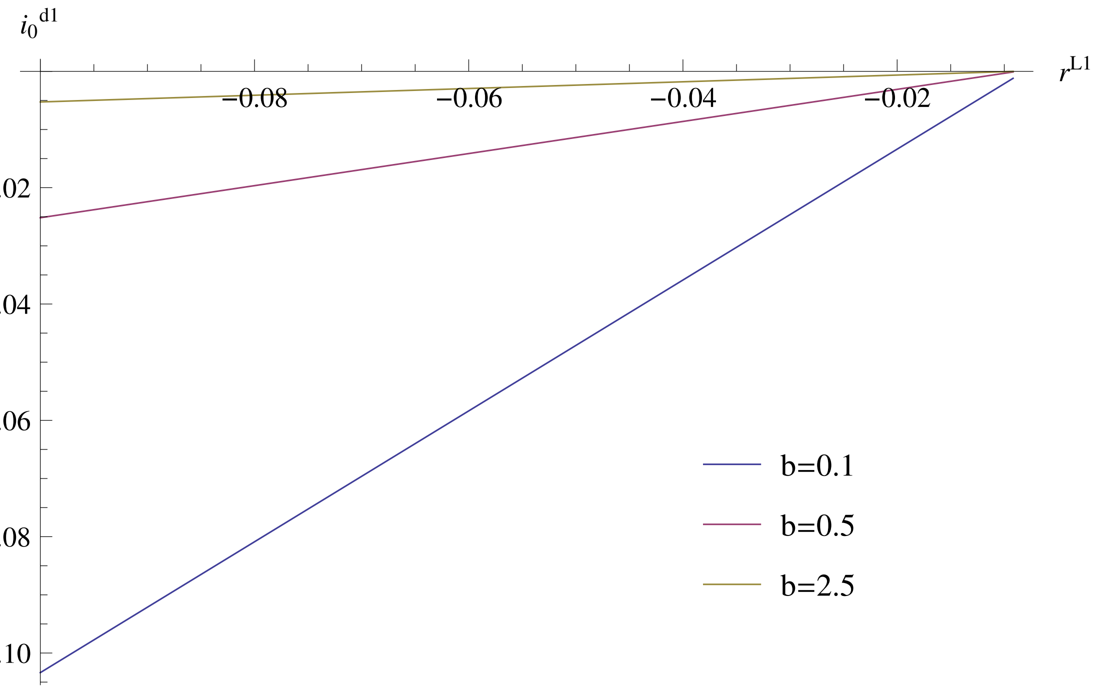

Figure B.14: The optimal nominal interest rate in function of r^{L1} for different values of b.

C Proofs for Chapter 5

C.1 Proof of Lemma 5.2

Combining Equations (5.11), (5.12) and (5.13) yields

$$\boldsymbol{Q^d} = (1-\delta)\boldsymbol{O^{-1}}\boldsymbol{Q^n_\tau} + \delta\boldsymbol{O^{-1}}\boldsymbol{Q^d} + \frac{r_L}{\sigma}\begin{pmatrix}\kappa\\1\end{pmatrix}. \tag{C.1}$$

Rearranging Equation (C.1) yields

$$\boldsymbol{Q^d} = (1-\delta)(\boldsymbol{I}-\delta\boldsymbol{O^{-1}})^{-1}\boldsymbol{O^{-1}}\boldsymbol{Q^n_\tau} + \frac{r_L}{\sigma}(\boldsymbol{I}-\delta\boldsymbol{O^{-1}})^{-1}\begin{pmatrix}\kappa\\1\end{pmatrix}, \tag{C.2}$$

where $\boldsymbol{I} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$.

Inserting Equation (5.10) in Equation (C.2) yields, after some algebraic manipulations:

$$\boldsymbol{Q^d} = \frac{1}{h(\delta)}\boldsymbol{\tilde{O}}\begin{pmatrix}\frac{b\sigma(1-\delta)}{\lambda+\kappa^2+b\sigma^2}(r_H - i^f_\tau)\\ r_L\end{pmatrix}, \tag{C.3}$$

where

$$\boldsymbol{\tilde{O}} = \begin{pmatrix}\phi_2 & \kappa\\ \phi_3 & 1-\beta\delta\end{pmatrix} \tag{C.4}$$

$$h(\delta) := \sigma\beta\delta^2 - \phi_1\delta + \sigma \tag{C.5}$$

$$\phi_1 := \sigma + \kappa + \sigma\beta \tag{C.6}$$

$$\phi_2 := \kappa(\sigma\beta + \phi_3) \tag{C.7}$$

$$\phi_3 := \kappa + \sigma - \sigma\beta\delta. \tag{C.8}$$

C.2 Proof of Proposition 5.1

In downturns, for given b, the central banker chooses i^f_τ to minimize the expected intertemporal losses

