Nonlinear Holography and Projection Moiré

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Preface

The present work was written at the Institute of Lightweight Structures and Ropeways (ILS) of the Swiss Federal Institute of Technology in Zurich (ETHZ) and was presented at the “Abteilung IIIA für Maschinenbau und Verfahrenstechnik” in relation to the lecture “Holographische Messmethoden”. The main scope was to review the basic concepts of nonlinear kinematics of deformation and to present new topics in the fields of holographic interferometry and of projection moiré together with a powerful tool, the intrinsic tensor calculus for engineers, which connect the aspects of optics and mechanics and also allows more flexibility by computing general geometries of optical setups. The author wishes to thank Prof. Dr. H.-R. Meyer-Piening, head of the ILS, for giving him a holographic laboratory and for the full support in this research.

Zurich, November 1998 Ph. Tatasciore

Contents

Summary

The main scope of the present work is to introduce new topics in holographic interferometry and projection moiré while reviewing the basic concepts of nonlinear kinematics of deformation.

In chapter 1, we recall some basic concepts of intrinsic tensor calculus and explicitly write some essential relations, which will be needed in all the following chapters. The scope of this chapter is to present the elements of vector and tensor analysis, which enable to understand the equations and computations performed in chapters 2, 3 and 4, and to introduce the reader to the intrinsic notation, which is used all over this work.

In chapter 2, we present some aspects related to the kinematics of deformation of 3-dimensional objects and of 2-dimensional curved surfaces in space. Because only the surface of an opaque object (not the interior of the body) can be recorded by means of holographic interferometry, emphasis is put on the nonlinear theory of the deformation of a curved surface in space. In the case of large deformation measurement of opaque bodies, or in the classical cases of deformation analysis of plates and shells, the dilatation, rotation and displacement terms often have different orders of magnitude. Practically, one may encounter small strains together with moderate rotations and large displacements. Thus, in order to properly analyze such deformations, we have developed the deformation up to higher order terms, by paying special attention on separating displacement, rotation and strain components. These relations are written in prevision of the applications in both chapters on holographic interferometry and projection moiré.

In chapter 3, we explain how to apply holographic interferometry to large deformation measurements and how to deal with the related problem of vanishing fringe patterns. In common industrial environment, large deformation measurements of opaque bodies by means of holographic interferometry are often related to the problem of decreasing fringe spacing and contrast, causing the loss of the interference fringe pattern, which contains the whole information on the corresponding deformation. Therefore, the only way to determine the surface strain, rotation and displacement components of a structure element under load relatively to the unloaded state is first to recover the interference fringes – at least locally – and then to use the correct adequate relations to process the recovered fringe pattern properly. We explicitly and quantitatively present the general equation system for a systematic fringe recovery procedure in the general case of a large unknown object deformation. The relations for the quantitative evaluation of the recovered fringes, i.e. the optical path difference and the exact fringe vector of the modified interference pattern, are also explicitly presented. All needed relations are given in form of general vector and tensor equations. Then, equations for fringe recovery are written in cartesian components and used within a quantitative practical experiment to demonstrate the reliability of the theory. These relations are general and may also be used in other application fields (with their related problems) of holographic interferometry, when the loss of fringe spacing and contrast should be compensated.

In chapter 4, we explain how to determine the shape of an object surface in the 3-dimensional space by applying the projection moiré technique. For plane surfaces, this process is obviously very trivial. For curved surfaces however, we need an optical method which allows accurate quantitative acquisition of the whole surface shape. The measurement result may then be used in holographic interferometry, where the unit normal to the surface plays an important role in the decomposition of the deformation. The main scope is to show that an optical shape measurement can be achieved by only applying the projection moiré technique, which means performing the calibration of the optical setup and the object

shape measurement without using other external techniques. Emphasis is put on relative moiré, which is used in most experiments, and on difference moiré, which is generally used to calibrate optical systems. The general tensor equations of projection moiré for all geometrical cases are explicitly written for the first time. The concept of sensitivity vector, which comes from holographic interferometry, is also introduced. Using a computer-based image processing system, a quantitative experimental verification of the theoretical equations is performed and shows the calibration process of an optical setup.

1. Basic concepts of tensor calculus for engineers

1.1 Vectors and tensors in space

We shall present here the basic concepts of vector and tensor calculus for engineers in the 3-dimensional Euclidean space. The intrinsic notation used here to represent tensor operations comes from the modern differential geometry and is free from any indices [1.1–1.11]. The rules used to combine vectors and tensors and to build derivatives come from the linear algebra and the analysis [1.12,1.13]. In most cases, scalars are represented by normal small or capital letter, vectors by bold small letter and tensors by bold capital letter.

Let us first consider a cartesian system (O, x, y, z) with the three coordinates x, y, z and the three orthogonal linear independent constant unit base vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$:

$$\begin{gathered}\text{x-axis} \mathbin{/\!/} \mathbf{i} \quad ; \quad \text{y-axis} \mathbin{/\!/} \mathbf{j} \quad ; \quad \text{z-axis} \mathbin{/\!/} \mathbf{k} \quad ; \qquad \mathbf{i}\,,\mathbf{j}\,,\mathbf{k} = \not{c} \quad \text{(constant)} \\ \mathbf{i} \perp \mathbf{j}\,, \quad \mathbf{j} \perp \mathbf{k}\,, \quad \mathbf{k} \perp \mathbf{i} \quad ; \quad |\mathbf{i}| = 1\,, \quad |\mathbf{j}| = 1\,, \quad |\mathbf{k}| = 1 \\ \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0 \quad ; \quad \mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1\end{gathered} \tag{1.1}$$

A vector (or tensor of rank 1) is defined as follows

$$\mathbf{u} = \sum_{i=1}^{3} \mathbf{u}_i = u_x \mathbf{i} + u_y \mathbf{j} + u_z \mathbf{k} \mathrel{\hat{=}} \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \begin{pmatrix} u_x & u_y & u_z \end{pmatrix} \qquad \Rightarrow \qquad \mathbf{u} \equiv \mathbf{u}^T \tag{1.2}$$

where all $\mathbf{u}_i$ are linear independent and where u_x, u_y, u_z are called the cartesian components of the vector $\mathbf{u}$. The sign $\hat{}$ over the equal sign $=$ draws attention to the fact that the base vectors are omitted in the matrix representation. The physical meaning of a vector written as a column or a line obviously remains the same, i.e. $\mathbf{u} = \mathbf{u}^T$. Therefore, it is not necessary to use the transposition sign ^{T} for vectors in the intrinsic notation.

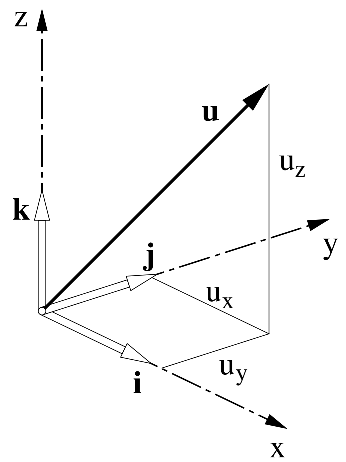

Fig. 1.1: Representation of a vector in the 3-dimensional space

A point P in the 3-dimensional space can be represented by its three cartesian coordinates x, y, z, which are the components of the position vector $\mathbf{r} = \overrightarrow{\mathrm{OP}}$ defined as follows

$$\mathrm{P}: \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \mathrel{\hat{=}} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{1.3}$$

The components x, y, z of the position vector can be written as function of the new three independent curvilinear coordinates $\theta^1, \theta^2, \theta^3$

$$\theta^1, \theta^2, \theta^3 \quad \rightarrow \quad \begin{aligned} x &= x(\theta^1, \theta^2, \theta^3) \\ y &= y(\theta^1, \theta^2, \theta^3) \\ z &= z(\theta^1, \theta^2, \theta^3) \end{aligned} \tag{1.4}$$

The position of point P can now be defined with the new variables $\theta^1, \theta^2, \theta^3$, which means that the position vector $\mathbf{r}$ is now a function of the curvilinear coordinates $\theta^1, \theta^2, \theta^3$. We have:

$$\theta^1, \theta^2, \theta^3 \quad \rightarrow \quad \mathbf{r} = \mathbf{r}(\theta^1, \theta^2, \theta^3) \tag{1.5}$$

By varying only one of the three variables θ^i (i=1,2,3) while keeping the two others constant, the position vector $\mathbf{r}$ describes a curved line in space, i.e. a line of curvilinear coordinate. By varying two of the three variables θ^i while keeping the other one constant, $\mathbf{r}$ describes a curved surface in space.

The total differential $\mathrm{d}\mathbf{r}$ of the position vector $\mathbf{r}$ can now be written as follows

$$\mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \theta^1}\mathrm{d}\theta^1 + \frac{\partial \mathbf{r}}{\partial \theta^2}\mathrm{d}\theta^2 + \frac{\partial \mathbf{r}}{\partial \theta^3}\mathrm{d}\theta^3 = \sum_{i=1}^{3} \frac{\partial \mathbf{r}}{\partial \theta^i}\mathrm{d}\theta^i = \frac{\partial \mathbf{r}}{\partial \theta^i}\mathrm{d}\theta^i = \mathbf{r}_{,i}\mathrm{d}\theta^i = \mathbf{g}_i\mathrm{d}\theta^i \tag{1.6}$$

where the repeated latin index i means a sum on i from 1 to 3. The vectors $\mathbf{g}_i = \partial\mathbf{r}/\partial\theta^i$ are called covariant base vectors and lie tangent to their corresponding lines of curvilinear coordinates. These base vectors are generally not constant and not perpendicular to each other. Note that in the physical space, the cartesian coordinates x, y, z and the curvilinear coordinates $\theta^1, \theta^2, \theta^3$ often have the dimension of a length (e.g. in m), whereas the base vectors are dimensionless.

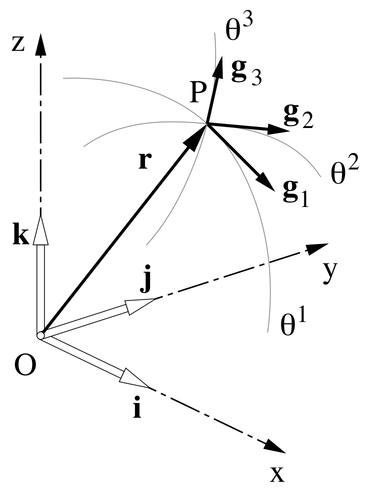

Fig. 1.2: Curvilinear coordinates

As example, the special case where $x = \theta^1$, $y = \theta^2$ and $z = \theta^3$ leads to

$$\frac{\partial \mathbf{r}}{\partial x} = \mathbf{i} \mathrel{\hat{=}} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \quad ; \quad \frac{\partial \mathbf{r}}{\partial y} = \mathbf{j} \mathrel{\hat{=}} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \quad ; \quad \frac{\partial \mathbf{r}}{\partial z} = \mathbf{k} \mathrel{\hat{=}} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \tag{1.7}$$

In a general case, a vector (tensor of rank 1) may be expressed relatively to the covariant base vectors as follows

$$\mathbf{u} = u^i \mathbf{g}_i = u^1 \mathbf{g}_1 + u^2 \mathbf{g}_2 + u^3 \mathbf{g}_3 \tag{1.8}$$

where the scalars u^i are the contravariant components of the vector $\mathbf{u}$. In a cartesian system, the scalar product of two vectors reads (*line* $\times$ *column*):

$$\mathbf{u} \cdot \mathbf{v} \mathrel{\hat{=}} \begin{pmatrix} u_x & u_y & u_z \end{pmatrix} \cdot \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} = u_x v_x + u_y v_y + u_z v_z \tag{1.9}$$

where the sign $\cdot$ represents a scalar product. The contravariant base vectors $\mathbf{g}^j$ are defined as follows

$$\mathbf{g}^j \cdot \mathbf{g}_i = \delta^j_{\ i} = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases} \quad ; \quad (i, j = 1, 2, 3) \tag{1.10}$$

with the so-called Kronecker symbol $\delta^j_{\ i}$. Thus a vector can also be expressed relatively to the contravariant base vectors as follows

$$\mathbf{u} = u_i \mathbf{g}^i = u_1 \mathbf{g}^1 + u_2 \mathbf{g}^2 + u_3 \mathbf{g}^3 \tag{1.11}$$

where the scalars u_i are the covariant components of the vector $\mathbf{u}$. All these different kinds of representation, i.e. with covariant, or contravariant, or cartesian base vectors, does not change the physical meaning. Consequently, we can choose an intrinsic notation without indices, containing both the components and the base vectors in a symbolic representation. The rules on how to use this notation are explained in the next lines. Practically, one needs to introduce a coordinate system only at the very end of an algebraic computation, that means just before numerical values are requested (e.g. in an experiment).

The scalar product of the two vectors $\mathbf{u} = u^i \mathbf{g}_i$ and $\mathbf{v} = v_j \mathbf{g}^j$ reads

$$\mathbf{u} \cdot \mathbf{v} = u^i \mathbf{g}_i \cdot \mathbf{g}^j v_j = u^i v_i \tag{1.12}$$

In a cartesian system, the tensor product of two vectors reads (*line* $\times$ *column*):

$$\mathbf{u} \otimes \mathbf{v} \mathrel{\hat{=}} \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} \otimes \begin{pmatrix} v_x & v_y & v_z \end{pmatrix} = \begin{bmatrix} u_x v_x & u_x v_y & u_x v_z \\ u_y v_x & u_y v_y & u_y v_z \\ u_z v_x & u_z v_y & u_z v_z \end{bmatrix} \tag{1.13}$$

where the sign $\otimes$ represents a tensor product (also often called a dyadic product). A tensor of rank 2 is defined as follows

$$\begin{aligned} \mathbf{T} &= \sum_{i=1}^{9} \mathbf{T}_i = \sum_{i=1}^{3} \sum_{j=1}^{3} \mathbf{p}_i \otimes \mathbf{q}_j \\ &= T_{xx} \mathbf{i} \otimes \mathbf{i} + T_{xy} \mathbf{i} \otimes \mathbf{j} + T_{xz} \mathbf{i} \otimes \mathbf{k} + T_{yx} \mathbf{j} \otimes \mathbf{i} + T_{yy} \mathbf{j} \otimes \mathbf{j} + T_{yz} \mathbf{j} \otimes \mathbf{k} \\ &\quad + T_{zx} \mathbf{k} \otimes \mathbf{i} + T_{zy} \mathbf{k} \otimes \mathbf{j} + T_{zz} \mathbf{k} \otimes \mathbf{k} \\ &\mathrel{\hat{=}} \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix} \end{aligned} \tag{1.14}$$

where all $\mathbf{T}_i$ are linear independent and where $T_{xx}, T_{xy}, \ldots, T_{zz}$ are called the cartesian components of the tensor $\mathbf{T}$. The vectors $\mathbf{p}_i$ and $\mathbf{q}_j$ are general linear independent vectors in space. A combination of tensor products of base vectors gives a tensor base. In a cartesian system, we have

$$\mathbf{i}\otimes\mathbf{i} \hat{=} \begin{Bmatrix}1\\0\\0\end{Bmatrix}\otimes(1\quad 0\quad 0) = \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} \quad ; \quad \mathbf{i}\otimes\mathbf{j} \hat{=} \begin{Bmatrix}1\\0\\0\end{Bmatrix}\otimes(0\quad 1\quad 0) = \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}$$
$$\mathbf{i}\otimes\mathbf{k} \hat{=} \begin{Bmatrix}1\\0\\0\end{Bmatrix}\otimes(0\quad 0\quad 1) = \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix} \quad ; \quad \mathbf{j}\otimes\mathbf{i} \hat{=} \begin{Bmatrix}0\\1\\0\end{Bmatrix}\otimes(1\quad 0\quad 0) = \begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}$$
$$\mathbf{j}\otimes\mathbf{j} \hat{=} \begin{Bmatrix}0\\1\\0\end{Bmatrix}\otimes(0\quad 1\quad 0) = \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix} \quad ; \quad \mathbf{j}\otimes\mathbf{k} \hat{=} \begin{Bmatrix}0\\1\\0\end{Bmatrix}\otimes(0\quad 0\quad 1) = \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}$$
$$\mathbf{k}\otimes\mathbf{i} \hat{=} \begin{Bmatrix}0\\0\\1\end{Bmatrix}\otimes(1\quad 0\quad 0) = \begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix} \quad ; \quad \mathbf{k}\otimes\mathbf{j} \hat{=} \begin{Bmatrix}0\\0\\1\end{Bmatrix}\otimes(0\quad 1\quad 0) = \begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}$$
$$\mathbf{k}\otimes\mathbf{k} \hat{=} \begin{Bmatrix}0\\0\\1\end{Bmatrix}\otimes(0\quad 0\quad 1) = \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} \tag{1.15}$$

In a general case, a tensor of rank 2 may be expressed as follows

$$\mathbf{T} = T^{ij}\mathbf{g}_i\otimes\mathbf{g}_j = T^i_{\ j}\mathbf{g}_i\otimes\mathbf{g}^j = T_i^{\ j}\mathbf{g}^i\otimes\mathbf{g}_j = T_{ij}\mathbf{g}^i\otimes\mathbf{g}^j \tag{1.16}$$

where T^{ij} are the contravariant, $T^i_{\ j}$ and $T_i^{\ j}$ the mixed and T_{ij} the covariant components of $\mathbf{T}$. The repeated indices i and j mean a sum on i and j from 1 to 3. Its transpose reads

$$\begin{aligned}\mathbf{T}^T &= T^{ij}\mathbf{g}_j\otimes\mathbf{g}_i = T^i_{\ j}\mathbf{g}^j\otimes\mathbf{g}_i = T_i^{\ j}\mathbf{g}_j\otimes\mathbf{g}^i = T_{ij}\mathbf{g}^j\otimes\mathbf{g}^i \\ &= T^{ji}\mathbf{g}_i\otimes\mathbf{g}_j = T^j_{\ i}\mathbf{g}^i\otimes\mathbf{g}_j = T_j^{\ i}\mathbf{g}_i\otimes\mathbf{g}^j = T_{ji}\mathbf{g}^i\otimes\mathbf{g}^j\end{aligned} \tag{1.17}$$

We say that $\mathbf{T}$ is symmetric if $\mathbf{T} = \mathbf{T}^T$ and antimetric if $\mathbf{T} = -\mathbf{T}^T$. In a general case, a tensor of rank 3 may be expressed as a combination of triadic products:

$$\boldsymbol{\mathcal{T}} = \sum_{i=1}^{27}\boldsymbol{\mathcal{T}}_i = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mathbf{p}_i\otimes\mathbf{q}_j\otimes\mathbf{s}_k = T^{ijk}\mathbf{g}_i\otimes\mathbf{g}_j\otimes\mathbf{g}_k = \ldots = T_{ijk}\mathbf{g}^i\otimes\mathbf{g}^j\otimes\mathbf{g}^k \tag{1.18}$$

where all $\boldsymbol{\mathcal{T}}_i$ are linear independent and where $\mathbf{p}_i$, $\mathbf{q}_j$ and $\mathbf{s}_k$ are general linear independent vectors in space. There are six possibilities to transpose a tensor of rank 3. For tensors of rank 4 and higher, we may apply analog definitions. A matrix representation for tensors of rank 3 and higher is not recommended.

Scalars (or tensors of rank 0), vectors and tensors may interact on each other in several ways. The most often encountered are the so-called linear transformations or mapping. For example, a tensor $\mathbf{T}$ may act as an operator and maps a vector $\mathbf{u}$ into a new vector $\mathbf{w} = \mathbf{T}\mathbf{u}$. With the two arbitrary vectors $\mathbf{p}$ and $\mathbf{q}$, we may show this mechanism in the following example

$$\mathbf{T} = \mathbf{p}\otimes\mathbf{q} \hat{=} \begin{Bmatrix}p_x\\p_y\\p_z\end{Bmatrix}\otimes(\,q_x\quad q_y\quad q_z\,) = \begin{bmatrix}p_xq_x & p_xq_y & p_xq_z\\ p_yq_x & p_yq_y & p_yq_z\\ p_zq_x & p_zq_y & p_zq_z\end{bmatrix} \tag{1.19a}$$

$$
\begin{aligned}
\mathbf{w} &= \mathbf{T}\mathbf{u} = (\mathbf{p}\otimes\mathbf{q})\mathbf{u} = \mathbf{p}(\mathbf{q}\cdot\mathbf{u}) \\
&\hat{=} \begin{bmatrix} p_xq_x & p_xq_y & p_xq_z \\ p_yq_x & p_yq_y & p_yq_z \\ p_zq_x & p_zq_y & p_zq_z \end{bmatrix} \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \begin{Bmatrix} p_x(q_xu_x+q_yu_y+q_zu_z) \\ p_y(q_xu_x+q_yu_y+q_zu_z) \\ p_z(q_xu_x+q_yu_y+q_zu_z) \end{Bmatrix} \\
&\hat{=} \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix} \left[\begin{pmatrix} q_x & q_y & q_z \end{pmatrix} \cdot \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} \right] = \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix} (q_xu_x+q_yu_y+q_zu_z)
\end{aligned}
\tag{1.19b}
$$

In our example, we often say that the vector $\mathbf{u}$ contracts with the vector $\mathbf{q}$ giving a vector parallel to $\mathbf{p}$. We also speak of applying the tensor $\mathbf{T}$ on the vector $\mathbf{u}$, keeping in mind that the vector $\mathbf{u}$ is on the right of the operator $\mathbf{T}$. With the arbitrary vector $\mathbf{v}$ and the scalars λ and μ, this transformation is linear because $\mathbf{T}(\lambda\mathbf{u}+\mu\mathbf{v}) = \lambda\mathbf{T}\mathbf{u}+\mu\mathbf{T}\mathbf{v}$ and is not commutative because generally $\mathbf{T}\mathbf{u} \neq \mathbf{u}\mathbf{T}$. On the other hand, we may write

$$
\mathbf{T}^T = \mathbf{q}\otimes\mathbf{p} \,\hat{=}\, \begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix} \otimes \begin{pmatrix} p_x & p_y & p_z \end{pmatrix} = \begin{bmatrix} q_xp_x & q_xp_y & q_xp_z \\ q_yp_x & q_yp_y & q_yp_z \\ q_zp_x & q_zp_y & q_zp_z \end{bmatrix}
\tag{1.20a}
$$

$$
\begin{aligned}
\mathbf{w} &= \mathbf{u}\mathbf{T}^T = \mathbf{u}(\mathbf{q}\otimes\mathbf{p}) = (\mathbf{u}\cdot\mathbf{q})\mathbf{p} \\
&\hat{=} \begin{pmatrix} u_x & u_y & u_z \end{pmatrix} \begin{bmatrix} q_xp_x & q_xp_y & q_xp_z \\ q_yp_x & q_yp_y & q_yp_z \\ q_zp_x & q_zp_y & q_zp_z \end{bmatrix} = \begin{Bmatrix} (u_xq_x+u_yq_y+u_zq_z)p_x \\ (u_xq_x+u_yq_y+u_zq_z)p_y \\ (u_xq_x+u_yq_y+u_zq_z)p_z \end{Bmatrix} \\
&\hat{=} \left[\begin{pmatrix} u_x & u_y & u_z \end{pmatrix} \cdot \begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix} \right] \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix} = (u_xq_x+u_yq_y+u_zq_z) \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix}
\end{aligned}
\tag{1.20b}
$$

which shows that $\mathbf{T}\mathbf{u} = \mathbf{u}\mathbf{T}^T$. The transposition rules of vectors, rank 2 and rank 3 tensors are the following:

$$
\mathbf{u} \equiv \mathbf{u}^T \quad ; \quad \begin{aligned} \mathbf{T} &= \mathbf{p}\otimes\mathbf{q} \\ \mathbf{T}^T &= \mathbf{q}\otimes\mathbf{p} \end{aligned} \quad ; \quad \begin{aligned} \boldsymbol{\mathcal{B}} &= \mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c} \\ \boldsymbol{\mathcal{B}})^T &= \mathbf{a}\otimes\mathbf{c}\otimes\mathbf{b} \\ {}^T(\boldsymbol{\mathcal{B}} &= \mathbf{b}\otimes\mathbf{a}\otimes\mathbf{c} \\ {}^T(\boldsymbol{\mathcal{B}}^T &= \mathbf{b}\otimes\mathbf{c}\otimes\mathbf{a} \\ \boldsymbol{\mathcal{B}}^T &= \mathbf{c}\otimes\mathbf{b}\otimes\mathbf{a} \\ \boldsymbol{\mathcal{B}}^T)^T &= \mathbf{c}\otimes\mathbf{a}\otimes\mathbf{b} \end{aligned}
\tag{1.21}
$$

With $\mathbf{R} = \mathbf{a}\otimes\mathbf{b}$, tensors of rank 2 may contract as follows

$$
\mathbf{T}\mathbf{R} = (\mathbf{p}\otimes\mathbf{q})(\mathbf{a}\otimes\mathbf{b}) = (\mathbf{q}\cdot\mathbf{a})\mathbf{p}\otimes\mathbf{b}
\tag{1.22a}
$$

$$
\mathbf{T}\cdot\mathbf{R} = (\mathbf{p}\otimes\mathbf{q})\cdot(\mathbf{a}\otimes\mathbf{b}) = (\mathbf{q}\cdot\mathbf{a})(\mathbf{p}\cdot\mathbf{b})
\tag{1.22b}
$$

As we can see, the first above equation (1.22a) represents the product of two tensors of rank 2 which gives a rank 2 tensor again (analog to a matrix product in the sense of linear algebra). The second above equation (1.22b) represents the double contraction of two tensors of rank 2 which gives a scalar. In these cases, one speaks about simple and double contraction of two tensors. The double contraction, which contracts the last with the first factor of corresponding tensors, should not be mistaken with the scalar product, where the same corresponding factors contract together. Thus, the scalar product of the tensors $\mathbf{T}$ and $\mathbf{R}$ reads

$$\mathbf{T}^T \cdot \mathbf{R} = (\mathbf{q} \otimes \mathbf{p}) \cdot (\mathbf{a} \otimes \mathbf{b}) = (\mathbf{p} \cdot \mathbf{a})(\mathbf{q} \cdot \mathbf{b}) \tag{1.23}$$

Let us also introduce the triple contraction, which reads with $\boldsymbol{\mathcal{T}} = \mathbf{p} \otimes \mathbf{q} \otimes \mathbf{s}$

$$\boldsymbol{\mathcal{B}} : \boldsymbol{\mathcal{T}} = (\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}) : (\mathbf{p} \otimes \mathbf{q} \otimes \mathbf{s}) = (\mathbf{c} \cdot \mathbf{p})(\mathbf{b} \cdot \mathbf{q})(\mathbf{a} \cdot \mathbf{s}) \tag{1.24}$$

As summary, the following rules may be deduced in a similar way

$$\begin{aligned}
&\mathbf{Tu} = \mathbf{uT}^T = (\mathbf{p} \otimes \mathbf{q})\mathbf{u} = \mathbf{p}(\mathbf{q} \cdot \mathbf{u}) = \mathbf{u}(\mathbf{q} \otimes \mathbf{p}) = (\mathbf{u} \cdot \mathbf{q})\mathbf{p} \\
&\mathbf{v} \cdot \mathbf{Tu} = \mathbf{vT} \cdot \mathbf{u} = \mathbf{v} \cdot (\mathbf{p} \otimes \mathbf{q})\mathbf{u} = (\mathbf{v} \cdot \mathbf{p})(\mathbf{q} \cdot \mathbf{u}) \\
&\mathbf{TR} = (\mathbf{p} \otimes \mathbf{q})(\mathbf{a} \otimes \mathbf{b}) = (\mathbf{q} \cdot \mathbf{a})\mathbf{p} \otimes \mathbf{b} \neq \mathbf{RT} \\
&\mathbf{T} \cdot \mathbf{R} = (\mathbf{p} \otimes \mathbf{q}) \cdot (\mathbf{a} \otimes \mathbf{b}) = (\mathbf{q} \cdot \mathbf{a})(\mathbf{p} \cdot \mathbf{b}) = \mathbf{R} \cdot \mathbf{T} = \mathbf{T}^T \cdot \mathbf{R}^T \\
&\boldsymbol{\mathcal{B}}\mathbf{u} = (\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c})\mathbf{u} = \mathbf{a} \otimes \mathbf{b}(\mathbf{c} \cdot \mathbf{u}) \\
&\boldsymbol{\mathcal{B}}\mathbf{T} = (\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c})(\mathbf{p} \otimes \mathbf{q}) = \mathbf{a} \otimes \mathbf{b} \otimes \mathbf{q}(\mathbf{c} \cdot \mathbf{p}) \\
&\boldsymbol{\mathcal{B}} \cdot \mathbf{T} = (\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}) \cdot (\mathbf{p} \otimes \mathbf{q}) = \mathbf{a}(\mathbf{b} \cdot \mathbf{q})(\mathbf{c} \cdot \mathbf{p})
\end{aligned} \tag{1.25}$$

An important tensor is the identity $\mathbf{I}$

$$\mathbf{I} = \mathbf{g}_i \otimes \mathbf{g}^i = \mathbf{g}^i \otimes \mathbf{g}_i = g^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = g_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = \mathbf{i} \otimes \mathbf{i} + \mathbf{j} \otimes \mathbf{j} + \mathbf{k} \otimes \mathbf{k} \,\hat{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.26}$$

where $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ and $g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j$ are the covariant and contravariant components respectively. Note that the double contraction $\mathbf{I} \cdot \mathbf{I} = 3$. Applying $\mathbf{I}$ on a vector $\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$, we get

$$\mathbf{u} = \mathbf{Iu} = (\mathbf{i} \otimes \mathbf{i} + \mathbf{j} \otimes \mathbf{j} + \mathbf{k} \otimes \mathbf{k})\mathbf{u} = (\mathbf{i} \cdot \mathbf{u})\mathbf{i} + (\mathbf{j} \cdot \mathbf{u})\mathbf{j} + (\mathbf{k} \cdot \mathbf{u})\mathbf{k} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k} \tag{1.27}$$

Proof:

$$\mathbf{g}_i = \mathbf{I}\mathbf{g}_i = (\mathbf{g}_j \otimes \mathbf{g}^j)\mathbf{g}_i = \mathbf{g}_j(\mathbf{g}^j \cdot \mathbf{g}_i) = \mathbf{g}_j\delta^j_{\ i} = \mathbf{g}_i \qquad \square \text{ qed} \tag{1.28}$$

The identity tensor allows the bridge between covariant and contravariant expressions. As example, we have for $\mathbf{u}$

$$\begin{aligned}
\mathbf{u} = \mathbf{uI} &= \mathbf{u}(\mathbf{g}^i \otimes \mathbf{g}_i) = (\mathbf{u} \cdot \mathbf{g}^i)\mathbf{g}_i = u^i\mathbf{g}_i \\
&= \mathbf{u}(\mathbf{g}_i \otimes \mathbf{g}^i) = (\mathbf{u} \cdot \mathbf{g}_i)\mathbf{g}^i = u_i\mathbf{g}^i
\end{aligned} \tag{1.29}$$

and for the base vectors

$$\begin{aligned}
\mathbf{g}^i &= \mathbf{g}^i\mathbf{I} = \mathbf{g}^i(\mathbf{g}^j \otimes \mathbf{g}_j) = (\mathbf{g}^i \cdot \mathbf{g}^j)\mathbf{g}_j = g^{ij}\mathbf{g}_j \\
\mathbf{g}_i &= \mathbf{g}_i\mathbf{I} = \mathbf{g}_i(\mathbf{g}_j \otimes \mathbf{g}^j) = (\mathbf{g}_i \cdot \mathbf{g}_j)\mathbf{g}^j = g_{ij}\mathbf{g}^j
\end{aligned} \tag{1.30}$$

A tensor $\mathbf{Q}$ is orthogonal in space if

$$\mathbf{Q}^{-1} = \mathbf{Q}^T \qquad ; \qquad \mathbf{QQ}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{1.31}$$

where $\mathbf{Q}^{-1}$ is called the inverse of the tensor $\mathbf{Q}$. Applied on a vector $\mathbf{u}$, the orthogonal tensor $\mathbf{Q}$ acts as a rotation. With $\mathbf{v} = \mathbf{Qu}$, we have

$$v^2 = \mathbf{v} \cdot \mathbf{v} = \mathbf{u}\mathbf{Q}^T \cdot \mathbf{Qu} = \mathbf{u} \cdot \mathbf{Q}^T\mathbf{Qu} = \mathbf{u} \cdot \mathbf{Iu} = \mathbf{u} \cdot \mathbf{u} = u^2 \tag{1.32}$$

which shows that the norm (length) of the vector remains unchanged. Each tensor $\mathbf{T}$ in space may be decomposed in a symmetric part $\mathbf{T}_S$ and an antimetric part $\mathbf{T}_A$ as follows

$$\mathbf{T} = \mathbf{T}_S + \mathbf{T}_A \qquad ; \qquad \begin{cases} \mathbf{T}_S = \mathbf{T}_S^T = \dfrac{1}{2}(\mathbf{T} + \mathbf{T}^T) \\ \mathbf{T}_A = -\mathbf{T}_A^T = \dfrac{1}{2}(\mathbf{T} - \mathbf{T}^T) \end{cases} \tag{1.33}$$

By introducing the 3-dimensional tensors $\mathbf{S} = \mathbf{S}^T$ (symmetric) and $\mathbf{A} = -\mathbf{A}^T$ (antimetric), we can write

$$\mathbf{S} \cdot \mathbf{A} = 0 \quad ; \quad \mathbf{T}_S \cdot \mathbf{T}_A = 0 \quad ; \quad \mathbf{T} \cdot \mathbf{S} = \mathbf{T}_S \cdot \mathbf{S} \quad ; \quad \mathbf{T} \cdot \mathbf{A} = \mathbf{T}_A \cdot \mathbf{A} \tag{1.34}$$

We shall often encounter the unit vector $\mathbf{n}$ defined as the unit normal of some plane or some curved surface in space. The normal projection onto that plane or onto the tangential plane of that curved surface is described by the normal projector $\mathbf{N}$:

$$\mathbf{N} = \mathbf{I} - \mathbf{n} \otimes \mathbf{n} = \mathbf{N}^T \qquad ; \qquad \mathbf{n} \cdot \mathbf{n} = 1 \tag{1.35}$$

Applied on a vector $\mathbf{u}$, the normal projector $\mathbf{N}$ acts as a normal projection of the vector $\mathbf{u}$ onto a plane normal to the direction $\mathbf{n}$, which means

$$\mathbf{Nu} = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\mathbf{u} = \mathbf{u} - (\mathbf{n} \cdot \mathbf{u})\mathbf{n} \tag{1.36}$$

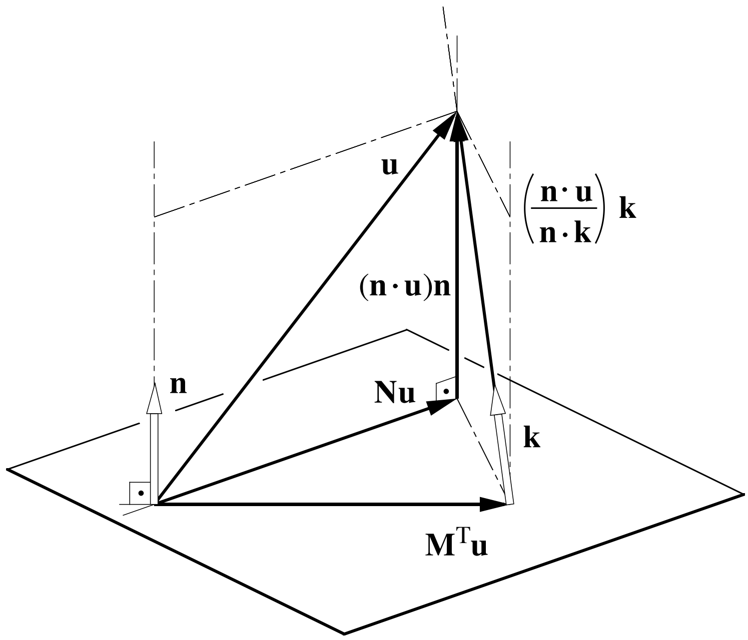

Fig. 1.3: Normal and oblique projections

An oblique projector $\mathbf{M}$ is defined as follows

$$\mathbf{M} = \mathbf{I} - \frac{\mathbf{n} \otimes \mathbf{k}}{\mathbf{n} \cdot \mathbf{k}} \neq \mathbf{M}^T \qquad ; \qquad \mathbf{k} \cdot \mathbf{k} = 1 \tag{1.37}$$

where $\mathbf{k}$ is some unit vector generally not parallel to $\mathbf{n}$. Applied on a vector $\mathbf{u}$, the oblique projector $\mathbf{M}$ acts as an oblique projection of the vector $\mathbf{u}$ along the direction $\mathbf{n}$ onto a plane normal to the direction $\mathbf{k}$. On the other hand, its transpose $\mathbf{M}^T$ acts as an oblique projection of the vector $\mathbf{u}$ along the direction $\mathbf{k}$ onto a plane normal to the direction $\mathbf{n}$. In both cases we have

$$\mathbf{M}\mathbf{u} = \left(\mathbf{I} - \frac{\mathbf{n} \otimes \mathbf{k}}{\mathbf{n} \cdot \mathbf{k}}\right)\mathbf{u} = \mathbf{u} - \left(\frac{\mathbf{k} \cdot \mathbf{u}}{\mathbf{n} \cdot \mathbf{k}}\right)\mathbf{n} = \mathbf{u}\mathbf{M}^T \tag{1.38a}$$

$$\mathbf{M}^T\mathbf{u} = \left(\mathbf{I} - \frac{\mathbf{k} \otimes \mathbf{n}}{\mathbf{k} \cdot \mathbf{n}}\right)\mathbf{u} = \mathbf{u} - \left(\frac{\mathbf{n} \cdot \mathbf{u}}{\mathbf{n} \cdot \mathbf{k}}\right)\mathbf{k} = \mathbf{u}\mathbf{M} \tag{1.38b}$$

Consequently, the arbitrary vector $\mathbf{u}$ may be decomposed in an interior part $\mathbf{N}\mathbf{u}$ and in an exterior part $(\mathbf{n} \cdot \mathbf{u})\mathbf{n}$ as follows

$$\mathbf{u} = \mathbf{I}\mathbf{u} = (\mathbf{N} + \mathbf{n} \otimes \mathbf{n})\mathbf{u} = \mathbf{N}\mathbf{u} + (\mathbf{n} \cdot \mathbf{u})\mathbf{n} \tag{1.39}$$

In a similar way, the 3-dimensional tensor $\mathbf{T}$ may decomposed in an interior part $\mathbf{NTN}$, in two semi-exterior parts $\mathbf{n} \otimes \mathbf{nTN}$ and $\mathbf{NTn} \otimes \mathbf{n}$ and in an exterior part $(\mathbf{n} \cdot \mathbf{Tn})\mathbf{n} \otimes \mathbf{n}$ as follows

$$\mathbf{T} = \mathbf{ITI} = (\mathbf{N} + \mathbf{n} \otimes \mathbf{n})\mathbf{T}(\mathbf{N} + \mathbf{n} \otimes \mathbf{n}) = \mathbf{NTN} + \mathbf{n} \otimes \mathbf{nTN} + \mathbf{NTn} \otimes \mathbf{n} + (\mathbf{n} \cdot \mathbf{Tn})\mathbf{n} \otimes \mathbf{n} \tag{1.40}$$

In a cartesian system, the vector product of two vectors is defined as follows

$$\mathbf{u} \times \mathbf{v} \mathrel{\hat{=}} \begin{vmatrix} u_x & v_x & \mathbf{i} \\ u_y & v_y & \mathbf{j} \\ u_z & v_z & \mathbf{k} \end{vmatrix} = \begin{Bmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{Bmatrix} \tag{1.41}$$

In a general case, the vector product of two vectors $\mathbf{u}$ and $\mathbf{v}$ reads

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = -\mathbf{u}\boldsymbol{\mathcal{E}}\mathbf{v} = \mathbf{v}\boldsymbol{\mathcal{E}}\mathbf{u} = -\mathbf{v} \times \mathbf{u} \tag{1.42}$$

where $\boldsymbol{\mathcal{E}}$ is the so-called 3-dimensional third-rank permutation tensor. This tensor is constant in the 3-dimensional space and is defined as follows

$$\begin{aligned}
\boldsymbol{\mathcal{E}} &= E_{ijk}\mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k \\
&= \mathbf{i} \otimes \mathbf{j} \otimes \mathbf{k} - \mathbf{i} \otimes \mathbf{k} \otimes \mathbf{j} + \mathbf{j} \otimes \mathbf{k} \otimes \mathbf{i} - \mathbf{j} \otimes \mathbf{i} \otimes \mathbf{k} + \mathbf{k} \otimes \mathbf{i} \otimes \mathbf{j} - \mathbf{k} \otimes \mathbf{j} \otimes \mathbf{i} \\
E_{ijk} &= \begin{cases} +\sqrt{g} & \text{for an even permutation of } ijk = 123 \\ -\sqrt{g} & \text{for an odd permutation of } ijk = 123 \\ \quad 0 & \text{if two indices are identical} \end{cases} \qquad ; \qquad (i, j, k = 1, 2, 3) \\
g &= \det g_{ij} = g_{11}g_{22}g_{33} - g_{11}g_{23}^2 - g_{22}g_{13}^2 - g_{33}g_{12}^2 + 2g_{12}g_{23}g_{13}
\end{aligned} \tag{1.43}$$

Note that the triple contraction $\boldsymbol{\mathcal{E}} : \boldsymbol{\mathcal{E}} = -6$. By applying $\boldsymbol{\mathcal{E}}$ on some unit vector in space, e.g. the unit normal $\mathbf{n}$, we get a 2-dimensional antimetric tensor, the so-called second-rank permutation tensor $\mathbf{E}$, which reads

$$\mathbf{E} = -\mathbf{E}^T = \boldsymbol{\mathcal{E}}\mathbf{n} \qquad ; \qquad \mathbf{E} \equiv \mathbf{NEN} \tag{1.44}$$

The third-rank permutation tensor $\boldsymbol{\mathcal{E}}$ may be decomposed with the second-rank permutation tensor $\mathbf{E}$ and the unit normal $\mathbf{n}$ in semi-exterior parts as follows

$$\boldsymbol{\mathcal{E}} = \mathbf{E} \otimes \mathbf{n} - \mathbf{E} \otimes \mathbf{n})^T + \mathbf{n} \otimes \mathbf{E} \tag{1.45}$$

where the sign $)^T$ represents a semi-transposition of the second and third factor in the corresponding triadic product. Applied on a vector $\mathbf{u}$, the tensor $\mathbf{E}$ acts first as a normal projection onto a plane normal to $\mathbf{n}$ and second as a rotation of $-\pi/2$ around the direction $\mathbf{n}$. Consequently we have $\mathbf{EE} = -\mathbf{N}$ and $\mathbf{E} \cdot \mathbf{E} = -2$.