$$\min_{i^f_\tau}\{\frac{0.5}{1-\beta\delta}[(\pi^d)^2+\lambda(x^d)^2]+\frac{0.5\beta(1-\delta)}{1-\beta\delta}[(\pi^n_\tau)^2+\lambda(x^n_\tau)^2+b(i^n_\tau-i^f_\tau)^2]\} \quad \text{(C.9)}$$

subject to Equations (5.9), (5.10) and (5.14).

Since the central banker would not forecast $i^f_\tau < -\frac{\lambda+\kappa^2}{b\sigma^2}r_H$, we only consider $i^f_\tau \geq -\frac{\lambda+\kappa^2}{b\sigma^2}r_H$.

Combining Equations (5.9) and (C.9) yields

$$\min_{i^f_\tau}\{(\pi^d_t)^2+\lambda(x^d_t)^2+\beta(1-\delta)[(\pi^n_\tau)^2+\lambda(x^n_\tau)^2+b\frac{(\lambda+\kappa^2)^2(r_H-i^f_\tau)^2}{(\lambda+\kappa^2+b\sigma^2)^2}]\}. \quad \text{(C.10)}$$

The first-order condition is

$$\pi^d_t\frac{\partial\pi^d_t}{\partial i^f_\tau}+\lambda x^d_t\frac{\partial x^d_t}{\partial i^f_\tau}+\beta(1-\delta)[\pi^n_\tau\frac{\partial\pi^n_\tau}{\partial i^f_\tau}+\lambda x^n_\tau\frac{\partial x^n_\tau}{\partial i^f_\tau}-b\frac{(\lambda+\kappa^2)^2(r_H-i^f_\tau)}{(\lambda+\kappa^2+b\sigma^2)^2}]=0. \quad \text{(C.11)}$$

Equation (5.10) implies

$$\frac{\partial \boldsymbol{Q^n_\tau}}{\partial i^f_\tau}=-\frac{b\sigma}{\lambda+\kappa^2+b\sigma^2}\begin{pmatrix}\kappa\\1\end{pmatrix}. \quad \text{(C.12)}$$

Combining Equations (5.10), (5.12) and (5.13) yields

$$\boldsymbol{Q^d_t}=(1-\delta)\frac{b\sigma(r_H-i^f_\tau)}{\lambda+\kappa^2+b\sigma^2}\boldsymbol{O^{-1}}\begin{pmatrix}\kappa\\1\end{pmatrix}+\delta\boldsymbol{O^{-1}Q^d_{t+1}}+\frac{r_L}{\sigma}\begin{pmatrix}\kappa\\1\end{pmatrix}. \quad \text{(C.13)}$$

Due to the fact that all periods in downturns are identical, we have $\boldsymbol{Q^d_{t+1}}=\boldsymbol{Q^d_t}=\boldsymbol{Q^d}$. Thus,

$$\frac{\partial\boldsymbol{Q^d}}{\partial i^f_\tau}=-\frac{1}{h(\delta)}\frac{b\sigma(1-\delta)}{\lambda+\kappa^2+b\sigma^2}\begin{pmatrix}\phi_2\\\phi_3\end{pmatrix}. \quad \text{(C.14)}$$

Combining Equations (C.11), (C.12) and (C.14) delivers

$$\phi_2\pi^d+\lambda\phi_3x^d+\beta h(\delta)\kappa\pi^n_\tau+\beta h(\delta)\lambda x^n_\tau+\beta h(\delta)\frac{(\lambda+\kappa^2)^2(r_H-i^f_\tau)}{\sigma(\lambda+\kappa^2+b\sigma^2)}=0. \quad \text{(C.15)}$$

Inserting Equations (4.9), (4.12) into Equation (C.15) yields

$$i_\tau^f = r_H + \frac{(\lambda + \kappa^2 + b\sigma^2)\phi_4}{\phi_5 + \phi_6 b} r_L, \tag{C.16}$$

where $\phi_4 = \sigma[\kappa\phi_2 + \lambda\phi_3(1 - \beta\delta)]$, $\phi_5 = h(\delta)^2\beta(\lambda + \kappa^2)^2$ and $\phi_6 = \sigma^2 h(\delta)^2\beta(\lambda + \kappa^2) + \sigma^2(1 - \delta)(\phi_2^2 + \lambda\phi_3^2)$.

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