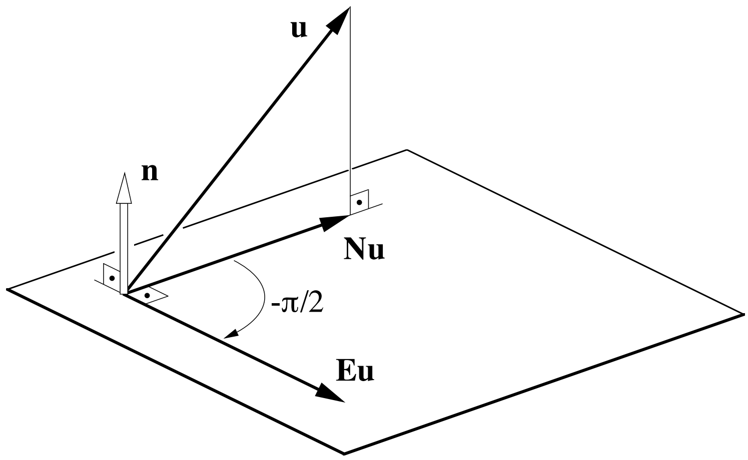

Fig. 1.4: Second-rank permutation tensor $\mathbf{E}$ applied on the vector $\mathbf{u}$

Example:

With the cartesian base vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, the second-rank permutation tensor $\mathbf{E}_z$ relatively to the direction $\mathbf{k}$ (i.e. z-axis) reads

$$\boldsymbol{\mathcal{E}}\mathbf{k} = \mathbf{i} \otimes \mathbf{j} - \mathbf{j} \otimes \mathbf{i} = \mathbf{E}_z \mathrel{\hat{=}} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ; \quad \mathbf{E}_z\mathbf{i} = -\mathbf{j} \quad ; \quad \mathbf{E}_z\mathbf{j} = \mathbf{i} \tag{1.46}$$

The three invariants of a general 3-dimensional tensor $\mathbf{T}$ are:

$$\begin{aligned} I_1 &= \operatorname{tr} \mathbf{T} = \mathbf{T} \cdot \mathbf{I} && : \ \textit{Trace} \text{ of } \mathbf{T} \\ I_2 &= \frac{1}{2} \mathbf{T} \cdot (\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}})^T \cdot \mathbf{T} && : \ \text{Sum of the } \textit{minor-determinants} \text{ of } \mathbf{T} \\ I_3 &= \det \mathbf{T} = \frac{1}{6} \mathbf{T} \cdot (\boldsymbol{\mathcal{E}}\mathbf{T}\boldsymbol{\mathcal{E}})^T \cdot \mathbf{T} && : \ \textit{Determinant} \text{ of } \mathbf{T} \end{aligned} \tag{1.47}$$

The three invariants I_1, I_2 and I_3 of the tensor $\mathbf{T}$ are the coefficients of the caracteristic equation

$$\det(\mathbf{T} - \sigma\mathbf{I}) = \frac{1}{6} (\mathbf{T} - \sigma\mathbf{I}) \cdot [\boldsymbol{\mathcal{E}}(\mathbf{T} - \sigma\mathbf{I})\boldsymbol{\mathcal{E}}]^T \cdot (\mathbf{T} - \sigma\mathbf{I}) = -\sigma^3 + I_1\sigma^2 - I_2\sigma + I_3 = 0 \tag{1.48}$$

because

$$
\begin{aligned}
&(\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}})^T \cdot \mathbf{I} = \mathbf{I} \cdot (\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}})^T = -(\boldsymbol{\mathcal{E}} \cdot \boldsymbol{\mathcal{E}})^T = -\boldsymbol{\mathcal{E}} \cdot \boldsymbol{\mathcal{E}} = 2\mathbf{I} \\
&\mathbf{I} \cdot (\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}})^T \cdot \mathbf{I} = 2\mathbf{I} \cdot \mathbf{I} = 6 \quad ; \qquad \mathbf{I} \cdot \mathbf{I} = 3 \\
&(\boldsymbol{\mathcal{E}}\mathbf{T}\boldsymbol{\mathcal{E}})^T \cdot \mathbf{I} = \mathbf{I} \cdot (\boldsymbol{\mathcal{E}}\mathbf{T}\boldsymbol{\mathcal{E}})^T = (\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}})^T \cdot \mathbf{T} = \mathbf{T} \cdot (\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}})^T \\
&\mathbf{I} \cdot (\boldsymbol{\mathcal{E}}\mathbf{T}\boldsymbol{\mathcal{E}})^T \cdot \mathbf{I} = \mathbf{T} \cdot (\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}})^T \cdot \mathbf{I} = \mathbf{I} \cdot (\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}})^T \cdot \mathbf{T} = 2\mathbf{T} \cdot \mathbf{I}
\end{aligned}
\tag{1.49}
$$

If $\mathbf{R} = \mathbf{NRN}$ is a general 2-dimensional tensor, its decomposition in a symmetric part $\mathbf{R}_S$ and an antimetric part $\mathbf{R}_A$ reads

$$
\mathbf{R} = \mathbf{NRN} = \mathbf{R}_S + \mathbf{R}_A \qquad ; \qquad
\begin{cases}
\mathbf{R}_S = \mathbf{R}_S^T = \mathbf{NR}_S\mathbf{N} = \dfrac{1}{2}(\mathbf{R} + \mathbf{R}^T) \\
\mathbf{R}_A = -\mathbf{R}_A^T = \mathbf{NR}_A\mathbf{N} = \dfrac{1}{2}(\mathbf{R} - \mathbf{R}^T) = \lambda\mathbf{E}
\end{cases}
\tag{1.50}
$$

where λ is a scalar. With the 3-dimensional tensors $\mathbf{S} = \mathbf{S}^T$ (symmetric) and $\mathbf{A} = -\mathbf{A}^T$ (antimetric), we have

$$
\mathbf{S} \cdot \mathbf{A} = 0 \quad ; \quad \mathbf{R}_S \cdot \mathbf{R}_A = 0 \quad ; \quad \mathbf{R} \cdot \mathbf{S} = \mathbf{R}_S \cdot \mathbf{S} \quad ; \quad \mathbf{R} \cdot \mathbf{A} = \mathbf{R}_A \cdot \mathbf{A} \tag{1.51}
$$

With the 2-dimensional tensors $\mathbf{N}$ and $\mathbf{E}$, we get

$$
\mathbf{R}_S \cdot \mathbf{E} = 0 \qquad ; \qquad \mathbf{R}_A \cdot \mathbf{N} = 0 \tag{1.52}
$$

The three invariants of the general 2-dimensional tensor $\mathbf{R} = \mathbf{NRN}$ are:

$$
\begin{aligned}
&I_1 = \operatorname{tr}\mathbf{R} = 2H_R = \mathbf{R} \cdot \mathbf{N} && : \ \textit{Trace} \text{ of } \mathbf{R} \\
&I_2 = \det(\mathbf{R} + \mathbf{n} \otimes \mathbf{n}) = K_R = -\frac{1}{2}\mathbf{R} \cdot \mathbf{E}\mathbf{R}^T\mathbf{E} && : \ \textit{(Minor-)determinant} \text{ of } \mathbf{R} \\
&I_3 = \det\mathbf{R} = 0 && : \ \textit{Determinant} \text{ of } \mathbf{R}
\end{aligned}
\tag{1.53}
$$

Note: The third invariant $I_3 = \det\mathbf{R}$ is zero because the 2-dimensional tensor $\mathbf{R}$ is considered in the 3-dimensional space. If $\mathbf{R}$ represents the curvature tensor of some curved surface in space, then H_R represents the mean curvature and K_R the Gaussian curvature of the surface.

1.2 Derivatives in the 3-dimensional space

We have seen before that the position vector $\mathbf{r}$ of a point P in space is a function of the curvilinear coordinates θ^i (i=1,2,3), which means

$$
\theta^1, \theta^2, \theta^3 \quad \rightarrow \quad \mathbf{r} = \mathbf{r}(\theta^1, \theta^2, \theta^3) \tag{1.54}
$$

Scalars, vectors and tensors may also be functions of the position vector $\mathbf{r}$, e.g. a scalar ϕ, a vector $\mathbf{u}$, a tensor $\mathbf{T}$ and higher-rank tensors $\boldsymbol{\mathcal{B}}$, etc ... can be written as function of the vector variable $\mathbf{r}(\theta^1, \theta^2, \theta^3)$

$$\mathrm{P} : \quad \theta^1, \theta^2, \theta^3 \quad \rightarrow \quad \mathbf{r} = \mathbf{r}(\theta^1, \theta^2, \theta^3) \quad \rightarrow \quad \begin{cases} \phi = \phi(\mathbf{r}) = \phi[\mathbf{r}(\theta^1, \theta^2, \theta^3)] \\ \mathbf{u} = \mathbf{u}(\mathbf{r}) = \mathbf{u}[\mathbf{r}(\theta^1, \theta^2, \theta^3)] \\ \mathbf{T} = \mathbf{T}(\mathbf{r}) = \mathbf{T}[\mathbf{r}(\theta^1, \theta^2, \theta^3)] \\ \boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{B}}(\mathbf{r}) = \boldsymbol{\mathcal{B}}[\mathbf{r}(\theta^1, \theta^2, \theta^3)] \\ \dots \end{cases} \tag{1.55}$$

In the neighborhood of point P, we consider another point $\bar{\mathrm{P}}$ of position vector $\bar{\mathbf{r}} = \mathbf{r} + \Delta\mathbf{r}$. We have

$$\begin{aligned} &\phi(\bar{\mathbf{r}}) = \phi(\mathbf{r}) + \mathrm{d}\phi + \frac{1}{2!}\mathrm{d}^2\phi + \frac{1}{3!}\mathrm{d}^3\phi + \dots \\ &\mathbf{u}(\bar{\mathbf{r}}) = \mathbf{u}(\mathbf{r}) + \mathrm{d}\mathbf{u} + \frac{1}{2!}\mathrm{d}^2\mathbf{u} + \frac{1}{3!}\mathrm{d}^3\mathbf{u} + \dots \\ &\mathbf{T}(\bar{\mathbf{r}}) = \mathbf{T}(\mathbf{r}) + \mathrm{d}\mathbf{T} + \frac{1}{2!}\mathrm{d}^2\mathbf{T} + \frac{1}{3!}\mathrm{d}^3\mathbf{T} + \dots \\ &\boldsymbol{\mathcal{B}}(\bar{\mathbf{r}}) = \boldsymbol{\mathcal{B}}(\mathbf{r}) + \mathrm{d}\boldsymbol{\mathcal{B}} + \frac{1}{2!}\mathrm{d}^2\boldsymbol{\mathcal{B}} + \frac{1}{3!}\mathrm{d}^3\boldsymbol{\mathcal{B}} + \dots \\ &\dots \end{aligned} \tag{1.56}$$

Of course this applies also to vector $\mathbf{r}$

$$\bar{\mathbf{r}} = \mathbf{r} + \Delta\mathbf{r} = \mathbf{r} + \mathrm{d}\mathbf{r} + \frac{1}{2!}\mathrm{d}^2\mathbf{r} + \frac{1}{3!}\mathrm{d}^3\mathbf{r} + \dots \tag{1.57}$$

The 3-dimensional derivative operator $\boldsymbol{\nabla}$ is defined as follows

$$\boldsymbol{\nabla} = \mathbf{g}^j \frac{\partial}{\partial\theta^j} \tag{1.58}$$

The derivative operator $\boldsymbol{\nabla}$, as is usual in analysis, always acts from the left to the right. The derivative of a scalar can be considered as trivial compared with the derivative of a vector or a tensor. To take the derivative of a vector or a tensor, it is necessary to derive not only the components, but also the associated base vectors and tensors. For example, one may consider the derivative of a contravariant base vector $\mathbf{g}^i$, which reads

$$\frac{\partial\mathbf{g}^i}{\partial\theta^j} = \mathbf{g}^i_{\ ,j} = -\Gamma^i_{j\ell}\mathbf{g}^\ell \qquad ; \qquad (i, j, k, \ell = 1, 2, 3) \tag{1.59}$$

The result is a new vector, which can be written as a linear combination of the contravariant base vectors $\mathbf{g}^\ell$ and which covariant components $-\Gamma^i_{j\ell}$ are called Christoffel symbols of second kind. By contracting that vector with $\mathbf{g}_k$, that means building the vector product, we get

$$\Gamma^i_{jk} = -\mathbf{g}^i_{\ ,j} \cdot \mathbf{g}_k = \mathbf{g}^i \cdot \mathbf{g}_{k,j} = \mathbf{g}^i \cdot \mathbf{g}_{j,k} = \Gamma^i_{kj} \quad ; \quad \begin{aligned} &\frac{\partial}{\partial\theta^j}(\mathbf{g}^i \cdot \mathbf{g}_k) = \mathbf{g}^i_{\ ,j} \cdot \mathbf{g}_k + \mathbf{g}^i \cdot \mathbf{g}_{k,j} = 0 \\ &\mathbf{g}_{k,j} = \frac{\partial\mathbf{g}_k}{\partial\theta^j} = \frac{\partial^2\mathbf{r}}{\partial\theta^j\partial\theta^k} = \frac{\partial\mathbf{g}_j}{\partial\theta^k} = \mathbf{g}_{j,k} \end{aligned} \tag{1.60}$$

The first and second-order total differentials of $\mathbf{r}$ read

$$\mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \theta^i}\mathrm{d}\theta^i = \mathbf{g}_i \mathrm{d}\theta^i \quad ; \quad \mathrm{d}^2\mathbf{r} = \frac{\partial^2 \mathbf{r}}{\partial \theta^i \partial \theta^j}\mathrm{d}\theta^i \mathrm{d}\theta^j = \mathbf{g}_{i,j}\mathrm{d}\theta^i \mathrm{d}\theta^j \quad ; \quad \ldots \tag{1.61}$$

Because we want to bypass the notation with indices, we introduce the following derivative rules

$$\begin{aligned}
\mathrm{d}\phi &= \frac{\partial \phi}{\partial \theta^i}\mathrm{d}\theta^i = \mathrm{d}\theta^i \mathbf{g}_i \cdot \mathbf{g}^j \frac{\partial \phi}{\partial \theta^j} = \mathrm{d}\mathbf{r} \cdot \boldsymbol{\nabla}\phi \\
\mathrm{d}\mathbf{u} &= \frac{\partial \mathbf{u}}{\partial \theta^i}\mathrm{d}\theta^i = \mathrm{d}\theta^i (\mathbf{g}_i \cdot \mathbf{g}^j)\frac{\partial \mathbf{u}}{\partial \theta^j} = \mathrm{d}\theta^i \mathbf{g}_i \left(\mathbf{g}^j \frac{\partial}{\partial \theta^j} \otimes \mathbf{u}\right) = \mathrm{d}\mathbf{r}(\boldsymbol{\nabla} \otimes \mathbf{u}) \\
\mathrm{d}\mathbf{T} &= \frac{\partial \mathbf{T}}{\partial \theta^i}\mathrm{d}\theta^i = \mathrm{d}\theta^i (\mathbf{g}_i \cdot \mathbf{g}^j)\frac{\partial \mathbf{T}}{\partial \theta^j} = \mathrm{d}\theta^i \mathbf{g}_i \left(\mathbf{g}^j \frac{\partial}{\partial \theta^j} \otimes \mathbf{T}\right) = \mathrm{d}\mathbf{r}(\boldsymbol{\nabla} \otimes \mathbf{T}) \\
&\ldots
\end{aligned} \tag{1.62}$$

For example, the tensor $\boldsymbol{\nabla} \otimes \mathbf{u}$ can be explicitly written with components and base vectors. With ${\mathbf{g}^i}_{,j} = -\Gamma^i_{jk}\mathbf{g}^k$ and $\mathbf{u} = u_i \mathbf{g}^i$ $(i, j, k = 1, 2, 3)$, we have

$$\begin{aligned}
\boldsymbol{\nabla} \otimes \mathbf{u} &= \mathbf{g}^j \frac{\partial}{\partial \theta^j} \otimes (u_i \mathbf{g}^i) = u_{i,j}\mathbf{g}^j \otimes \mathbf{g}^i + u_i \mathbf{g}^j \otimes {\mathbf{g}^i}_{,j} \\
&= (u_{j,i} - \Gamma^k_{ij}u_k)\mathbf{g}^i \otimes \mathbf{g}^j = u_{j;i}\mathbf{g}^i \otimes \mathbf{g}^j
\end{aligned} \tag{1.63}$$

where $u_{j;i} = u_{j,i} - \Gamma^k_{ij}u_k$ are the covariant derivatives of the components of the vector $\mathbf{u}$ in space (not to be confused with the partial derivative $u_{j,i} = \partial u_j / \partial \theta^i$). With the "basic rules" (1.62), we write

$$\begin{aligned}
&\mathrm{d}^2\phi = \mathrm{d}(\mathrm{d}\phi) = \mathrm{d}(\mathrm{d}\mathbf{r} \cdot \boldsymbol{\nabla}\phi) = \mathrm{d}^2\mathbf{r} \cdot \boldsymbol{\nabla}\phi + \mathrm{d}\mathbf{r} \cdot \mathrm{d}(\boldsymbol{\nabla}\phi) = \mathrm{d}\mathbf{r} \cdot (\boldsymbol{\nabla} \otimes \boldsymbol{\nabla}\phi)\mathrm{d}\mathbf{r} + \mathrm{d}^2\mathbf{r} \cdot \boldsymbol{\nabla}\phi \\
&\mathrm{d}^2\mathbf{u} = \mathrm{d}(\mathrm{d}\mathbf{u}) = \mathrm{d}[\mathrm{d}\mathbf{r}(\boldsymbol{\nabla} \otimes \mathbf{u})] = (\mathrm{d}\mathbf{r} \otimes \mathrm{d}\mathbf{r}) \cdot (\boldsymbol{\nabla} \otimes \boldsymbol{\nabla} \otimes \mathbf{u}) + \mathrm{d}^2\mathbf{r}(\boldsymbol{\nabla} \otimes \mathbf{u}) \\
&\mathrm{d}^2\mathbf{T} = \mathrm{d}(\mathrm{d}\mathbf{T}) = \mathrm{d}[\mathrm{d}\mathbf{r}(\boldsymbol{\nabla} \otimes \mathbf{T})] = (\mathrm{d}\mathbf{r} \otimes \mathrm{d}\mathbf{r}) \cdot (\boldsymbol{\nabla} \otimes \boldsymbol{\nabla} \otimes \mathbf{T}) + \mathrm{d}^2\mathbf{r}(\boldsymbol{\nabla} \otimes \mathbf{T}) \\
&\ldots
\end{aligned} \tag{1.64}$$

Derivatives of higher-order are deduced in a similar way. Here, we only prove the $\mathrm{d}^2\phi$ derivative. Because the θ^i are independent variables, we have $\mathrm{d}^2\theta^i = \mathrm{d}^3\theta^i = \ldots = 0$ and

$$\begin{aligned}
\mathrm{d}^2\phi &= \mathrm{d}(\mathrm{d}\phi) = \mathrm{d}\left(\frac{\partial \phi}{\partial \theta^i}\right)\mathrm{d}\theta^i + \underbrace{\frac{\partial \phi}{\partial \theta^i}\mathrm{d}^2\theta^i}_{=0} = \underline{\underline{\frac{\partial^2 \phi}{\partial \theta^i \partial \theta^j}\mathrm{d}\theta^i \mathrm{d}\theta^j}} \\
&= \underbrace{\mathrm{d}\mathbf{r}}_{\mathbf{g}_\ell \mathrm{d}\theta^\ell} \cdot \underbrace{(\boldsymbol{\nabla} \otimes \boldsymbol{\nabla}\phi)}_{\mathbf{g}^i \frac{\partial}{\partial \theta^i} \otimes \left(\mathbf{g}^j \frac{\partial \phi}{\partial \theta^j}\right)} \underbrace{\mathrm{d}\mathbf{r}}_{\mathbf{g}_k \mathrm{d}\theta^k} + \underbrace{\mathrm{d}^2\mathbf{r}}_{\mathbf{g}_{i,k}\mathrm{d}\theta^i \mathrm{d}\theta^k} \cdot \underbrace{\boldsymbol{\nabla}\phi}_{\mathbf{g}^j \frac{\partial \phi}{\partial \theta^j}} \\
&= \underbrace{\left({\mathbf{g}^j}_{,i}\frac{\partial \phi}{\partial \theta^j} + \mathbf{g}^j \frac{\partial^2 \phi}{\partial \theta^i \partial \theta^j}\right) \cdot \mathbf{g}_k}_{-\Gamma^j_{ik}\phi_{,j} + {\delta^j}_k \frac{\partial^2 \phi}{\partial \theta^i \partial \theta^j}} \mathrm{d}\theta^i \mathrm{d}\theta^k + \underbrace{\mathbf{g}_{i,k} \cdot \mathbf{g}^j}_{\Gamma^j_{ik}} \underbrace{\frac{\partial \phi}{\partial \theta^j}}_{\phi_{,j}} \mathrm{d}\theta^i \mathrm{d}\theta^k \\
&= \underline{\underline{\frac{\partial^2 \phi}{\partial \theta^i \partial \theta^j}\mathrm{d}\theta^i \mathrm{d}\theta^j}} \qquad \square \text{ qed}
\end{aligned} \tag{1.65}$$

The role of the position vector $\mathbf{r}$, also called vector coordinate, is very important. As we can see, we have replaced the three independent scalar variables θ^i with the vector variable $\mathbf{r}$. This implies that scalar, vector and tensor functions in space can now be written as function of $\mathbf{r}$. Thus, the vector coordinate $\mathbf{r}$ makes the bridge between the original variables θ^i and the different tensor functions in space. These rules also apply to the position vector $\mathbf{r}$. We can write:

$$\mathrm{d}\mathbf{r} = \mathrm{d}\mathbf{r}(\boldsymbol{\nabla} \otimes \mathbf{r}) \qquad \Rightarrow \qquad \boldsymbol{\nabla} \otimes \mathbf{r} = \mathbf{I} \tag{1.66}$$

Proof:

$$\boldsymbol{\nabla} \otimes \mathbf{r} = \mathbf{g}^i \frac{\partial}{\partial \theta^i} \otimes \mathbf{r} = \mathbf{g}^i \otimes \frac{\partial \mathbf{r}}{\partial \theta^i} = \mathbf{g}^i \otimes \mathbf{g}_i = \mathbf{I} \qquad \square \text{ qed} \tag{1.67}$$

Because $\mathbf{I}$ and $\boldsymbol{\mathcal{E}}$ are constant in space, their derivatives read:

$$\boldsymbol{\nabla} \otimes \mathbf{I} = 0 \qquad ; \qquad \boldsymbol{\nabla} \otimes \boldsymbol{\mathcal{E}} = 0 \tag{1.68}$$

Example

In a cartesian system with constant base vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, we may choose $x = \theta^1$, $y = \theta^2$ and $z = \theta^3$. Writing in components, we get

$$\mathbf{r} = \mathbf{r}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \hat{=} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \quad ; \quad \frac{\partial \mathbf{r}}{\partial x} = \mathbf{i} \quad ; \quad \frac{\partial \mathbf{r}}{\partial y} = \mathbf{j} \quad ; \quad \frac{\partial \mathbf{r}}{\partial z} = \mathbf{k} \tag{1.69}$$

$$\phi = \phi(\mathbf{r}) = \phi(x,y,z) \qquad ; \qquad \mathbf{u} = \mathbf{u}(\mathbf{r}) = \mathbf{u}(x,y,z) \hat{=} \begin{Bmatrix} u_x(x,y,z) \\ u_y(x,y,z) \\ u_z(x,y,z) \end{Bmatrix} \tag{1.70}$$

$$\mathrm{d}\mathbf{r} = \mathrm{d}x\mathbf{i} + \mathrm{d}y\mathbf{j} + \mathrm{d}z\mathbf{k} = \frac{\partial \mathbf{r}}{\partial x}\mathrm{d}x + \frac{\partial \mathbf{r}}{\partial y}\mathrm{d}y + \frac{\partial \mathbf{r}}{\partial z}\mathrm{d}z \hat{=} \begin{Bmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{Bmatrix} \tag{1.71}$$

$$\boldsymbol{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \hat{=} \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{Bmatrix} \tag{1.72}$$

$$\mathrm{d}\phi = \mathrm{d}\mathbf{r} \cdot \boldsymbol{\nabla}\phi \hat{=} \begin{pmatrix} \mathrm{d}x & \mathrm{d}y & \mathrm{d}z \end{pmatrix} \cdot \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{Bmatrix} \phi = \frac{\partial \phi}{\partial x}\mathrm{d}x + \frac{\partial \phi}{\partial y}\mathrm{d}y + \frac{\partial \phi}{\partial z}\mathrm{d}z \tag{1.73}$$

$$
\begin{aligned}
\mathrm{d}\mathbf{u} = \mathrm{d}\mathbf{r}(\boldsymbol{\nabla}\otimes\mathbf{u}) \,\hat{=}\, \begin{Bmatrix} \mathrm{d}u_x \\ \mathrm{d}u_y \\ \mathrm{d}u_z \end{Bmatrix} &= (\,\mathrm{d}x \quad \mathrm{d}y \quad \mathrm{d}z\,) \left[\begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{Bmatrix} \otimes (\,u_x \quad u_y \quad u_z\,) \right] \\
\hat{=}\, (\,\mathrm{d}x \quad \mathrm{d}y \quad \mathrm{d}z\,) \begin{bmatrix} \dfrac{\partial u_x}{\partial x} & \dfrac{\partial u_y}{\partial x} & \dfrac{\partial u_z}{\partial x} \\ \dfrac{\partial u_x}{\partial y} & \dfrac{\partial u_y}{\partial y} & \dfrac{\partial u_z}{\partial y} \\ \dfrac{\partial u_x}{\partial z} & \dfrac{\partial u_y}{\partial z} & \dfrac{\partial u_z}{\partial z} \end{bmatrix} &= \begin{Bmatrix} \dfrac{\partial u_x}{\partial x}\mathrm{d}x + \dfrac{\partial u_x}{\partial y}\mathrm{d}y + \dfrac{\partial u_x}{\partial z}\mathrm{d}z \\ \dfrac{\partial u_y}{\partial x}\mathrm{d}x + \dfrac{\partial u_y}{\partial y}\mathrm{d}y + \dfrac{\partial u_y}{\partial z}\mathrm{d}z \\ \dfrac{\partial u_z}{\partial x}\mathrm{d}x + \dfrac{\partial u_z}{\partial y}\mathrm{d}y + \dfrac{\partial u_z}{\partial z}\mathrm{d}z \end{Bmatrix}
\end{aligned} \tag{1.74}
$$

$$
\mathbf{I} = \boldsymbol{\nabla}\otimes\mathbf{r} \,\hat{=}\, \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{Bmatrix} \otimes (\,x \quad y \quad z\,) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.75}
$$

Derivative rules for standard tensor expressions

Scalar, vector and other general tensor expressions in space are often build on other scalar, vector and tensor functions. For example, a scalar can be build on a scalar product of two vectors. To derive such expressions one may use the following rules

$$
\begin{aligned}
&\underline{\boldsymbol{\nabla}(\text{scalar}) \rightarrow \text{vector}} \\
&\boldsymbol{\nabla}(\mathbf{u}\cdot\mathbf{v}) = (\boldsymbol{\nabla}\otimes\mathbf{u})\mathbf{v} + (\boldsymbol{\nabla}\otimes\mathbf{v})\mathbf{u} \\
&\boldsymbol{\nabla}(\mathbf{T}\cdot\mathbf{R}) = (\boldsymbol{\nabla}\otimes\mathbf{T})\cdot\mathbf{R} + (\boldsymbol{\nabla}\otimes\mathbf{R})\cdot\mathbf{T}
\end{aligned} \tag{1.76}
$$

$$
\begin{aligned}
&\underline{\boldsymbol{\nabla}\cdot(\text{vector}) \rightarrow \text{scalar}} \\
&\boldsymbol{\nabla}\cdot(\phi\mathbf{u}) = \boldsymbol{\nabla}\phi\cdot\mathbf{u} + \phi\boldsymbol{\nabla}\cdot\mathbf{u} \\
&\boldsymbol{\nabla}\cdot(\mathbf{T}\mathbf{u}) = \boldsymbol{\nabla}\mathbf{T}\cdot\mathbf{u} + \mathbf{T}^T\cdot(\boldsymbol{\nabla}\otimes\mathbf{u})
\end{aligned} \tag{1.77}
$$

$$
\begin{aligned}
&\underline{\boldsymbol{\nabla}\otimes(\text{vector}) \rightarrow \text{tensor}} \\
&\boldsymbol{\nabla}\otimes(\phi\mathbf{u}) = \boldsymbol{\nabla}\phi\otimes\mathbf{u} + \phi\boldsymbol{\nabla}\otimes\mathbf{u} \\
&\boldsymbol{\nabla}\otimes(\mathbf{T}\mathbf{u}) = (\boldsymbol{\nabla}\otimes\mathbf{T})\mathbf{u} + (\boldsymbol{\nabla}\otimes\mathbf{u})\mathbf{T}^T
\end{aligned} \tag{1.78}
$$

$$
\begin{aligned}
&\underline{\boldsymbol{\nabla}(\text{tensor}) \rightarrow \text{vector}} \\
&\boldsymbol{\nabla}(\phi\mathbf{T}) = (\boldsymbol{\nabla}\phi)\mathbf{T} + \phi\boldsymbol{\nabla}\mathbf{T} \\
&\boldsymbol{\nabla}(\mathbf{u}\otimes\mathbf{v}) = (\boldsymbol{\nabla}\cdot\mathbf{u})\mathbf{v} + \mathbf{u}(\boldsymbol{\nabla}\otimes\mathbf{v}) \\
&\boldsymbol{\nabla}(\mathbf{T}\mathbf{R}) = (\boldsymbol{\nabla}\mathbf{T})\mathbf{R} + \mathbf{T}^T\cdot(\boldsymbol{\nabla}\otimes\mathbf{R}) \\
&\boldsymbol{\nabla}(\boldsymbol{\mathcal{B}}\mathbf{u}) = (\boldsymbol{\nabla}\boldsymbol{\mathcal{B}})\mathbf{u} + (\boldsymbol{\nabla}\otimes\mathbf{u})^T\cdot\boldsymbol{\mathcal{B}})^T
\end{aligned} \tag{1.79}
$$

$$
\begin{array}{l}
\underline{\boldsymbol{\nabla}\otimes(\text{second-rank tensor})\rightarrow\text{third-rank tensor}}\\
\boldsymbol{\nabla}\otimes(\phi\mathbf{T})=\boldsymbol{\nabla}\phi\otimes\mathbf{T}+\phi\boldsymbol{\nabla}\otimes\mathbf{T}\\
\boldsymbol{\nabla}\otimes(\mathbf{u}\otimes\mathbf{v})=\boldsymbol{\nabla}\otimes\mathbf{u}\otimes\mathbf{v}+\boldsymbol{\nabla}\otimes\mathbf{v}\otimes\mathbf{u})^T\\
\boldsymbol{\nabla}\otimes(\mathbf{TR})=(\boldsymbol{\nabla}\otimes\mathbf{T})\mathbf{R}+\boldsymbol{\nabla}\otimes\mathbf{R})^T\mathbf{T}^T]^T\\
\boldsymbol{\nabla}\otimes(\boldsymbol{\mathcal{B}}\mathbf{u})=(\boldsymbol{\nabla}\otimes\boldsymbol{\mathcal{B}})\mathbf{u}+(\boldsymbol{\nabla}\otimes\mathbf{u})\boldsymbol{\mathcal{B}}^T)^T
\end{array}
\tag{1.80}
$$

$$
\begin{array}{l}
\underline{\boldsymbol{\nabla}\otimes(\text{third-rank tensor})\rightarrow\text{fourth-rank tensor}}\\
\boldsymbol{\nabla}\otimes(\boldsymbol{\mathcal{B}}\mathbf{T})=(\boldsymbol{\nabla}\otimes\boldsymbol{\mathcal{B}})\mathbf{T}+\boldsymbol{\nabla}\otimes\mathbf{T})^T\boldsymbol{\mathcal{B}}^T]]^T\\
\boldsymbol{\nabla}\otimes(\mathbf{T}\boldsymbol{\mathcal{B}})=(\boldsymbol{\nabla}\otimes\mathbf{T})\boldsymbol{\mathcal{B}}+\boldsymbol{\nabla}\otimes\boldsymbol{\mathcal{B}}))^T\mathbf{T}^T]]^T
\end{array}
\tag{1.81}
$$

where the signs $))^T$ and $]]^T$ represent a semi-transposition between the second and fourth factor in the above quadriadic products, as shown by the transposition rule $\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}\otimes\mathbf{d}))^T=\mathbf{a}\otimes\mathbf{d}\otimes\mathbf{c}\otimes\mathbf{b}$ for fourth-rank tensors. With the shortcut $\partial_i=\partial/\partial\theta^i$, we only prove here the following derivative rule:

$$
\begin{aligned}
\boldsymbol{\nabla}\otimes(\mathbf{Tu})&=\mathbf{g}^i\partial_i\otimes(\mathbf{Tu})=\mathbf{g}^i\otimes\partial_i(\mathbf{Tu})=\mathbf{g}^i\otimes\partial_i(\mathbf{u}\mathbf{T}^T)\\
&=\mathbf{g}^i\otimes\partial_i\mathbf{T}\mathbf{u}+\mathbf{g}^i\otimes\partial_i\mathbf{u}\mathbf{T}^T=(\mathbf{g}^i\otimes\partial_i\mathbf{T})\mathbf{u}+(\mathbf{g}^i\otimes\partial_i\mathbf{u})\mathbf{T}^T\\
&=(\boldsymbol{\nabla}\otimes\mathbf{T})\mathbf{u}+(\boldsymbol{\nabla}\otimes\mathbf{u})\mathbf{T}^T \qquad \square\ \text{qed}
\end{aligned}
\tag{1.82}
$$

Derivatives of lengths, directions and projectors

Derivatives of lengths, directions (always represented by a unit vector) and projectors relatively to some collineation center (i.e a fixed point in space) may also be calculated. With the fixed point P_0 and the variable point P in space, we define

$$
\begin{array}{lclcl}
\mathbf{r}_0=\overrightarrow{\mathrm{OP}_0}=\not{c} & ; & \mathbf{p}=p\mathbf{h}=\mathbf{r}-\mathbf{r}_0 & ; & \mathbf{H}=\mathbf{I}-\mathbf{h}\otimes\mathbf{h}\\
\mathbf{r}=\overrightarrow{\mathrm{OP}} & & p=\mathbf{p}\cdot\mathbf{h} & & \mathbf{h}\cdot\mathbf{h}=1
\end{array}
\tag{1.83}
$$

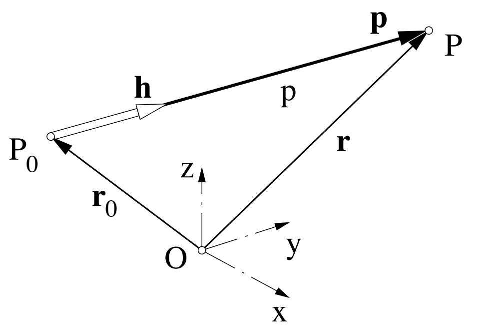

Fig. 1.5: Derivatives of a length and a direction in space

With

$$\boldsymbol{\nabla}(\mathbf{h}\cdot\mathbf{h})=2(\boldsymbol{\nabla}\otimes\mathbf{h})\mathbf{h}=0$$

$$\boldsymbol{\nabla} \otimes \mathbf{p} = \boldsymbol{\nabla} \otimes \mathbf{r} - \boldsymbol{\nabla} \otimes \mathbf{r}_0 = \boldsymbol{\nabla} \otimes \mathbf{r} = \mathbf{I} \tag{1.84}$$

the derivatives of the length p, the direction $\mathbf{h}$ and the normal projector $\mathbf{H}$ read

$$\boldsymbol{\nabla} p = \boldsymbol{\nabla}(\mathbf{p}\cdot\mathbf{h}) = (\boldsymbol{\nabla}\otimes\mathbf{p})\mathbf{h} + (\boldsymbol{\nabla}\otimes\mathbf{h})\mathbf{p} = \mathbf{I}\mathbf{h} + p(\boldsymbol{\nabla}\otimes\mathbf{h})\mathbf{h} = \mathbf{h} \tag{1.85}$$

$$\begin{aligned}\boldsymbol{\nabla}\otimes\boldsymbol{\nabla} p &= \boldsymbol{\nabla}\otimes\mathbf{h} = \boldsymbol{\nabla}\otimes\left(\frac{1}{p}\mathbf{p}\right) = \boldsymbol{\nabla}\frac{1}{p}\otimes\mathbf{p} + \frac{1}{p}\boldsymbol{\nabla}\otimes\mathbf{p}\\ &= -\frac{1}{p^2}\boldsymbol{\nabla} p\otimes\mathbf{p} + \frac{1}{p}\mathbf{I} = \frac{1}{p}(\mathbf{I} - \mathbf{h}\otimes\mathbf{h}) = \frac{1}{p}\mathbf{H}\end{aligned} \tag{1.86}$$

$$\begin{aligned}\boldsymbol{\nabla}\otimes\boldsymbol{\nabla}\otimes\boldsymbol{\nabla} p &= \boldsymbol{\nabla}\otimes\boldsymbol{\nabla}\otimes\mathbf{h} = \boldsymbol{\nabla}\otimes\left(\frac{1}{p}\mathbf{H}\right) = \boldsymbol{\nabla}\frac{1}{p}\otimes\mathbf{H} + \frac{1}{p}\boldsymbol{\nabla}\otimes\mathbf{H}\\ &= -\frac{1}{p^2}\boldsymbol{\nabla} p\otimes\mathbf{H} + \frac{1}{p}\boldsymbol{\nabla}\otimes\mathbf{I} - \frac{1}{p}\boldsymbol{\nabla}\otimes(\mathbf{h}\otimes\mathbf{h})\\ &= -\frac{1}{p^2}\mathbf{h}\otimes\mathbf{H} - \frac{1}{p}\boldsymbol{\nabla}\otimes\mathbf{h}\otimes\mathbf{h} - \frac{1}{p}\boldsymbol{\nabla}\otimes\mathbf{h}\otimes\mathbf{h})^T\\ &= -\frac{1}{p^2}[\mathbf{h}\otimes\mathbf{H} + \mathbf{H}\otimes\mathbf{h})^T + \mathbf{H}\otimes\mathbf{h}] = -\frac{1}{p^2}\boldsymbol{\mathcal{H}}\end{aligned} \tag{1.87}$$

$$\begin{aligned}\boldsymbol{\nabla}\otimes\boldsymbol{\nabla}\otimes\boldsymbol{\nabla}\otimes\boldsymbol{\nabla} p &= \boldsymbol{\nabla}\otimes\boldsymbol{\nabla}\otimes\boldsymbol{\nabla}\otimes\mathbf{h} = \boldsymbol{\nabla}\otimes\boldsymbol{\nabla}\otimes\left(\frac{1}{p}\mathbf{H}\right)\\ &= \boldsymbol{\nabla}\otimes\left(-\frac{1}{p^2}\boldsymbol{\mathcal{H}}\right) = \frac{1}{p^3}\boldsymbol{H}\end{aligned} \tag{1.88}$$

where $\boldsymbol{\mathcal{H}} = \mathbf{h}\otimes\mathbf{H} + \mathbf{H}\otimes\mathbf{h})^T + \mathbf{H}\otimes\mathbf{h}$ is a so-called superprojector (third-rank tensor) and $\boldsymbol{H} = \mathbf{h}\otimes\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{H}}\otimes\mathbf{h} + {}^T(\mathbf{h}\otimes\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{H}}\otimes\mathbf{h})^T - \mathbf{H}\otimes\mathbf{H} - (\mathbf{H}\otimes\mathbf{H})^T - \mathbf{H}\otimes\mathbf{H}))^T$ is a so-called hyperprojector (fourth-rank tensor). Both tensors $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{H}$ are symmetric relatively to all factors. For example, by applying the unit vector $\mathbf{h}$ on the superprojector $\boldsymbol{\mathcal{H}}$, one get the normal projector $\mathbf{H}$. Thus, the derivative of a length gives a direction parallel to the length, the derivative of a direction gives a normal projector, the derivative of a normal projector a superprojector and so on.

1.3 Derivatives on a 2-dimensional curved surface in space

By keeping the third independent variable θ^3 constant, the position vector $\mathbf{r} = \mathbf{r}(\theta^1, \theta^2)$ of a variable point P describes a curved surface $\mathbb{A}^2$ in the space $\mathbb{R}^3$:

$$\mathrm{P}\in\mathbb{A}^2 \; : \qquad \theta^1, \theta^2, \theta^3\Big|_{\theta^3 = ¢} \qquad \rightarrow \qquad \mathbf{r} = \mathbf{r}(\theta^1, \theta^2, \theta^3)\Big|_{\theta^3 = ¢} = \mathbf{r}(\theta^1, \theta^2) \tag{1.89}$$

Thus, a curved surface in space can be described by two independent curvilinear coordinates θ^1 and θ^2. Scalar, vector and tensor functions in the space $\mathbb{R}^3$ may be written either as function of $\mathbf{r}(\theta^1, \theta^2, \theta^3)$ or as function of $\mathbf{r}(\theta^1, \theta^2)$ on the corresponding curved surface $\mathbb{A}^2$. Because surfaces are located in space, tensor functions described on 2-dimensional surfaces may also be considered from "outside", which means from the 3-dimensional space and only thereafter, by a suitable "projection", be again described on the surface. This allows a great flexibility and enables to clarify complicated calculations on curved surfaces. Let us now explain how to deal with derivatives on curved surface.

We consider a curved surface $\mathbb{A}^2$ in space, which is given by the position vector $\mathbf{r} = \mathbf{r}(\theta^1, \theta^2)$. We know that the position vector was originally dependent of the three variables θ^i $(i = 1, 2, 3)$, but is now only dependent of both variables θ^α $(\alpha = 1, 2)$, because $\theta^3 = \not{c}$ is constant on the surface.

$$\mathbb{A}^2 \; : \qquad \theta^1, \theta^2 \qquad \rightarrow \qquad \mathbf{r} = \mathbf{r}(\theta^1, \theta^2) \tag{1.90}$$

The total differential reads

$$\mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \theta^1}\mathrm{d}\theta^1 + \frac{\partial \mathbf{r}}{\partial \theta^2}\mathrm{d}\theta^2 = \sum_{\alpha=1}^{2} \frac{\partial \mathbf{r}}{\partial \theta^\alpha}\mathrm{d}\theta^\alpha = \frac{\partial \mathbf{r}}{\partial \theta^\alpha}\mathrm{d}\theta^\alpha = \mathbf{r}_{,\alpha}\mathrm{d}\theta^\alpha = \mathbf{a}_\alpha \mathrm{d}\theta^\alpha \tag{1.91}$$

where the repeated greek index α means a sum on α from 1 to 2. The variable vectors $\mathbf{a}_\alpha = \partial \mathbf{r} / \partial \theta^\alpha$ are covariant base vectors (only dependent of θ^1 and θ^2) and lie on the plane tangential to the curved surface in point P (the so-called tangential plane).

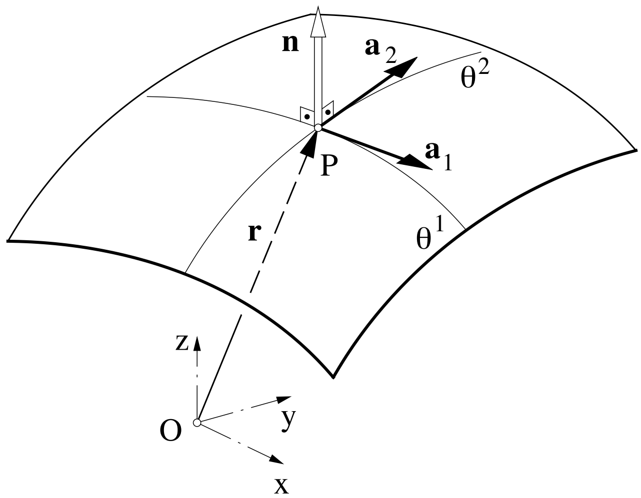

Fig. 1.6: Curved surface in space

If we again consider the position vector $\mathbf{r}$ with the additional variable θ^3, we can write the following covariant base vectors $\mathbf{g}_i$ (all dependent of θ^1, θ^2 and θ^3)

$$\mathbf{g}_\alpha = \frac{\partial \mathbf{r}}{\partial \theta^\alpha} \qquad ; \qquad \mathbf{g}_3 = \frac{\partial \mathbf{r}}{\partial \theta^3} \tag{1.92}$$

Obviously, we have $\mathbf{g}_\alpha = \mathbf{a}_\alpha$ on the surface $\mathbb{A}^2$. The associated contravariant base vectors $\mathbf{g}^i$ (dependent of θ^1, θ^2 and θ^3 with $i = 1, 2, 3$) are still defined in the usual way

$$\mathbf{g}^\beta \cdot \mathbf{g}_\alpha = \delta^\beta_\alpha = \begin{cases} 1 & \text{if } \beta = \alpha \\ 0 & \text{if } \beta \neq \alpha \end{cases} \quad ; \quad \begin{array}{l} \mathbf{g}^\beta \cdot \mathbf{g}_3 = \mathbf{g}^3 \cdot \mathbf{g}_\alpha = 0 \\ \mathbf{g}^3 \cdot \mathbf{g}_3 = 1 \end{array} \quad ; \quad (\alpha, \beta = 1, 2) \tag{1.93}$$

On the surface $\mathbb{A}^2$, the third contravariant base vector $\mathbf{g}^3$ is defined perpendicular to the tangential plane because

$$\mathbf{g}^3 \cdot \mathbf{g}_\alpha = \mathbf{g}^3 \cdot \mathbf{a}_\alpha = 0 \tag{1.94}$$

Important note: Because of the choice of the curvilinear coordinates θ^i, which means the choice of the functions $x = x(\theta^1, \theta^2, \theta^3)$, $y = y(\theta^1, \theta^2, \theta^3)$ and $z = z(\theta^1, \theta^2, \theta^3)$, the direction of the third covariant base vector $\mathbf{g}_3$ is generally not perpendicular to the base vectors $\mathbf{a}_\alpha$. Therefore, the contravariant base vectors $\mathbf{g}^\beta$ are not situated in the tangential plane of the curved surface $\mathbb{A}^2$. Consequently, a suitable definition must be now introduced for the contravariant base vectors $\mathbf{a}^\beta$ (only dependent of θ^1 and θ^2) in order that they always remain in the tangential plane of the curved surface $\mathbb{A}^2$, together with the covariant base vectors $\mathbf{a}_\alpha$.

Thus, we first define the unit normal $\mathbf{n}$ to the surface as follows

$$\mathbf{n} \cdot \mathbf{a}_\alpha = 0 \qquad ; \qquad \mathbf{n} \cdot \mathbf{n} = 1 \qquad ; \qquad \mathbf{n} = \mathbf{n}(\theta^1, \theta^2) \tag{1.95}$$

Obviously, the unit normal $\mathbf{n}$ is perpendicular to the tangential plane of the surface. On the surface $\mathbb{A}^2$, we have $\mathbf{n} /\!/ \mathbf{g}^3$, but the directions may be different. The unit normal $\mathbf{n}$ is often defined in the literature as $\mathbf{n} = \mathbf{a}_1 \times \mathbf{a}_2 / |\mathbf{a}_1 \times \mathbf{a}_2|$. For our purposes, we choose the following definitions and conditions

$$\begin{array}{l} \mathbf{n} = \mathbf{n}(\theta^1, \theta^2, \theta^3) = n_3 \mathbf{g}^3 \\ \mathbf{n} \cdot \mathbf{n} = n_3^2 \mathbf{g}^3 \cdot \mathbf{g}^3 = n_3^2 g^{33} = 1 \end{array} \quad ; \quad n_3 = \frac{1}{\pm\sqrt{g^{33}}} \quad ; \quad \begin{array}{l} n_3 = n_3(\theta^1, \theta^2, \theta^3) \\ \mathbf{g}^3 = \mathbf{g}^3(\theta^1, \theta^2, \theta^3) \end{array} \tag{1.96}$$

In this general case, the unit normal $\mathbf{n}$ is defined everywhere in the 3-dimensional space, because it is a function of the three variables θ^i. On the curved surface $\mathbb{A}^2$, where $\theta^3 = ¢$ (constant), one get $\mathbf{n} = \mathbf{n}(\theta^1, \theta^2)$. This allows a continuous transition between $\mathbb{A}^2$ and $\mathbb{R}^3$. The contravariant base vectors $\mathbf{a}^\beta$ may now be defined as follows

$$\mathbf{a}^\beta \cdot \mathbf{a}_\alpha = \delta^\beta_\alpha = \begin{cases} 1 & \text{if } \beta = \alpha \\ 0 & \text{if } \beta \neq \alpha \end{cases} \quad ; \quad \mathbf{a}^\beta \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{a}_\alpha = 0 \quad ; \quad (\alpha, \beta = 1, 2) \tag{1.97}$$

Note that we have $\mathbf{a}_\alpha = \mathbf{g}_\alpha$ but $\mathbf{a}^\beta \neq \mathbf{g}^\beta$ on $\mathbb{A}^2$. The normal projector $\mathbf{N}$, also called metric tensor of the surface, reads

$$\mathbf{N} = \mathbf{a}_\alpha \otimes \mathbf{a}^\alpha = \mathbf{a}^\alpha \otimes \mathbf{a}_\alpha = a^{\alpha\beta} \mathbf{a}_\alpha \otimes \mathbf{a}_\beta = a_{\alpha\beta} \mathbf{a}^\alpha \otimes \mathbf{a}^\beta = \mathbf{I} - \mathbf{n} \otimes \mathbf{n} \tag{1.98}$$

where $a^{\alpha\beta} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta$ and $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$. The first fundamental form of the surface is written

$$(ds)^2 = \mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{r} = a_{\alpha\beta} \mathrm{d}\theta^\alpha \mathrm{d}\theta^\beta \tag{1.99}$$

On the surface $\mathbb{A}^2$ where $\theta^3 = ¢$, we have

$$\mathrm{d}\mathbf{r} \equiv \mathbf{N}\mathrm{d}\mathbf{r} \quad ; \quad \mathbf{N}\mathbf{n} = 0 \quad ; \quad \mathbf{N}\mathbf{g}^\beta = \mathbf{a}^\beta \quad ; \quad \mathbf{N}\mathbf{g}^3 = 0 \quad ; \quad \mathbf{N}\mathbf{g}_3 \neq 0 \tag{1.100}$$

Projecting the 3-dimensional total differential $\mathrm{d}\mathbf{r}$ in space onto the surface while setting $\mathrm{d}\theta^3 = 0$ gives the 2-dimensional total differential $\mathrm{d}\mathbf{r} \equiv \mathbf{N}\mathrm{d}\mathbf{r}$ on the surface. We write

$$\begin{array}{lll} \mathbb{R}^3 & : & \mathrm{d}\mathbf{r} = \mathbf{g}_i \mathrm{d}\theta^i = \mathbf{g}_\alpha \mathrm{d}\theta^\alpha + \mathbf{g}_3 \mathrm{d}\theta^3 \\ \mathbb{A}^2 & : & \mathrm{d}\mathbf{r} \equiv \mathbf{N}\mathrm{d}\mathbf{r}\Big|_{\mathrm{d}\theta^3 = 0} = \underbrace{\mathbf{N}\mathbf{g}_\alpha}_{\equiv \mathbf{a}_\alpha} \underbrace{\mathrm{d}\theta^\alpha}_{\neq 0} + \underbrace{\mathbf{N}\mathbf{g}_3}_{\neq 0} \underbrace{\mathrm{d}\theta^3}_{\equiv 0} = \mathbf{a}_\alpha \mathrm{d}\theta^\alpha \end{array} \tag{1.101}$$

The proof of $\mathbf{N}\mathbf{g}^\beta = \mathbf{a}^\beta$ reads

$$\mathbf{N}\mathbf{g}^\beta = (\mathbf{a}^\alpha \otimes \mathbf{a}_\alpha)\mathbf{g}^\beta = (\mathbf{a}^\alpha \otimes \mathbf{g}_\alpha)\mathbf{g}^\beta = \mathbf{a}^\beta \qquad \square \text{ qed} \tag{1.102}$$

The 2-dimensional antimetric second-rank permutation tensor $\mathbf{E}$ reads

$$\begin{aligned} &\mathbf{E} = \boldsymbol{\mathcal{E}}\mathbf{n} = E_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta \\ &E_{\alpha\beta} = \begin{cases} +\sqrt{a} & \text{for an even permutation of } \alpha\beta = 12 \\ -\sqrt{a} & \text{for an odd permutation of } \alpha\beta = 12 \\ 0 & \text{if } \alpha = \beta \end{cases} \\ &a = \det a_{\alpha\beta} = a_{11}a_{22} - a_{12}a_{21} \end{aligned} \tag{1.103}$$

To build derivatives on a 2-dimensional curved surface, we need the so-called 2-dimensional derivative operator $\boldsymbol{\nabla}_n$, which is defined as follows

$$\boldsymbol{\nabla}_n = \mathbf{N}\boldsymbol{\nabla} = \mathbf{a}^\beta \frac{\partial}{\partial\theta^\beta} \tag{1.104}$$

The index n reminds us that $\boldsymbol{\nabla}_n$ is always perpendicular to the unit normal $\mathbf{n}$. We then have $\boldsymbol{\nabla}_n \equiv \mathbf{N}\boldsymbol{\nabla}_n$. On $\mathbb{A}^2$, the 2-dimensional derivative operator $\boldsymbol{\nabla}_n$ is the normal projection of the 3-dimensional derivative operator $\boldsymbol{\nabla}$ onto the tangential plane. Proof:

$$\begin{aligned} \boldsymbol{\nabla}_n = \mathbf{N}\boldsymbol{\nabla} &= (\mathbf{a}^\alpha \otimes \mathbf{a}_\alpha)\left(\mathbf{g}^\beta \frac{\partial}{\partial\theta^\beta} + \mathbf{g}^3 \frac{\partial}{\partial\theta^3}\right) \\ &= \mathbf{a}^\alpha \underbrace{(\mathbf{a}_\alpha \cdot \mathbf{g}^\beta)}_{= \mathbf{g}_\alpha \cdot \mathbf{g}^\beta = \delta_\alpha^\beta} \frac{\partial}{\partial\theta^\beta} + \mathbf{a}^\alpha \underbrace{(\mathbf{a}_\alpha \cdot \mathbf{g}^3)}_{=0} \frac{\partial}{\partial\theta^3} = \mathbf{a}^\beta \frac{\partial}{\partial\theta^\beta} \qquad \square \text{ qed} \end{aligned} \tag{1.105}$$

With $\alpha, \beta, \gamma = 1, 2$, the derivative of the unit normal $\mathbf{n}$ reads

$$\frac{\partial\mathbf{n}}{\partial\theta^\beta} = \mathbf{n}_{,\beta} = \frac{\partial}{\partial\theta^\beta}(n_3\mathbf{g}^3) = n_{3,\beta}\mathbf{g}^3 - n_3\Gamma^3_{\beta\gamma}\mathbf{g}^\gamma - \Gamma^3_{\beta3}\mathbf{n} = -n_3\Gamma^3_{\beta\gamma}\mathbf{a}^\gamma \tag{1.106}$$

because

$$\begin{aligned} &\mathbf{g}^\gamma = \mathbf{I}\mathbf{g}^\gamma = (\mathbf{N} + \mathbf{n} \otimes \mathbf{n})\mathbf{g}^\gamma = \mathbf{a}^\gamma + (\mathbf{n} \cdot \mathbf{g}^\gamma)\mathbf{n} = \mathbf{a}^\gamma + n_3 g^{3\gamma}\mathbf{n} \\ &\mathbf{n} \cdot \mathbf{g}^\gamma = n_3\mathbf{g}^3 \cdot \mathbf{g}^\gamma = n_3 g^{3\gamma} \\ &\frac{\partial}{2\partial\theta^\beta}(\mathbf{n} \cdot \mathbf{n}) = \mathbf{n} \cdot \frac{\partial\mathbf{n}}{\partial\theta^\beta} = n_{3,\beta}n_3 g^{33} - n_3\Gamma^3_{\beta\gamma}(\mathbf{n} \cdot \mathbf{g}^\gamma) - \Gamma^3_{\beta3} = 0 \\ &\Rightarrow \qquad n_{3,\beta} = n_3 \frac{g^{3\gamma}}{g^{33}}\Gamma^3_{\beta\gamma} + \frac{\Gamma^3_{\beta3}}{n_3 g^{33}} \end{aligned} \tag{1.107}$$

With

$$\mathbf{a}^\alpha = \mathbf{N}\mathbf{g}^\alpha = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\mathbf{g}^\alpha = \mathbf{g}^\alpha - (\mathbf{n} \cdot \mathbf{g}^\alpha)\mathbf{n} = \mathbf{g}^\alpha - n_3 g^{3\alpha}\mathbf{n}$$

$$
\begin{aligned}
&\frac{\partial \mathbf{a}^\alpha}{\partial \theta^\beta} = \mathbf{a}^\alpha{}_{,\beta} = \frac{\partial \mathbf{g}^\alpha}{\partial \theta^\beta} - \frac{\partial}{\partial \theta^\beta}[(\mathbf{n}\cdot\mathbf{g}^\alpha)\mathbf{n}] \\
&\frac{\partial \mathbf{g}^\alpha}{\partial \theta^\beta} = -\Gamma^\alpha_{\beta i}\mathbf{g}^i = -\Gamma^\alpha_{\beta\gamma}\mathbf{g}^\gamma - \Gamma^\alpha_{\beta 3}\mathbf{g}^3 \\
&\frac{\partial}{\partial \theta^\beta}[(\mathbf{n}\cdot\mathbf{g}^\alpha)\mathbf{n}] = \left(\mathbf{g}^\alpha\cdot\frac{\partial \mathbf{n}}{\partial \theta^\beta}\right)\mathbf{n} + \left(\mathbf{n}\cdot\frac{\partial \mathbf{g}^\alpha}{\partial \theta^\beta}\right)\mathbf{n} + (\mathbf{n}\cdot\mathbf{g}^\alpha)\frac{\partial \mathbf{n}}{\partial \theta^\beta} \\
&\left(\mathbf{g}^\alpha\cdot\frac{\partial \mathbf{n}}{\partial \theta^\beta}\right)\mathbf{n} = -n_3 a^{\alpha\gamma}\Gamma^3_{\beta\gamma}\mathbf{n} \\
&\left(\mathbf{n}\cdot\frac{\partial \mathbf{g}^\alpha}{\partial \theta^\beta}\right)\mathbf{n} = -(n_3 g^{3\gamma}\Gamma^\alpha_{\beta\gamma} + n_3 g^{33}\Gamma^\alpha_{\beta 3})\mathbf{n} \\
&(\mathbf{n}\cdot\mathbf{g}^\alpha)\frac{\partial \mathbf{n}}{\partial \theta^\beta} = -n_3^2 g^{3\alpha}\Gamma^3_{\beta\gamma}\mathbf{a}^\gamma
\end{aligned}
\tag{1.108}
$$

the derivative of the contravariant base vector $\mathbf{a}^\alpha$ reads

$$
\begin{aligned}
\frac{\partial \mathbf{a}^\alpha}{\partial \theta^\beta} = \mathbf{a}^\alpha{}_{,\beta} &= -(\Gamma^\alpha_{\beta\gamma} - n_3^2 g^{3\alpha}\Gamma^3_{\beta\gamma})\mathbf{a}^\gamma + n_3 a^{\alpha\gamma}\Gamma^3_{\beta\gamma}\mathbf{n} \\
&= -(\Gamma^\alpha_{\beta\gamma} - \frac{g^{3\alpha}}{g^{33}}\Gamma^3_{\beta\gamma})\mathbf{a}^\gamma + \frac{a^{\alpha\gamma}}{g^{33}}\Gamma^3_{\beta\gamma}\mathbf{g}^3
\end{aligned}
\tag{1.109}
$$

where

$$
\Gamma^\alpha_{\beta\gamma} - n_3^2 g^{3\alpha}\Gamma^3_{\beta\gamma} = -\mathbf{a}^\alpha{}_{,\beta}\cdot\mathbf{a}_\gamma = \mathbf{a}^\alpha\cdot\mathbf{a}_{\gamma,\beta} = \mathbf{a}^\alpha\cdot\mathbf{a}_{\beta,\gamma} = \Gamma^\alpha_{\gamma\beta} - n_3^2 g^{3\alpha}\Gamma^3_{\gamma\beta} \tag{1.110a}
$$

$$
\frac{\partial}{\partial \theta^\beta}(\mathbf{a}^\alpha\cdot\mathbf{a}_\gamma) = \mathbf{a}^\alpha{}_{,\beta}\cdot\mathbf{a}_\gamma + \mathbf{a}^\alpha\cdot\mathbf{a}_{\gamma,\beta} = 0 \quad ; \quad \mathbf{a}_{\gamma,\beta} = \frac{\partial \mathbf{a}_\gamma}{\partial \theta^\beta} = \frac{\partial^2 \mathbf{r}}{\partial \theta^\beta \partial \theta^\gamma} = \frac{\partial \mathbf{a}_\beta}{\partial \theta^\gamma} = \mathbf{a}_{\beta,\gamma}
$$

$$
n_3 a^{\alpha\gamma}\Gamma^3_{\beta\gamma} = \mathbf{a}^\alpha{}_{,\beta}\cdot\mathbf{n} = -\mathbf{a}^\alpha\cdot\mathbf{n}_{,\beta} \quad ; \quad a^{\alpha\gamma}\Gamma^3_{\beta\gamma} = \mathbf{a}^\alpha{}_{,\beta}\cdot\mathbf{g}^3 = -\mathbf{a}^\alpha\cdot\mathbf{g}^3{}_{,\beta} \tag{1.110b}
$$

$$
\frac{\partial}{\partial \theta^\beta}(\mathbf{a}^\alpha\cdot\mathbf{n}) = \mathbf{a}^\alpha{}_{,\beta}\cdot\mathbf{n} + \mathbf{a}^\alpha\cdot\mathbf{n}_{,\beta} = 0 \quad ; \quad \frac{\partial}{\partial \theta^\beta}(\mathbf{a}^\alpha\cdot\mathbf{g}^3) = \mathbf{a}^\alpha{}_{,\beta}\cdot\mathbf{g}^3 + \mathbf{a}^\alpha\cdot\mathbf{g}^3{}_{,\beta} = 0
$$

Thus, the derivative of an interior base vector has an interior and an exterior part. We may encounter two kinds of functions in space, namely scalar, vector and tensor functions of $\theta^1, \theta^2, \theta^3$ or functions of only θ^1, θ^2 on a surface. Both kinds can be derivated on a surface $\mathbb{A}^2$ as follows

$$
\begin{aligned}
&\mathrm{d}\phi = \frac{\partial \phi}{\partial \theta^\alpha}\mathrm{d}\theta^\alpha = \mathrm{d}\theta^\alpha\mathbf{a}_\alpha\cdot\mathbf{a}^\beta\frac{\partial \phi}{\partial \theta^\beta} = \mathrm{d}\mathbf{r}\cdot\boldsymbol{\nabla}_n\phi \\
&\mathrm{d}\mathbf{u} = \frac{\partial \mathbf{u}}{\partial \theta^\alpha}\mathrm{d}\theta^\alpha = \mathrm{d}\theta^\alpha(\mathbf{a}_\alpha\cdot\mathbf{a}^\beta)\frac{\partial \mathbf{u}}{\partial \theta^\beta} = \mathrm{d}\theta^\alpha\mathbf{a}_\alpha\left(\mathbf{a}^\beta\frac{\partial}{\partial \theta^\beta}\otimes\mathbf{u}\right) = \mathrm{d}\mathbf{r}(\boldsymbol{\nabla}_n\otimes\mathbf{u}) \\
&\mathrm{d}\mathbf{T} = \frac{\partial \mathbf{T}}{\partial \theta^\alpha}\mathrm{d}\theta^\alpha = \mathrm{d}\theta^\alpha(\mathbf{a}_\alpha\cdot\mathbf{a}^\beta)\frac{\partial \mathbf{T}}{\partial \theta^\beta} = \mathrm{d}\theta^\alpha\mathbf{a}_\alpha\left(\mathbf{a}^\beta\frac{\partial}{\partial \theta^\beta}\otimes\mathbf{T}\right) = \mathrm{d}\mathbf{r}(\boldsymbol{\nabla}_n\otimes\mathbf{T}) \\
&\dots
\end{aligned}
\tag{1.111}
$$

We briefly show here, that these rules are also valid for functions $\phi, \mathbf{u}, \mathbf{T}, \dots$ of the three variables θ^i. In case of a scalar ϕ, we become the projection of the 3-dimensional gradient $\boldsymbol{\nabla}\phi$ on $\mathrm{d}\mathbf{r}$. Proof:

$$
\mathrm{d}\phi = \mathrm{d}\mathbf{r}\cdot\boldsymbol{\nabla}_n\phi = \mathrm{d}\mathbf{r}\cdot\mathbf{N}\boldsymbol{\nabla}\phi = \mathrm{d}\theta^\alpha\mathbf{a}_\alpha\cdot\left(\mathbf{g}^\beta\frac{\partial \phi}{\partial \theta^\beta} + \mathbf{g}^3\frac{\partial \phi}{\partial \theta^3}\right) = \frac{\partial \phi}{\partial \theta^\alpha}\mathrm{d}\theta^\alpha \qquad \square\ \text{qed} \tag{1.112}
$$

The total differential of $\mathbf{r}$ reads

$$\mathrm{d}\mathbf{r} = \mathbf{N}\mathrm{d}\mathbf{r} = \mathrm{d}\mathbf{r}(\boldsymbol{\nabla}_n \otimes \mathbf{r}) \qquad \Rightarrow \qquad \mathbf{N} = \mathbf{N}^T = \boldsymbol{\nabla}_n \otimes \mathbf{r} \tag{1.113}$$

Proof:

$$\begin{aligned} \mathbf{a}^\alpha &= a^{\alpha\beta}\mathbf{a}_\beta \quad ; \quad a^{\alpha\beta} = a^{\beta\alpha} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta \\ \boldsymbol{\nabla}_n \otimes \mathbf{r} &= \mathbf{a}^\alpha \frac{\partial}{\partial\theta^\alpha} \otimes \mathbf{r} = \mathbf{a}^\alpha \otimes \frac{\partial \mathbf{r}}{\partial\theta^\alpha} = \mathbf{a}^\alpha \otimes \mathbf{a}_\alpha \\ &= a^{\alpha\beta}\mathbf{a}_\beta \otimes \mathbf{a}_\alpha = a^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta = \mathbf{N} = \mathbf{N}^T \qquad \square \text{ qed} \end{aligned} \tag{1.114}$$

The total differential of $\mathbf{n}$ reads

$$\mathrm{d}\mathbf{n} = \mathrm{d}\mathbf{r}(\boldsymbol{\nabla}_n \otimes \mathbf{n}) = -\mathrm{d}\mathbf{r}\mathbf{B} = -\mathbf{B}\mathrm{d}\mathbf{r} \tag{1.115}$$

where

$$\mathbf{B} = \mathbf{B}^T = \mathbf{N}\mathbf{B}\mathbf{N} = -\boldsymbol{\nabla}_n \otimes \mathbf{n} = B_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta \qquad ; \qquad B_{\alpha\beta} = n_3\Gamma^3_{\alpha\beta} = \mathbf{n} \cdot \mathbf{a}_{\alpha,\beta} \tag{1.116}$$

is the so-called 2-dimensional curvature tensor of the surface $\mathbb{A}^2$. This second-rank tensor is symmetric and interior, which means that it is situated in the tangential plane of the surface and do not contain any exterior part. Proof:

$$\begin{aligned} &\mathbf{N}\mathbf{a}^\alpha = \mathbf{a}^\alpha \qquad ; \qquad \Gamma^3_{\alpha\beta} = \Gamma^3_{\beta\alpha} \\ &\boldsymbol{\nabla}_n = \mathbf{N}\boldsymbol{\nabla}_n \\ &\boldsymbol{\nabla}_n(\mathbf{n} \cdot \mathbf{n}) = 2(\boldsymbol{\nabla}_n \otimes \mathbf{n})\mathbf{n} = 0 \\ &\boldsymbol{\nabla}_n \otimes \mathbf{n} = (\boldsymbol{\nabla}_n \otimes \mathbf{n})\mathbf{I} = (\boldsymbol{\nabla}_n \otimes \mathbf{n})\mathbf{N} + (\boldsymbol{\nabla}_n \otimes \mathbf{n})\mathbf{n} \otimes \mathbf{n} = (\boldsymbol{\nabla}_n \otimes \mathbf{n})\mathbf{N} = \mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{n})\mathbf{N} \\ &\qquad = \mathbf{N}(\mathbf{a}^\beta \otimes \mathbf{n}_{,\beta})\mathbf{N} = -n_3\Gamma^3_{\beta\alpha}\mathbf{N}(\mathbf{a}^\beta \otimes \mathbf{a}^\alpha)\mathbf{N} = -n_3\Gamma^3_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta \\ &(\boldsymbol{\nabla}_n \otimes \mathbf{n})^T = -n_3\Gamma^3_{\alpha\beta}\mathbf{a}^\beta \otimes \mathbf{a}^\alpha = -n_3\Gamma^3_{\beta\alpha}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta = -n_3\Gamma^3_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta \qquad \square \text{ qed} \end{aligned} \tag{1.117}$$

With $\mathrm{d}(\mathrm{d}\mathbf{r} \cdot \mathbf{n}) = \mathrm{d}^2\mathbf{r} \cdot \mathbf{n} + \mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{n} = 0$, the second fundamental form of the surface reads

$$B\mathrm{d}s^2 = \mathrm{d}^2\mathbf{r} \cdot \mathbf{n} = -\mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{n} = -\mathrm{d}\mathbf{r} \cdot (\boldsymbol{\nabla}_n \otimes \mathbf{n})\mathrm{d}\mathbf{r} = \mathrm{d}\mathbf{r} \cdot \mathbf{B}\mathrm{d}\mathbf{r} \tag{1.118}$$

The total differential of $\mathbf{N}$ reads

$$\mathrm{d}\mathbf{N} = \mathrm{d}\mathbf{r}(\boldsymbol{\nabla}_n \otimes \mathbf{N}) \qquad ; \qquad \begin{aligned} \boldsymbol{\nabla}_n \otimes \mathbf{N} &= \boldsymbol{\nabla}_n \otimes \mathbf{I} - \boldsymbol{\nabla}_n \otimes \mathbf{n} \otimes \mathbf{n} - \boldsymbol{\nabla}_n \otimes \mathbf{n} \otimes \mathbf{n})^T \\ &= \mathbf{B} \otimes \mathbf{n} + \mathbf{B} \otimes \mathbf{n})^T \end{aligned} \tag{1.119}$$

The total differential of $\mathbf{E}$ reads with $\boldsymbol{\mathcal{E}} = ¢$ in $\mathbb{R}^3$

$$\mathrm{d}\mathbf{E} = \mathrm{d}\mathbf{r}(\boldsymbol{\nabla}_n \otimes \mathbf{E}) \qquad ; \qquad \begin{aligned} \boldsymbol{\nabla}_n \otimes \mathbf{E} &= \boldsymbol{\nabla}_n \otimes (\boldsymbol{\mathcal{E}}\mathbf{n}) = (\boldsymbol{\nabla}_n \otimes \boldsymbol{\mathcal{E}})\mathbf{n} + (\boldsymbol{\nabla}_n \otimes \mathbf{n})\boldsymbol{\mathcal{E}} \\ &= -\mathbf{B}\boldsymbol{\mathcal{E}} = -\mathbf{B}[\mathbf{E} \otimes \mathbf{n} - \mathbf{E} \otimes \mathbf{n})^T + \mathbf{n} \otimes \mathbf{E}] \\ &= \mathbf{B}\mathbf{E} \otimes \mathbf{n})^T - \mathbf{B}\mathbf{E} \otimes \mathbf{n} \end{aligned} \tag{1.120}$$

With the “basic rules” above, we write on a curved surface $\mathbb{A}^2$

$$\begin{aligned}\mathrm{d}^2\phi &= \mathrm{d}(\mathrm{d}\phi) = \mathrm{d}(\mathrm{d}\mathbf{r}\cdot\boldsymbol{\nabla}_n\phi) = \mathrm{d}(\mathbf{N}\mathrm{d}\mathbf{r}\cdot\boldsymbol{\nabla}_n\phi) = \mathrm{d}(\mathbf{N}\mathrm{d}\mathbf{r})\cdot\boldsymbol{\nabla}_n\phi + \mathbf{N}\mathrm{d}\mathbf{r}\cdot\mathrm{d}(\boldsymbol{\nabla}_n\phi)\\ &= (\mathrm{d}\mathbf{N}\mathrm{d}\mathbf{r} + \mathbf{N}\mathrm{d}^2\mathbf{r})\cdot\boldsymbol{\nabla}_n\phi + \mathrm{d}\mathbf{r}\cdot[\mathrm{d}\mathbf{r}(\boldsymbol{\nabla}_n\otimes\boldsymbol{\nabla}_n\phi)]\\ &= [\mathrm{d}\mathbf{r}(\boldsymbol{\nabla}_n\otimes\mathbf{N})\mathrm{d}\mathbf{r} + \mathbf{N}\mathrm{d}^2\mathbf{r}]\cdot\boldsymbol{\nabla}_n\phi + (\mathrm{d}\mathbf{r}\otimes\mathrm{d}\mathbf{r})\cdot(\boldsymbol{\nabla}_n\otimes\boldsymbol{\nabla}_n\phi)\\ &= [(\mathrm{d}\mathbf{r}\cdot\mathbf{B}\mathrm{d}\mathbf{r})\mathbf{n} + \mathbf{N}\mathrm{d}^2\mathbf{r}]\cdot\mathbf{N}\boldsymbol{\nabla}_n\phi + (\mathrm{d}\mathbf{r}\otimes\mathrm{d}\mathbf{r})\cdot(\boldsymbol{\nabla}_n\otimes\boldsymbol{\nabla}_n\phi)\\ &= \mathrm{d}^2\mathbf{r}\cdot\boldsymbol{\nabla}_n\phi + \mathrm{d}\mathbf{r}\cdot\mathrm{d}(\boldsymbol{\nabla}_n\phi) = \mathrm{d}\mathbf{r}\cdot(\boldsymbol{\nabla}_n\otimes\boldsymbol{\nabla}_n\phi)\mathrm{d}\mathbf{r} + \mathrm{d}^2\mathbf{r}\cdot\boldsymbol{\nabla}_n\phi\\ \mathrm{d}^2\mathbf{u} &= \mathrm{d}(\mathrm{d}\mathbf{u}) = \mathrm{d}[\mathrm{d}\mathbf{r}(\boldsymbol{\nabla}_n\otimes\mathbf{u})] = (\mathrm{d}\mathbf{r}\otimes\mathrm{d}\mathbf{r})\cdot(\boldsymbol{\nabla}_n\otimes\boldsymbol{\nabla}_n\otimes\mathbf{u}) + \mathrm{d}^2\mathbf{r}(\boldsymbol{\nabla}_n\otimes\mathbf{u})\\ \mathrm{d}^2\mathbf{T} &= \mathrm{d}(\mathrm{d}\mathbf{T}) = \mathrm{d}[\mathrm{d}\mathbf{r}(\boldsymbol{\nabla}_n\otimes\mathbf{T})] = (\mathrm{d}\mathbf{r}\otimes\mathrm{d}\mathbf{r})\cdot(\boldsymbol{\nabla}_n\otimes\boldsymbol{\nabla}_n\otimes\mathbf{T}) + \mathrm{d}^2\mathbf{r}(\boldsymbol{\nabla}_n\otimes\mathbf{T})\\ &\dots\end{aligned} \tag{1.121}$$

where

$$\mathrm{d}^2\mathbf{r} = \mathbf{N}\mathrm{d}^2\mathbf{r} + (\mathrm{d}\mathbf{r}\cdot\mathbf{B}\mathrm{d}\mathbf{r})\mathbf{n} \tag{1.122}$$

Let us now look at the decomposition of the derivative of a 3-dimensional vector $\mathbf{u}$. We have

$$\mathbf{u} = \mathbf{I}\mathbf{u} = (\mathbf{N} + \mathbf{n}\otimes\mathbf{n})\mathbf{u} = \mathbf{N}\mathbf{u} + (\mathbf{n}\cdot\mathbf{u})\mathbf{n} = \mathbf{v} + w\mathbf{n} \tag{1.123}$$

where $\mathbf{v} = \mathbf{N}\mathbf{u}$ and $w = \mathbf{n}\cdot\mathbf{u}$. Thus, the derivative of $\mathbf{u}$ read

$$\begin{aligned}\boldsymbol{\nabla}_n\otimes\mathbf{u} &= \boldsymbol{\nabla}_n\otimes(\mathbf{v} + w\mathbf{n})\\ &= \boldsymbol{\nabla}_n\otimes\mathbf{v} + \boldsymbol{\nabla}_n w\otimes\mathbf{n} - w\boldsymbol{\nabla}_n\otimes\mathbf{n}\\ &= \boldsymbol{\nabla}_n\otimes\mathbf{v} + \boldsymbol{\nabla}_n w\otimes\mathbf{n} - \mathbf{B}w\end{aligned} \tag{1.124}$$

Because $\mathbf{B}$ is interior, we have

$$(\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{N} = (\boldsymbol{\nabla}_n\otimes\mathbf{v})\mathbf{N} - \mathbf{B}w \tag{1.125}$$

with

$$\begin{aligned}\boldsymbol{\nabla}_n\otimes\mathbf{v} &= \boldsymbol{\nabla}_n\otimes(\mathbf{N}\mathbf{v}) = (\boldsymbol{\nabla}_n\otimes\mathbf{v})\mathbf{N} + (\boldsymbol{\nabla}_n\otimes\mathbf{N})\mathbf{v}\\ &= (\boldsymbol{\nabla}_n\otimes\mathbf{v})\mathbf{N} + [\mathbf{B}\otimes\mathbf{n} + \mathbf{B}\otimes\mathbf{n})^T]\mathbf{v} = (\boldsymbol{\nabla}_n\otimes\mathbf{v})\mathbf{N} + \mathbf{B}\mathbf{v}\otimes\mathbf{n}\end{aligned} \tag{1.126}$$

which gives

$$\boldsymbol{\nabla}_n\otimes\mathbf{u} = (\boldsymbol{\nabla}_n\otimes\mathbf{v})\mathbf{N} - \mathbf{B}w + (\mathbf{B}\mathbf{v} + \boldsymbol{\nabla}_n w)\otimes\mathbf{n} \tag{1.127}$$

For example, the tensor $\boldsymbol{\nabla}_n\otimes\mathbf{v}$ can be explicitly written with components and base vectors. With $\mathbf{a}^\alpha_{,\beta} = -(\Gamma^\alpha_{\beta\gamma} - n_3^2 g^{3\alpha}\Gamma^3_{\beta\gamma})\mathbf{a}^\gamma + n_3 a^{\alpha\gamma}\Gamma^3_{\beta\gamma}\mathbf{n}$ and $\mathbf{v} = v_\alpha\mathbf{a}^\alpha$ $(\alpha,\beta,\gamma = 1,2)$, we have

$$\begin{aligned}\boldsymbol{\nabla}_n\otimes\mathbf{v} &= \mathbf{a}^\beta\frac{\partial}{\partial\theta^\beta}\otimes(v_\alpha\mathbf{a}^\alpha) = v_{\alpha,\beta}\mathbf{a}^\beta\otimes\mathbf{a}^\alpha + v_\alpha\mathbf{a}^\beta\otimes\mathbf{a}^\alpha_{,\beta}\\ &= (v_{\beta,\alpha} - \Gamma^\gamma_{\alpha\beta}v_\gamma + n_3^2 g^{3\gamma}\Gamma^3_{\alpha\beta}v_\gamma)\mathbf{a}^\alpha\otimes\mathbf{a}^\beta + n_3 a^{\gamma\beta}\Gamma^3_{\alpha\beta}v_\gamma\mathbf{a}^\alpha\otimes\mathbf{n}\\ &= v_{\beta;\alpha}\mathbf{a}^\alpha\otimes\mathbf{a}^\beta + n_3 a^{\gamma\beta}\Gamma^3_{\alpha\beta}v_\gamma\mathbf{a}^\alpha\otimes\mathbf{n}\end{aligned} \tag{1.128}$$

where $v_{\beta;\alpha} = v_{\beta,\alpha} - \Gamma^\gamma_{\alpha\beta}v_\gamma + n_3^2 g^{3\gamma}\Gamma^3_{\alpha\beta}v_\gamma$ are the covariant derivatives of the components of the vector $\mathbf{v} = \mathbf{N}\mathbf{u}$ on the curved surface, which represent the components of the interior part of the tensor $\boldsymbol{\nabla}_n\otimes\mathbf{v}$.

Special case where the curvilinear coordinate θ^3 is perpendicular to the curved surface

In the special case where $\mathbf{g}_3 = \mathbf{g}^3 = \mathbf{n}$ with $n_3 = 1$, the contravariant base vectors $\mathbf{g}^\beta$ are normal to $\mathbf{n}$, which means that $\mathbf{N}\mathbf{g}^\beta = \mathbf{g}^\beta = \mathbf{a}^\beta$ and $g^{3\gamma} = \mathbf{g}^3 \cdot \mathbf{g}^\gamma = \mathbf{n} \cdot \mathbf{a}^\gamma = 0$. Therefore, we have $v_{\beta;\alpha} = v_{\beta,\alpha} - \Gamma^\gamma_{\alpha\beta} v_\gamma$ and $-\Gamma^\gamma_{\alpha 3} = \mathbf{g}^\gamma_{,\alpha} \cdot \mathbf{g}_3 = \mathbf{a}^\gamma_{,\alpha} \cdot \mathbf{n} = a^{\gamma\beta}\Gamma^3_{\alpha\beta}$. The tensor $\boldsymbol{\nabla}_n \otimes \mathbf{v}$ then reads

$$\boldsymbol{\nabla}_n \otimes \mathbf{v} = (v_{\beta,\alpha} - \Gamma^\gamma_{\alpha\beta} v_\gamma)\mathbf{a}^\alpha \otimes \mathbf{a}^\beta - \Gamma^\gamma_{\alpha 3} v_\gamma \mathbf{a}^\alpha \otimes \mathbf{n} \tag{1.129}$$

Analytical example

Let us consider the position vector $\mathbf{r}$ of a point $\widetilde{\mathrm{P}}$ in the 3-dimensional space $\mathbb{R}^3$

$$\widetilde{\mathrm{P}} \quad : \qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \theta^1\mathbf{i} + \theta^2\mathbf{j} + c_1\left(1 - \frac{(\theta^1)^2}{a^2} - \frac{(\theta^2)^2}{b^2}\right)\mathbf{k} + c_2\theta^3\mathbf{k} \tag{1.130}$$

with the constants c_1, c_2 and the curvilinear coordinates $\theta^1, \theta^2, \theta^3$ such that

$$x = \theta^1 \quad ; \quad y = \theta^2 \quad ; \quad z = c_1\left(1 - \frac{(\theta^1)^2}{a^2} - \frac{(\theta^2)^2}{b^2}\right) + c_2\theta^3 \tag{1.131}$$

and where the cartesian base vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, corresponding to the x, y, and z-axes respectively, are constant. By setting $\theta^3 = 0$, we get the position vector

$$\mathrm{P} \quad : \qquad \mathbf{r}(\theta^1, \theta^2, \theta^3)\Big|_{\theta^3 = 0} = \mathbf{r}(\theta^1, \theta^2) = \mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + c_1\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)\mathbf{k} \tag{1.132}$$

which describes the position of point P on a 2-dimensional parabolic surface $\mathbb{A}^2$ in space. On this surface, $\mathbf{r}$ is only function of the two curvilinear coordinates $\theta^1 = x$ and $\theta^2 = y$.

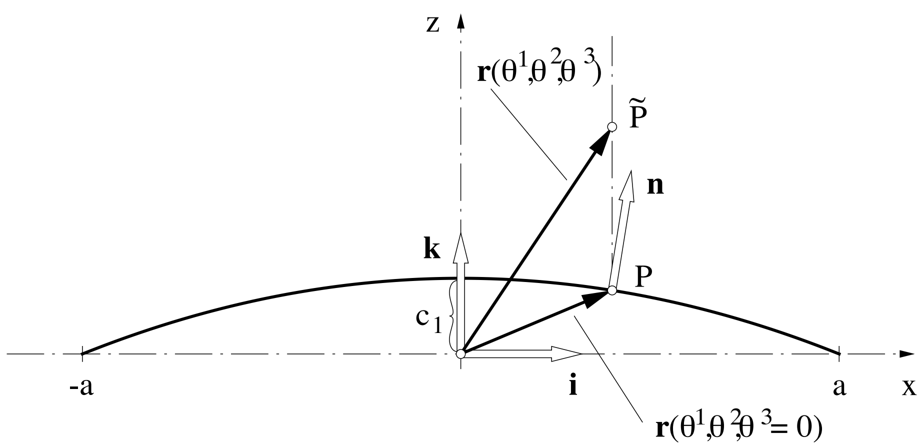

Fig. 1.7: Definition of a curved surface in space

The covariant base vectors in space and on the curved surface (with their derivatives) are

$$\begin{aligned} \mathbf{g}_1 &= \mathbf{a}_1 = \frac{\partial \mathbf{r}}{\partial \theta^1} = \frac{\partial \mathbf{r}}{\partial x} = \mathbf{i} - \frac{2c_1 x}{a^2}\mathbf{k} \\ \mathbf{g}_2 &= \mathbf{a}_2 = \frac{\partial \mathbf{r}}{\partial \theta^2} = \frac{\partial \mathbf{r}}{\partial y} = \mathbf{j} - \frac{2c_1 y}{b^2}\mathbf{k} \\ \mathbf{g}_3 &= \frac{\partial \mathbf{r}}{\partial \theta^3} = c_2 \mathbf{k} \end{aligned} \quad ; \quad \begin{aligned} \mathbf{a}_{1,1} &= -\frac{2c_1}{a^2}\mathbf{k} \quad ; \quad \mathbf{a}_{1,2} = 0 \\ \mathbf{a}_{2,1} &= 0 \quad ; \quad \mathbf{a}_{2,2} = -\frac{2c_1}{b^2}\mathbf{k} \end{aligned} \tag{1.133}$$

where $\mathbf{g}_3$ is generally not perpendicular to the surface. The corresponding contravariant base vectors are

$$\mathbf{g}^1 = \mathbf{i} \quad ; \quad \mathbf{g}^2 = \mathbf{j} \quad ; \quad \mathbf{g}^3 = \frac{1}{c_2}\left(\frac{2c_1 x}{a^2}\mathbf{i} + \frac{2c_1 y}{b^2}\mathbf{j} + \mathbf{k}\right) \tag{1.134}$$

where $\mathbf{g}^3$ is perpendicular to the surface and with

$$\begin{aligned} \mathbf{g}^1 \cdot \mathbf{g}_1 = 1 \quad ; \quad \mathbf{g}^1 \cdot \mathbf{g}_2 = 0 \quad ; \quad \mathbf{g}^1 \cdot \mathbf{g}_3 = 0 \\ \mathbf{g}^2 \cdot \mathbf{g}_1 = 0 \quad ; \quad \mathbf{g}^2 \cdot \mathbf{g}_2 = 1 \quad ; \quad \mathbf{g}^2 \cdot \mathbf{g}_3 = 0 \\ \mathbf{g}^3 \cdot \mathbf{g}_1 = 0 \quad ; \quad \mathbf{g}^3 \cdot \mathbf{g}_2 = 0 \quad ; \quad \mathbf{g}^3 \cdot \mathbf{g}_3 = 1 \end{aligned} \tag{1.135}$$

The unit normal $\mathbf{n}$ to the surface reads with $\mathbf{n} \cdot \mathbf{n} = 1$

$$\mathbf{n} = n_3 \mathbf{g}^3 = \left[1 + \left(\frac{2c_1 x}{a^2}\right)^2 + \left(\frac{2c_1 y}{b^2}\right)^2\right]^{-\frac{1}{2}} \left(\frac{2c_1 x}{a^2}\mathbf{i} + \frac{2c_1 y}{b^2}\mathbf{j} + \mathbf{k}\right) \tag{1.136}$$

and allows to compute the contravariant base vectors $\mathbf{a}^1$ and $\mathbf{a}^2$ on the surface, which read

$$\begin{aligned} \mathbf{a}^1 &= \mathbf{i} - \frac{2c_1 x}{a^2}\left[1 + \left(\frac{2c_1 x}{a^2}\right)^2 + \left(\frac{2c_1 y}{b^2}\right)^2\right]^{-1} \left(\frac{2c_1 x}{a^2}\mathbf{i} + \frac{2c_1 y}{b^2}\mathbf{j} + \mathbf{k}\right) \\ &= \mathbf{i} - \frac{2c_1 x}{a^2}\left[1 + \left(\frac{2c_1 x}{a^2}\right)^2 + \left(\frac{2c_1 y}{b^2}\right)^2\right]^{-\frac{1}{2}} \mathbf{n} \end{aligned} \tag{1.137a}$$

$$\begin{aligned} \mathbf{a}^2 &= \mathbf{j} - \frac{2c_1 y}{b^2}\left[1 + \left(\frac{2c_1 x}{a^2}\right)^2 + \left(\frac{2c_1 y}{b^2}\right)^2\right]^{-1} \left(\frac{2c_1 x}{a^2}\mathbf{i} + \frac{2c_1 y}{b^2}\mathbf{j} + \mathbf{k}\right) \\ &= \mathbf{j} - \frac{2c_1 y}{b^2}\left[1 + \left(\frac{2c_1 x}{a^2}\right)^2 + \left(\frac{2c_1 y}{b^2}\right)^2\right]^{-\frac{1}{2}} \mathbf{n} \end{aligned} \tag{1.137b}$$

with

$$\begin{aligned} \mathbf{a}^1 \cdot \mathbf{a}_1 = 1 \quad ; \quad \mathbf{a}^1 \cdot \mathbf{a}_2 = 0 \quad ; \quad \mathbf{a}^1 \cdot \mathbf{n} = 0 \\ \mathbf{a}^2 \cdot \mathbf{a}_1 = 0 \quad ; \quad \mathbf{a}^2 \cdot \mathbf{a}_2 = 1 \quad ; \quad \mathbf{a}^2 \cdot \mathbf{n} = 0 \end{aligned} \tag{1.138}$$

Note that $\mathbf{a}^1 \cdot \mathbf{a}^2 \neq 0$ in our case. The 3-dimensional derivative operator $\boldsymbol{\nabla}$ in space and the 2-dimensional derivative operator $\boldsymbol{\nabla}_n$ on the surface respectively read

$$\begin{aligned} \boldsymbol{\nabla} &= \mathbf{g}^1 \frac{\partial}{\partial \theta^1} + \mathbf{g}^2 \frac{\partial}{\partial \theta^2} + \mathbf{g}^3 \frac{\partial}{\partial \theta^3} = \mathbf{g}^1 \frac{\partial}{\partial x} + \mathbf{g}^2 \frac{\partial}{\partial y} + \mathbf{g}^3 \frac{\partial}{\partial \theta^3} \\ &= \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \frac{1}{c_2}\left(\frac{2c_1 x}{a^2}\mathbf{i} + \frac{2c_1 y}{b^2}\mathbf{j} + \mathbf{k}\right)\frac{\partial}{\partial \theta^3} \end{aligned} \tag{1.139}$$

$$\begin{aligned}
\boldsymbol{\nabla}_n &= \mathbf{N}\boldsymbol{\nabla} = \mathbf{a}^1\frac{\partial}{\partial\theta^1} + \mathbf{a}^2\frac{\partial}{\partial\theta^2} = \mathbf{a}^1\frac{\partial}{\partial x} + \mathbf{a}^2\frac{\partial}{\partial y} \\
&= \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} - \mathbf{n}\left[1+\left(\frac{2c_1x}{a^2}\right)^2+\left(\frac{2c_1y}{b^2}\right)^2\right]^{-\frac{1}{2}}\left[\left(\frac{2c_1x}{a^2}\right)\frac{\partial}{\partial x}+\left(\frac{2c_1y}{b^2}\right)\frac{\partial}{\partial y}\right] \qquad (1.140)\\
&= \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} - \left(\frac{2c_1x}{a^2}\mathbf{i}+\frac{2c_1y}{b^2}\mathbf{j}+\mathbf{k}\right)\left[1+\left(\frac{2c_1x}{a^2}\right)^2+\left(\frac{2c_1y}{b^2}\right)^2\right]^{-1}\left[\left(\frac{2c_1x}{a^2}\right)\frac{\partial}{\partial x}+\left(\frac{2c_1y}{b^2}\right)\frac{\partial}{\partial y}\right]
\end{aligned}$$

where the normal projector $\mathbf{N} = \mathbf{I} - \mathbf{n}\otimes\mathbf{n} = \mathbf{a}^1\otimes\mathbf{a}_1 + \mathbf{a}^2\otimes\mathbf{a}_2$, which projects any vector onto the tangential plane of the surface, is also called the metric tensor of the surface. With

$$\begin{aligned}
\frac{\partial\mathbf{n}}{\partial x} &= \frac{2c_1}{a^2}\left[1+\left(\frac{2c_1x}{a^2}\right)^2+\left(\frac{2c_1y}{b^2}\right)^2\right]^{-\frac{1}{2}}\mathbf{a}^1 \\
\frac{\partial\mathbf{n}}{\partial y} &= \frac{2c_1}{b^2}\left[1+\left(\frac{2c_1x}{a^2}\right)^2+\left(\frac{2c_1y}{b^2}\right)^2\right]^{-\frac{1}{2}}\mathbf{a}^2 \qquad (1.141)
\end{aligned}$$

the corresponding curvature tensor $\mathbf{B}$ of the surface reads

$$\begin{aligned}
\mathbf{B} &= -\boldsymbol{\nabla}_n\otimes\mathbf{n} = -\left(\mathbf{a}^1\otimes\frac{\partial\mathbf{n}}{\partial x}+\mathbf{a}^2\otimes\frac{\partial\mathbf{n}}{\partial y}\right) \\
&= -\left[1+\left(\frac{2c_1x}{a^2}\right)^2+\left(\frac{2c_1y}{b^2}\right)^2\right]^{-\frac{1}{2}}\left(\frac{2c_1}{a^2}\mathbf{a}^1\otimes\mathbf{a}^1+\frac{2c_1}{b^2}\mathbf{a}^2\otimes\mathbf{a}^2\right) \qquad (1.142)
\end{aligned}$$

which is in agreement with the definition $\mathbf{B} = n_3\Gamma^3_{\alpha\beta}\mathbf{a}^\alpha\otimes\mathbf{a}^\beta$ where $n_3\Gamma^3_{\alpha\beta} = \mathbf{n}\cdot\mathbf{a}_{\alpha,\beta}$. In point $\mathrm{P}_0(0,0,c_1)$, which means for $x = y = 0$, we get

$$\mathbf{B}_0 = -\left(\frac{2c_1}{a^2}\mathbf{i}\otimes\mathbf{i}+\frac{2c_1}{b^2}\mathbf{j}\otimes\mathbf{j}\right) \hat{=} \begin{bmatrix} -\dfrac{2c_1}{a^2} & 0 & 0 \\ 0 & -\dfrac{2c_1}{b^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (1.143)$$

1.4 Developments of lengths, directions and normal projectors in space

If the relations $x = x(\theta^1,\theta^2,\theta^3)$, $y = y(\theta^1,\theta^2,\theta^3)$ and $z = z(\theta^1,\theta^2,\theta^3)$ are linear functions of the three independent variables θ^i, then the base vectors $\mathbf{g}_i$ and $\mathbf{g}^i$ are constant in space. In that case, we have

$$\bar{\mathbf{r}} = \mathbf{r} + \Delta\mathbf{r} = \mathbf{r} + \mathrm{d}\mathbf{r} \qquad \text{where} \qquad \begin{cases} \mathrm{d}\mathbf{r} = \Delta\mathbf{r} \\ \mathrm{d}^2\mathbf{r} = \mathrm{d}^3\mathbf{r} = \ldots = 0 \end{cases} \qquad (1.144)$$

Thus, we write

$$
\begin{aligned}
&\bar{\phi} = \phi(\bar{\mathbf{r}}) = \phi(\mathbf{r} + \Delta\mathbf{r}) = \phi(\mathbf{r}) + \Delta\mathbf{r} \cdot \boldsymbol{\nabla}\phi + \frac{1}{2!}\Delta\mathbf{r}(\boldsymbol{\nabla} \otimes \boldsymbol{\nabla}\phi)\Delta\mathbf{r} + \ldots \\
&\bar{\mathbf{u}} = \mathbf{u}(\bar{\mathbf{r}}) = \mathbf{u}(\mathbf{r} + \Delta\mathbf{r}) = \mathbf{u}(\mathbf{r}) + \Delta\mathbf{r}(\boldsymbol{\nabla} \otimes \mathbf{u}) + \frac{1}{2!}(\Delta\mathbf{r} \otimes \Delta\mathbf{r}) \cdot (\boldsymbol{\nabla} \otimes \boldsymbol{\nabla} \otimes \mathbf{u}) + \ldots \\
&\bar{\mathbf{T}} = \mathbf{T}(\bar{\mathbf{r}}) = \mathbf{T}(\mathbf{r} + \Delta\mathbf{r}) = \mathbf{T}(\mathbf{r}) + \Delta\mathbf{r}(\boldsymbol{\nabla} \otimes \mathbf{T}) + \frac{1}{2!}(\Delta\mathbf{r} \otimes \Delta\mathbf{r}) \cdot (\boldsymbol{\nabla} \otimes \boldsymbol{\nabla} \otimes \mathbf{T}) + \ldots \\
&\ldots
\end{aligned}
\tag{1.145}
$$

Let us now consider the points P_0, P and $\bar{\mathrm{P}}$ with

$$
\begin{aligned}
&\mathbf{r}_0 = \overrightarrow{\mathrm{OP}_0} && \mathbf{p} = p\mathbf{h} = \mathbf{r} - \mathbf{r}_0 && \bar{\mathbf{p}} = \bar{p}\bar{\mathbf{h}} = \mathbf{p} + \Delta\mathbf{r} = \bar{\mathbf{r}} - \mathbf{r}_0 \\
&\mathbf{r} = \overrightarrow{\mathrm{OP}} && \mathbf{h} \cdot \mathbf{h} = 1 && \bar{\mathbf{h}} \cdot \bar{\mathbf{h}} = 1 \\
&\bar{\mathbf{r}} = \overrightarrow{\mathrm{O\bar{P}}} \quad ; && p = \mathbf{p} \cdot \mathbf{h} \quad ; && \bar{p} = \bar{\mathbf{p}} \cdot \bar{\mathbf{h}} \\
&\Delta\mathbf{r} = \bar{\mathbf{r}} - \mathbf{r} && \mathbf{H} = \mathbf{I} - \mathbf{h} \otimes \mathbf{h} && \bar{\mathbf{H}} = \mathbf{I} - \bar{\mathbf{h}} \otimes \bar{\mathbf{h}}
\end{aligned}
\tag{1.146}
$$

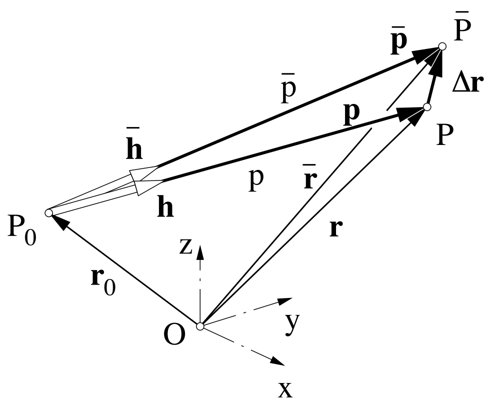

Fig. 1.8: Developments in space

For $|\Delta\mathbf{r}| \ll |\mathbf{p}|$, the developments (Taylor series) of the functions p, $\mathbf{h}$ and $\mathbf{H} = \mathbf{I} - \mathbf{h} \otimes \mathbf{h}$ in the neighborhood of point P are written as follows

$$
\begin{aligned}
&\bar{p} = p + \Delta\mathbf{r} \cdot \boldsymbol{\nabla}p + \frac{1}{2!}\Delta\mathbf{r} \cdot (\boldsymbol{\nabla} \otimes \boldsymbol{\nabla}p)\Delta\mathbf{r} + \ldots = p + \Delta\mathbf{r} \cdot \mathbf{h} + \frac{1}{2p}\Delta\mathbf{r} \cdot \mathbf{H}\Delta\mathbf{r} + \ldots \\
&\bar{\mathbf{h}} = \mathbf{h} + \Delta\mathbf{r}(\boldsymbol{\nabla} \otimes \mathbf{h}) + \frac{1}{2!}(\Delta\mathbf{r} \otimes \Delta\mathbf{r}) \cdot (\boldsymbol{\nabla} \otimes \boldsymbol{\nabla} \otimes \mathbf{h}) + \ldots = \mathbf{h} + \frac{1}{p}\mathbf{H}\Delta\mathbf{r} - \frac{1}{2p^2}\Delta\mathbf{r}\boldsymbol{\mathcal{H}}\Delta\mathbf{r} + \ldots \\
&\frac{1}{\bar{p}}\bar{\mathbf{H}} = \frac{1}{p}\mathbf{H} + \Delta\mathbf{r}\left[\boldsymbol{\nabla} \otimes \left(\frac{1}{p}\mathbf{H}\right)\right] + \frac{1}{2!}(\Delta\mathbf{r} \otimes \Delta\mathbf{r}) \cdot \left[\boldsymbol{\nabla} \otimes \boldsymbol{\nabla} \otimes \left(\frac{1}{p}\mathbf{H}\right)\right] + \ldots \\
&\qquad = \frac{1}{p}\mathbf{H} - \frac{1}{p^2}\boldsymbol{\mathcal{H}}\Delta\mathbf{r} + \frac{1}{2p^3}\Delta\mathbf{r}\boldsymbol{H}\Delta\mathbf{r} + \ldots \\
&\ldots
\end{aligned}
\tag{1.147}
$$

1.5 Affine connections

Affine connections make the bridge between different curved surfaces in space under the condition that the surfaces are described by the same set of curvilinear coordinates (convected coordinates). With the two vector coordinates $\mathbf{r}$ and $\widehat{\mathbf{r}}$ of both curved surfaces $\mathbb{A}^2$ and $\widehat{\mathbb{A}}^2$ respectively, we write

$$\theta^1, \theta^2 \qquad \rightarrow \qquad \begin{array}{c} \mathbf{r} = \mathbf{r}(\theta^1, \theta^2) \\ \widehat{\mathbf{r}} = \widehat{\mathbf{r}}(\theta^1, \theta^2) \end{array} \tag{1.148}$$

The position vectors $\mathbf{r}$ and $\widehat{\mathbf{r}}$ respectively give the position of the points P and $\widehat{\mathrm{P}}$ on the corresponding curved surfaces. Consequently, the whole set of points $\{\mathrm{P}\}$ can be mapped on the set of points $\{\widehat{\mathrm{P}}\}$, which means that the position vector $\widehat{\mathbf{r}}$ can be written as a function of the position vector $\mathbf{r}$. Thus, we have

$$\theta^1, \theta^2 \qquad \rightarrow \qquad \mathbf{r} = \mathbf{r}(\theta^1, \theta^2) \qquad \rightarrow \qquad \widehat{\mathbf{r}} = \widehat{\mathbf{r}}(\mathbf{r}) = \widehat{\mathbf{r}}[\mathbf{r}(\theta^1, \theta^2)] = \widehat{\mathbf{r}}(\theta^1, \theta^2) \tag{1.149}$$

The first and second total differentials $\mathrm{d}\widehat{\mathbf{r}}$ and $\mathrm{d}^2\widehat{\mathbf{r}}$ are generally written as follows

$$\mathrm{d}\widehat{\mathbf{r}} = \mathrm{d}\mathbf{r}(\boldsymbol{\nabla}_n \otimes \widehat{\mathbf{r}}) \qquad ; \qquad \mathrm{d}^2\widehat{\mathbf{r}} = (\mathrm{d}\mathbf{r} \otimes \mathrm{d}\mathbf{r}) \cdot (\boldsymbol{\nabla}_n \otimes \boldsymbol{\nabla}_n \otimes \widehat{\mathbf{r}}) + \mathrm{d}^2\mathbf{r}(\boldsymbol{\nabla}_n \otimes \widehat{\mathbf{r}}) \tag{1.150}$$

For the first total differential, the tensor $(\boldsymbol{\nabla}_n \otimes \widehat{\mathbf{r}})$ is applied on the vector $\mathrm{d}\mathbf{r}$ and acts as a linear transformation. This means that the vector $\mathrm{d}\mathbf{r}$ is mapped onto the vector $\mathrm{d}\widehat{\mathbf{r}}$. For the second total differential, the tensor $(\boldsymbol{\nabla}_n \otimes \widehat{\mathbf{r}})$ is applied onto the vector $\mathrm{d}^2\mathbf{r}$ and also acts as a linear transformation. The third-rank tensor $(\boldsymbol{\nabla}_n \otimes \boldsymbol{\nabla}_n \otimes \widehat{\mathbf{r}})$ is applied onto the second-rank tensor $(\mathrm{d}\mathbf{r} \otimes \mathrm{d}\mathbf{r})$ in the sense of a double contraction.

There are a lot of different kinds of affine connections between surfaces in the 3-dimensional space. In holographic interferometry, we often encounter affine connections between several curved surfaces (up to twelve and more). An important case is when considering a bundle of rays coming from a single point and going through different surfaces. Such single fixed points in space are called collineation center and are often physically represented by light sources or projection centers. Another case is when considering a deformed object surface relatively to its undeformed configuration; in that case, both surfaces are related by an affine connection called object surface deformation. Other affine connections may describe e. g. the connection between the corresponding wavefronts at recording and at reconstruction (holography) or the connection between the parallel surfaces of a shell (thin-shell theory).

Here, we only treat some essential cases of interest, which will be encountered in the next sections without giving an exhaustive listing of other possibilities of affine connections in space.

Affine connections in case of a collineation center

The purpose of this section is to explain, how two curved surfaces $\mathbb{A}^2$ and $\widehat{\mathbb{A}}^2$ may be connected by a single fixed collineation center C. We write

$$\widehat{\mathbf{r}} = \mathbf{r}_c + \widehat{p}\mathbf{k} \tag{1.151}$$

where $\mathbf{r}_c = ¢$ is the constant position vector of the fixed collineation center C, $\widehat{p}$ is the length from C to $\widehat{\mathrm{P}}$ and $\mathbf{k}$ is the direction parallel to the vector $\overrightarrow{\mathrm{CP}}$ (Fig. 1.9). All three points C, P and $\widehat{\mathrm{P}}$ are located on the same straight line. We have with $\boldsymbol{\nabla}_n \otimes \mathbf{r}_c = 0$

$$\begin{aligned} &\boldsymbol{\nabla}_n \otimes \widehat{\mathbf{r}} = \boldsymbol{\nabla}_n \otimes (\mathbf{r}_c + \widehat{p}\mathbf{k}) = \boldsymbol{\nabla}_n \otimes (\widehat{p}\mathbf{k}) = \boldsymbol{\nabla}_n \widehat{p} \otimes \mathbf{k} + \widehat{p}\boldsymbol{\nabla}_n \otimes \mathbf{k} = \boldsymbol{\nabla}_n \widehat{p} \otimes \mathbf{k} + \frac{\widehat{p}}{p}\mathbf{NK} \\ &\boldsymbol{\nabla}_n \otimes \mathbf{k} = \mathbf{N}\boldsymbol{\nabla} \otimes \mathbf{k} = \frac{1}{p}\mathbf{NK} \quad ; \quad \mathbf{N} = \mathbf{I} - \mathbf{n} \otimes \mathbf{n} \quad ; \quad \mathbf{K} = \mathbf{I} - \mathbf{k} \otimes \mathbf{k} \end{aligned} \tag{1.152}$$

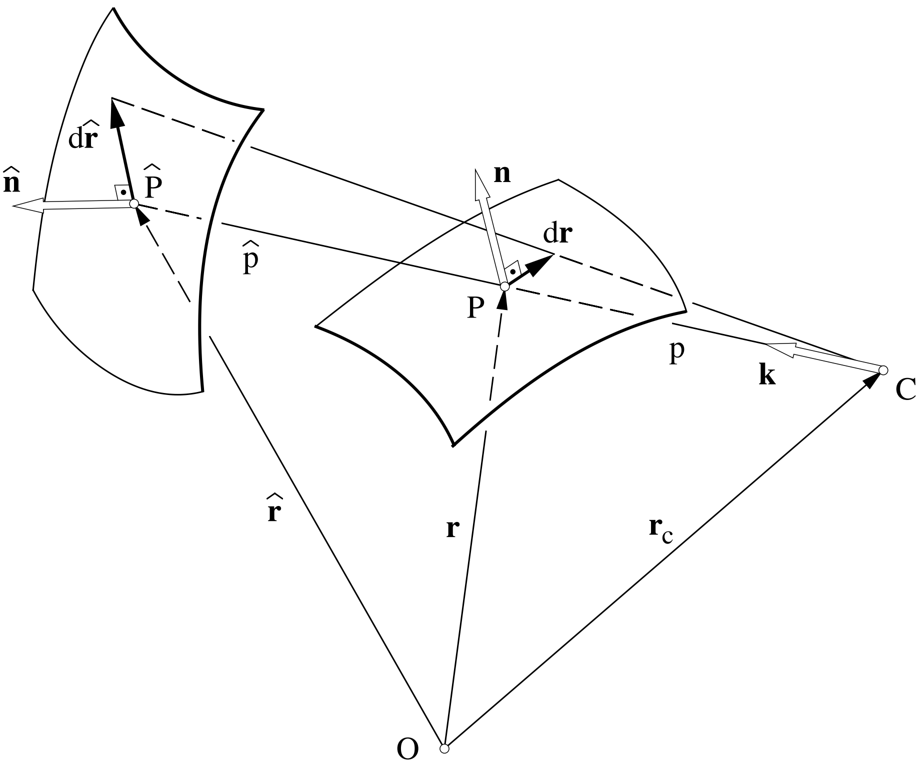

Fig. 1.9: Affine connection between two surfaces with collineation center C

With the unit normal $\widehat{\mathbf{n}}$ of the surface $\widehat{\mathbb{A}}^2$, the gradient $\boldsymbol{\nabla}_n \widehat{p}$ of the length $\widehat{p}$ can be explicitly calculated by using the condition of normality $\mathrm{d}\widehat{\mathbf{r}} \cdot \widehat{\mathbf{n}} = 0$, which reads $\forall\, \mathrm{d}\mathbf{r}$

$$0 = \mathrm{d}\widehat{\mathbf{r}} \cdot \widehat{\mathbf{n}} = \mathrm{d}\mathbf{r} \cdot (\boldsymbol{\nabla}_n \otimes \widehat{\mathbf{r}})\widehat{\mathbf{n}} = \mathrm{d}\mathbf{r} \cdot \left[\boldsymbol{\nabla}_n \widehat{p}(\mathbf{k} \cdot \widehat{\mathbf{n}}) + \frac{\widehat{p}}{p}\mathbf{NK}\widehat{\mathbf{n}}\right] \quad \Rightarrow \quad \boldsymbol{\nabla}_n \widehat{p} = -\frac{\widehat{p}}{p(\mathbf{k} \cdot \widehat{\mathbf{n}})}\mathbf{NK}\widehat{\mathbf{n}} \tag{1.153}$$

We then become

$$\begin{aligned} &\boldsymbol{\nabla}_n \otimes \widehat{\mathbf{r}} = \frac{\widehat{p}}{p}\mathbf{NK}\left(\mathbf{I} - \frac{\widehat{\mathbf{n}} \otimes \mathbf{k}}{\widehat{\mathbf{n}} \cdot \mathbf{k}}\right) = \frac{\widehat{p}}{p}\mathbf{NK}\widehat{\mathbf{M}} = \frac{\widehat{p}}{p}\mathbf{N}\widehat{\mathbf{M}} \quad ; \quad \widehat{\mathbf{M}} = \mathbf{I} - \frac{\widehat{\mathbf{n}} \otimes \mathbf{k}}{\widehat{\mathbf{n}} \cdot \mathbf{k}} \\ &\Rightarrow \quad \underline{\underline{\mathrm{d}\widehat{\mathbf{r}} = \mathrm{d}\mathbf{r}(\boldsymbol{\nabla}_n \otimes \widehat{\mathbf{r}}) = \frac{\widehat{p}}{p}\mathrm{d}\mathbf{r}\mathbf{N}\widehat{\mathbf{M}} = \frac{\widehat{p}}{p}\widehat{\mathbf{M}}^T \mathrm{d}\mathbf{r}}} \end{aligned} \tag{1.154}$$

Equation (1.154) describes a linear mapping of $\mathrm{d}\mathbf{r}$ onto $\mathrm{d}\widehat{\mathbf{r}}$. The tensor $\widehat{\mathbf{M}}$ is an oblique projector (not symmetric) which acts as an oblique projection along the direction $\widehat{\mathbf{n}}$ onto a plane normal to the direction $\mathbf{k}$. Thus both vectors $\mathrm{d}\mathbf{r}$ and $\mathrm{d}\widehat{\mathbf{r}}$ are related by an oblique projection and a proportionality factor.

By switching the role of $\mathbf{r}$ and $\widehat{\mathbf{r}}$, we get similarly

$$\begin{aligned} &\mathbf{r} = \mathbf{r}(\widehat{\mathbf{r}}) = \mathbf{r}_c + p\mathbf{k} \quad ; \quad \mathrm{d}\mathbf{r} = \mathrm{d}\widehat{\mathbf{r}}(\boldsymbol{\nabla}_{\hat{n}} \otimes \mathbf{r}) \quad ; \quad \mathbf{M} = \mathbf{I} - \frac{\mathbf{n}\otimes\mathbf{k}}{\mathbf{n}\cdot\mathbf{k}} \\ &\boldsymbol{\nabla}_{\hat{n}} p = -\frac{p}{\widehat{p}(\mathbf{k}\cdot\mathbf{n})}\widehat{\mathbf{N}}\mathbf{K}\mathbf{n} \quad ; \quad \boldsymbol{\nabla}_{\hat{n}} \otimes \mathbf{r} = \frac{p}{\widehat{p}}\widehat{\mathbf{N}}\mathbf{M} \quad ; \quad \underline{\underline{\mathrm{d}\mathbf{r} = \frac{p}{\widehat{p}}\mathbf{M}^T\mathrm{d}\widehat{\mathbf{r}}}} \end{aligned} \tag{1.155}$$

where $\boldsymbol{\nabla}_{\hat{n}} = \widehat{\mathbf{N}}\boldsymbol{\nabla}_{\hat{n}} = \widehat{\mathbf{N}}\boldsymbol{\nabla} = \widehat{\mathbf{a}}^{\beta}\partial/\partial\theta^{\beta}$ is the 2-dimensional derivative operator on the surface $\widehat{\mathbb{A}}^2$ and where $\widehat{\mathbf{N}} = \mathbf{I} - \widehat{\mathbf{n}}\otimes\widehat{\mathbf{n}}$ is the corresponding normal projector. Because both surfaces are described by the same curvilinear coordinates, we have

$$\mathrm{d}\mathbf{r}\cdot\boldsymbol{\nabla}_n = \mathrm{d}\theta^{\alpha}\mathbf{a}_{\alpha}\cdot\mathbf{a}^{\beta}\frac{\partial}{\partial\theta^{\beta}} = \mathrm{d}\theta^{\alpha}\frac{\partial}{\partial\theta^{\alpha}} = \mathrm{d}\theta^{\alpha}\widehat{\mathbf{a}}_{\alpha}\cdot\widehat{\mathbf{a}}^{\beta}\frac{\partial}{\partial\theta^{\beta}} = \mathrm{d}\widehat{\mathbf{r}}\cdot\boldsymbol{\nabla}_{\hat{n}} \tag{1.156}$$

which gives with $\mathrm{d}\mathbf{r} = \mathbf{N}\mathrm{d}\mathbf{r}$ and $\mathrm{d}\widehat{\mathbf{r}} = \widehat{\mathbf{N}}\mathrm{d}\widehat{\mathbf{r}}$

$$\mathrm{d}\mathbf{r}\cdot\boldsymbol{\nabla}_n = \mathrm{d}\widehat{\mathbf{r}}\cdot\frac{p}{\widehat{p}}\widehat{\mathbf{N}}\mathbf{M}\boldsymbol{\nabla}_n = \mathrm{d}\widehat{\mathbf{r}}\cdot\boldsymbol{\nabla}_{\hat{n}} = \mathrm{d}\mathbf{r}\cdot\frac{\widehat{p}}{p}\mathbf{N}\widehat{\mathbf{M}}\boldsymbol{\nabla}_{\hat{n}} \qquad , \qquad \forall\, \mathrm{d}\mathbf{r}, \mathrm{d}\widehat{\mathbf{r}} \tag{1.157}$$

Consequently, both 2-dimensional derivative operators $\boldsymbol{\nabla}_n$ and $\boldsymbol{\nabla}_{\hat{n}}$ are also related by a linear mapping, namely an oblique projection and a proportionality factor. We have

$$\underline{\underline{\boldsymbol{\nabla}_n = \frac{\widehat{p}}{p}\mathbf{N}\widehat{\mathbf{M}}\boldsymbol{\nabla}_{\hat{n}}}} \qquad ; \qquad \underline{\underline{\boldsymbol{\nabla}_{\hat{n}} = \frac{p}{\widehat{p}}\widehat{\mathbf{N}}\mathbf{M}\boldsymbol{\nabla}_n}} \tag{1.158}$$

For the second total differential $\mathrm{d}^2\widehat{\mathbf{r}}$, we must first perform some calculations

$$\begin{aligned} &\boldsymbol{\nabla}_n \otimes \boldsymbol{\nabla}_n \otimes \widehat{\mathbf{r}} = \boldsymbol{\nabla}_n \otimes \left(\frac{\widehat{p}}{p}\mathbf{N}\widehat{\mathbf{M}}\right) = \boldsymbol{\nabla}_n\widehat{p}\otimes\frac{1}{p}\mathbf{N}\widehat{\mathbf{M}} + \widehat{p}\boldsymbol{\nabla}_n\otimes\left(\frac{1}{p}\mathbf{N}\widehat{\mathbf{M}}\right) \\ &\boldsymbol{\nabla}_n \otimes \left(\frac{1}{p}\mathbf{N}\widehat{\mathbf{M}}\right) = \boldsymbol{\nabla}_n\frac{1}{p}\otimes\mathbf{N}\widehat{\mathbf{M}} + \frac{1}{p}(\boldsymbol{\nabla}_n\otimes\mathbf{N})\widehat{\mathbf{M}} + \frac{1}{p}\boldsymbol{\nabla}_n\otimes\widehat{\mathbf{M}})^T\mathbf{N})^T \\ &\boldsymbol{\nabla}_n\frac{1}{p} = -\frac{1}{p^2}\boldsymbol{\nabla}_n p = -\frac{1}{p^2}\mathbf{N}\mathbf{k} \\ &(\boldsymbol{\nabla}_n\otimes\mathbf{N})\widehat{\mathbf{M}} = \mathbf{B}\otimes\mathbf{n}\widehat{\mathbf{M}} + \mathbf{B}\widehat{\mathbf{M}}\otimes\mathbf{n})^T \\ &\boldsymbol{\nabla}_n\otimes\widehat{\mathbf{M}} = \boldsymbol{\nabla}_n\otimes\left(\mathbf{I} - \frac{\widehat{\mathbf{n}}\otimes\mathbf{k}}{\widehat{\mathbf{n}}\cdot\mathbf{k}}\right) = -\boldsymbol{\nabla}_n\otimes\left(\frac{\widehat{\mathbf{n}}\otimes\mathbf{k}}{\widehat{\mathbf{n}}\cdot\mathbf{k}}\right) \\ &\qquad = -\boldsymbol{\nabla}_n\left(\frac{1}{\widehat{\mathbf{n}}\cdot\mathbf{k}}\right)\otimes\widehat{\mathbf{n}}\otimes\mathbf{k} - \frac{1}{\widehat{\mathbf{n}}\cdot\mathbf{k}}\boldsymbol{\nabla}_n\otimes\widehat{\mathbf{n}}\otimes\mathbf{k} - \frac{1}{\widehat{\mathbf{n}}\cdot\mathbf{k}}\boldsymbol{\nabla}_n\otimes\mathbf{k}\otimes\widehat{\mathbf{n}})^T \\ &\boldsymbol{\nabla}_n\left(\frac{1}{\widehat{\mathbf{n}}\cdot\mathbf{k}}\right) = -\frac{1}{(\widehat{\mathbf{n}}\cdot\mathbf{k})^2}\boldsymbol{\nabla}_n(\widehat{\mathbf{n}}\cdot\mathbf{k}) = -\frac{1}{(\widehat{\mathbf{n}}\cdot\mathbf{k})^2}[(\boldsymbol{\nabla}_n\otimes\widehat{\mathbf{n}})\mathbf{k} + (\boldsymbol{\nabla}_n\otimes\mathbf{k})\widehat{\mathbf{n}}] \end{aligned} \tag{1.159}$$

$$\boldsymbol{\nabla}_n \otimes \widehat{\mathbf{n}} = \frac{\widehat{p}}{p}\mathbf{N}\widehat{\mathbf{M}}\boldsymbol{\nabla}_{\hat{n}} \otimes \widehat{\mathbf{n}} = -\frac{\widehat{p}}{p}\mathbf{N}\widehat{\mathbf{M}}\widehat{\mathbf{B}}$$

where $\widehat{\mathbf{B}} = -\boldsymbol{\nabla}_{\hat{n}} \otimes \widehat{\mathbf{n}}$ is the 2-dimensional curvature tensor of the surface $\widehat{\mathbb{A}}^2$. Introducing these above equations in the expression for $\mathrm{d}^2\widehat{\mathbf{r}}$, we get after some calculations

$$\mathrm{d}^2\widehat{\mathbf{r}} = \frac{\widehat{p}}{p^2}\mathrm{d}\mathbf{r}\left(\frac{\widehat{p}\widehat{\mathbf{M}}\widehat{\mathbf{B}}\widehat{\mathbf{M}}^T \otimes \mathbf{k})^T - 2\widehat{\mathbf{M}} \otimes \widehat{\mathbf{n}}}{\widehat{\mathbf{n}} \cdot \mathbf{k}}\right)\mathrm{d}\mathbf{r} + \frac{\widehat{p}}{p}\widehat{\mathbf{M}}^T\mathrm{d}^2\mathbf{r} = \frac{\widehat{p}}{p^2}\mathrm{d}\mathbf{r}\widehat{\boldsymbol{\mathcal{M}}}^T\mathrm{d}\mathbf{r} + \frac{\widehat{p}}{p}\widehat{\mathbf{M}}^T\mathrm{d}^2\mathbf{r} \quad (1.160)$$

where $\widehat{\boldsymbol{\mathcal{M}}} = [\widehat{p}\widehat{\mathbf{M}}\widehat{\mathbf{B}}\widehat{\mathbf{M}}^T \otimes \mathbf{k})^T - 2\widehat{\mathbf{n}} \otimes \widehat{\mathbf{M}}^T]/(\widehat{\mathbf{n}} \cdot \mathbf{k})$ is a third-rank tensor (not symmetric).

1.6 Derivatives of functions, which depend on several vector variables in space

Here, we shall present the derivative rules of tensor functions, which are dependent of several vector variables in space. As we will see, these rules will be very usefull when applied to the fields of holographic interferometry and projection moiré in the next sections.

1.6.1 Derivatives in space

Without restriction of the generality, we present here the case of tensor functions, which are dependent of two independent vector variables $\mathbf{r}$ and $\widehat{\mathbf{r}}$ in space. With the independent curvilinear coordinates θ^i and $\hat{\theta}^i$ $(i = 1, 2, 3)$, we have

$$\begin{matrix} \theta^1, \theta^2, \theta^3 \\ \hat{\theta}^1, \hat{\theta}^2, \hat{\theta}^3 \end{matrix} \quad \rightarrow \quad \begin{matrix} \mathbf{r} = \mathbf{r}(\theta^1, \theta^2, \theta^3) \\ \widehat{\mathbf{r}} = \widehat{\mathbf{r}}(\hat{\theta}^1, \hat{\theta}^2, \hat{\theta}^3) \end{matrix} \quad \rightarrow \quad \begin{cases} \phi = \phi(\mathbf{r}, \widehat{\mathbf{r}}) \\ \mathbf{u} = \mathbf{u}(\mathbf{r}, \widehat{\mathbf{r}}) \\ \mathbf{T} = \mathbf{T}(\mathbf{r}, \widehat{\mathbf{r}}) \\ \boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{B}}(\mathbf{r}, \widehat{\mathbf{r}}) \\ \ldots \end{cases} \quad (1.161)$$

where

$$\begin{aligned} \mathbf{r} &= x(\theta^1, \theta^2, \theta^3)\mathbf{i} + y(\theta^1, \theta^2, \theta^3)\mathbf{j} + z(\theta^1, \theta^2, \theta^3)\mathbf{k} \\ \widehat{\mathbf{r}} &= \widehat{x}(\hat{\theta}^1, \hat{\theta}^2, \hat{\theta}^3)\mathbf{i} + \widehat{y}(\hat{\theta}^1, \hat{\theta}^2, \hat{\theta}^3)\mathbf{j} + \widehat{z}(\hat{\theta}^1, \hat{\theta}^2, \hat{\theta}^3)\mathbf{k} \end{aligned} \quad (1.162)$$

We have

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial \theta^i} \quad , \quad \mathrm{d}\mathbf{r} = \mathbf{g}_i \mathrm{d}\theta^i \quad ; \quad \widehat{\mathbf{g}}_i = \frac{\partial \widehat{\mathbf{r}}}{\partial \hat{\theta}^i} \quad , \quad \mathrm{d}\widehat{\mathbf{r}} = \widehat{\mathbf{g}}_i \mathrm{d}\hat{\theta}^i \quad (1.163)$$

The 3-dimensional partial derivative operators $\boldsymbol{\partial}$ and $\widehat{\boldsymbol{\partial}}$ are defined as follows

$$\boldsymbol{\partial} = \mathbf{g}^j \frac{\partial}{\partial \theta^j} \quad ; \quad \widehat{\boldsymbol{\partial}} = \widehat{\mathbf{g}}^j \frac{\partial}{\partial \hat{\theta}^j} \quad ; \quad (j = 1, 2, 3) \quad (1.164)$$

where $\mathbf{g}^j \cdot \mathbf{g}_i = \widehat{\mathbf{g}}^j \cdot \widehat{\mathbf{g}}_i = \delta^j_{\ i}$. The total differentials of the tensor functions

$$\phi = \phi(\mathbf{r}, \widehat{\mathbf{r}}) = \phi(\theta^1, \theta^2, \theta^3, \hat{\theta}^1, \hat{\theta}^2, \hat{\theta}^3)$$

$$\begin{aligned}
&\mathbf{u} = \mathbf{u}(\mathbf{r},\widehat{\mathbf{r}}) = \mathbf{u}(\theta^1,\theta^2,\theta^3,\hat{\theta}^1,\hat{\theta}^2,\hat{\theta}^3)\\
&\mathbf{T} = \mathbf{T}(\mathbf{r},\widehat{\mathbf{r}}) = \mathbf{T}(\theta^1,\theta^2,\theta^3,\hat{\theta}^1,\hat{\theta}^2,\hat{\theta}^3)\\
&\boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{B}}(\mathbf{r},\widehat{\mathbf{r}}) = \boldsymbol{\mathcal{B}}(\theta^1,\theta^2,\theta^3,\hat{\theta}^1,\hat{\theta}^2,\hat{\theta}^3)\\
&\ldots
\end{aligned} \tag{1.165}$$

are then written as follows

$$\begin{aligned}
\mathrm{d}\phi &= \frac{\partial\phi}{\partial\theta^i}\mathrm{d}\theta^i + \frac{\partial\phi}{\partial\hat{\theta}^i}\mathrm{d}\hat{\theta}^i = \mathrm{d}\mathbf{r}\cdot\boldsymbol{\partial}\phi + \mathrm{d}\widehat{\mathbf{r}}\cdot\widehat{\boldsymbol{\partial}}\phi\\
\mathrm{d}\mathbf{u} &= \frac{\partial\mathbf{u}}{\partial\theta^i}\mathrm{d}\theta^i + \frac{\partial\mathbf{u}}{\partial\hat{\theta}^i}\mathrm{d}\hat{\theta}^i = \mathrm{d}\mathbf{r}(\boldsymbol{\partial}\otimes\mathbf{u}) + \mathrm{d}\widehat{\mathbf{r}}(\widehat{\boldsymbol{\partial}}\otimes\mathbf{u})\\
\mathrm{d}\mathbf{T} &= \frac{\partial\mathbf{T}}{\partial\theta^i}\mathrm{d}\theta^i + \frac{\partial\mathbf{T}}{\partial\hat{\theta}^i}\mathrm{d}\hat{\theta}^i = \mathrm{d}\mathbf{r}(\boldsymbol{\partial}\otimes\mathbf{T}) + \mathrm{d}\widehat{\mathbf{r}}(\widehat{\boldsymbol{\partial}}\otimes\mathbf{T})\\
\mathrm{d}\boldsymbol{\mathcal{B}} &= \frac{\partial\boldsymbol{\mathcal{B}}}{\partial\theta^i}\mathrm{d}\theta^i + \frac{\partial\boldsymbol{\mathcal{B}}}{\partial\hat{\theta}^i}\mathrm{d}\hat{\theta}^i = \mathrm{d}\mathbf{r}(\boldsymbol{\partial}\otimes\boldsymbol{\mathcal{B}}) + \mathrm{d}\widehat{\mathbf{r}}(\widehat{\boldsymbol{\partial}}\otimes\boldsymbol{\mathcal{B}})\\
&\ldots
\end{aligned} \tag{1.166}$$

1.6.2 Derivatives of lengths, directions and normal projectors in space

Lengths, directions and normal projectors may also be dependent of more than one vector variable in space. Here we present the case of tensor functions of two vector variables in space.

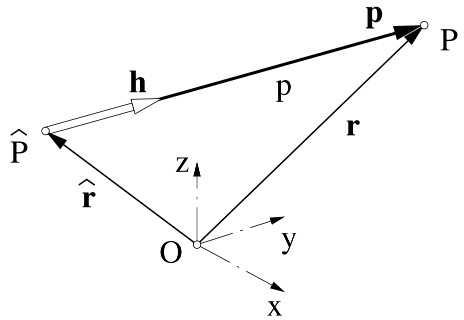

Fig. 1.10: Derivatives of tensor functions of two vector variables

In this case, we write

$$\begin{aligned}
&\boldsymbol{\partial}p = \mathbf{h}\\
&\boldsymbol{\partial}\otimes\boldsymbol{\partial}p = \boldsymbol{\partial}\otimes\mathbf{h} = \frac{1}{p}\mathbf{H}\\
&\boldsymbol{\partial}\otimes\boldsymbol{\partial}\otimes\boldsymbol{\partial}p = \boldsymbol{\partial}\otimes\boldsymbol{\partial}\otimes\mathbf{h} = \boldsymbol{\partial}\otimes\left(\frac{1}{p}\mathbf{H}\right) = -\frac{1}{p^2}\boldsymbol{\mathcal{H}}\\
&\boldsymbol{\partial}\otimes\boldsymbol{\partial}\otimes\boldsymbol{\partial}\otimes\boldsymbol{\partial}p = \boldsymbol{\partial}\otimes\boldsymbol{\partial}\otimes\boldsymbol{\partial}\otimes\mathbf{h} = \boldsymbol{\partial}\otimes\boldsymbol{\partial}\otimes\left(\frac{1}{p}\mathbf{H}\right) = -\boldsymbol{\partial}\otimes\left(\frac{1}{p^2}\boldsymbol{\mathcal{H}}\right) = \frac{1}{p^3}\boldsymbol{H}
\end{aligned} \tag{1.167}$$

$$
\begin{aligned}
&\dots\\
&\widehat{\boldsymbol{\partial}} p = -\mathbf{h}\\
&\widehat{\boldsymbol{\partial}} \otimes \widehat{\boldsymbol{\partial}} p = -\widehat{\boldsymbol{\partial}} \otimes \mathbf{h} = \frac{1}{p}\mathbf{H}\\
&\widehat{\boldsymbol{\partial}} \otimes \widehat{\boldsymbol{\partial}} \otimes \widehat{\boldsymbol{\partial}} p = -\widehat{\boldsymbol{\partial}} \otimes \widehat{\boldsymbol{\partial}} \otimes \mathbf{h} = \widehat{\boldsymbol{\partial}} \otimes \left(\frac{1}{p}\mathbf{H}\right) = \frac{1}{p^2}\boldsymbol{\mathcal{H}}\\
&\widehat{\boldsymbol{\partial}} \otimes \widehat{\boldsymbol{\partial}} \otimes \widehat{\boldsymbol{\partial}} \otimes \widehat{\boldsymbol{\partial}} p = -\widehat{\boldsymbol{\partial}} \otimes \widehat{\boldsymbol{\partial}} \otimes \widehat{\boldsymbol{\partial}} \otimes \mathbf{h} = \widehat{\boldsymbol{\partial}} \otimes \widehat{\boldsymbol{\partial}} \otimes \left(\frac{1}{p}\mathbf{H}\right) = \widehat{\boldsymbol{\partial}} \otimes \left(\frac{1}{p^2}\boldsymbol{\mathcal{H}}\right) = \frac{1}{p^3}\boldsymbol{H}\\
&\dots
\end{aligned}
\tag{1.168}
$$

1.6.3 Derivatives on curved surfaces

By keeping the third independent variables θ^3 and $\hat{\theta}^3$ constant, the position vectors $\mathbf{r}$ and $\widehat{\mathbf{r}}$ describe two curved surfaces in space. Thus, we have (with $\alpha = 1, 2$):

$$
\mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial \theta^\alpha} \quad , \quad \mathrm{d}\mathbf{r} = \mathbf{N}\mathrm{d}\mathbf{r} = \mathbf{a}_\alpha \mathrm{d}\theta^\alpha \qquad ; \qquad \widehat{\mathbf{a}}_\alpha = \frac{\partial \widehat{\mathbf{r}}}{\partial \hat{\theta}^\alpha} \quad , \quad \mathrm{d}\widehat{\mathbf{r}} = \widehat{\mathbf{N}}\mathrm{d}\widehat{\mathbf{r}} = \widehat{\mathbf{a}}_\alpha \mathrm{d}\hat{\theta}^\alpha \tag{1.169}
$$

The 2-dimensional partial derivative operators $\boldsymbol{\partial}_n$ and $\boldsymbol{\partial}_{\hat{n}}$ on the surfaces $\mathbb{A}^2$ and $\widehat{\mathbb{A}}^2$ are defined as follows

$$
\boldsymbol{\partial}_n = \mathbf{a}^\beta \frac{\partial}{\partial \theta^\beta} = \mathbf{N}\boldsymbol{\partial} \qquad ; \qquad \boldsymbol{\partial}_{\hat{n}} = \widehat{\mathbf{a}}^\beta \frac{\partial}{\partial \hat{\theta}^\beta} = \widehat{\mathbf{N}}\widehat{\boldsymbol{\partial}} \qquad ; \qquad (\beta = 1, 2) \tag{1.170}
$$

where $\mathbf{a}^\beta \cdot \mathbf{a}_\alpha = \widehat{\mathbf{a}}^\beta \cdot \widehat{\mathbf{a}}_\alpha = \delta^\beta_{\ \alpha}$. The total differentials of the corresponding tensor functions read

$$
\begin{aligned}
&\mathrm{d}\phi = \mathrm{d}\mathbf{r} \cdot \boldsymbol{\partial}_n \phi + \mathrm{d}\widehat{\mathbf{r}} \cdot \boldsymbol{\partial}_{\hat{n}} \phi\\
&\mathrm{d}\mathbf{u} = \mathrm{d}\mathbf{r}(\boldsymbol{\partial}_n \otimes \mathbf{u}) + \mathrm{d}\widehat{\mathbf{r}}(\boldsymbol{\partial}_{\hat{n}} \otimes \mathbf{u})\\
&\mathrm{d}\mathbf{T} = \mathrm{d}\mathbf{r}(\boldsymbol{\partial}_n \otimes \mathbf{T}) + \mathrm{d}\widehat{\mathbf{r}}(\boldsymbol{\partial}_{\hat{n}} \otimes \mathbf{T})\\
&\mathrm{d}\boldsymbol{\mathcal{B}} = \mathrm{d}\mathbf{r}(\boldsymbol{\partial}_n \otimes \boldsymbol{\mathcal{B}}) + \mathrm{d}\widehat{\mathbf{r}}(\boldsymbol{\partial}_{\hat{n}} \otimes \boldsymbol{\mathcal{B}})\\
&\dots
\end{aligned}
\tag{1.171}
$$

2. Kinematics of deformation

The purpose of the first part of this chapter is to present, in a simplified way, the kinematics of deformation of a 3-dimensional object and of a 2-dimensional curved surface in space. Emphasis is put on the linear theory and on the bridge between the 3-dimensional and 2-dimensional deformation. The second part deal with the some more complicated case of nonlinear kinematics of deformation [2.1–2.9]. The relations presented here will be very usefull in the following section on holographic interferometry.

2.1 Deformation of a 3-dimensional object in space

In case of a 3-dimensional object deformation in space, we have

$$\theta^1, \theta^2, \theta^3 \qquad \rightarrow \qquad \begin{cases} \mathbf{r} = \mathbf{r}(\theta^1, \theta^2, \theta^3) \\ \mathbf{r}' = \mathbf{r}'(\theta^1, \theta^2, \theta^3) \end{cases} \tag{2.1}$$

where $\mathbf{r}$ represents the vector coordinate of a point P of the undeformed object and $\mathbf{r}'$ the vector coordinate of a point P$'$ of the deformed object.

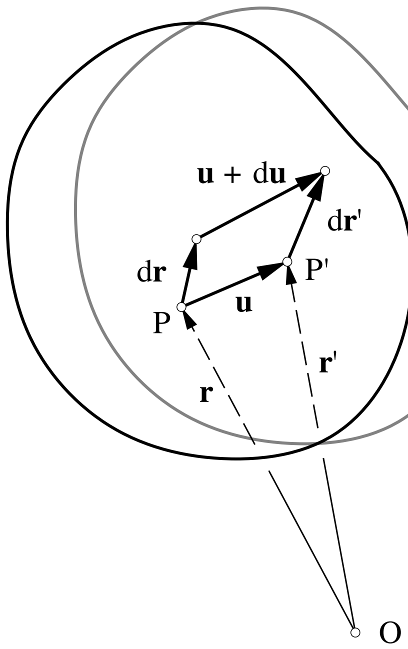

Fig. 2.1: Deformation of a 3-dimensional object in space

2.1.1 Lagrangean representation

The description of the object deformation in a Lagrangean representation may be written as follows

$$\begin{aligned} \mathbf{r} &\quad\rightarrow\quad \mathbf{u} = \mathbf{u}(\mathbf{r}) \\ \mathbf{r} &\quad\rightarrow\quad \mathbf{r}' = \mathbf{r}'(\mathbf{r}) = \mathbf{r} + \mathbf{u}(\mathbf{r}) \end{aligned} \tag{2.2}$$

where $\mathbf{u}$ describes the displacement of point P in the new position P$'$. If no dislocations are present, the total differential $\mathrm{d}\mathbf{r}'$ reads

$$\mathrm{d}\mathbf{r}' = \mathrm{d}\mathbf{r}(\boldsymbol{\nabla} \otimes \mathbf{r}') = \mathrm{d}\mathbf{r} + \mathrm{d}\mathbf{u} = \mathrm{d}\mathbf{r} + \mathrm{d}\mathbf{r}(\boldsymbol{\nabla} \otimes \mathbf{u}) = \mathbf{F}\mathrm{d}\mathbf{r} \tag{2.3}$$

where $\mathbf{F} = \mathbf{I} + (\boldsymbol{\nabla} \otimes \mathbf{u})^T$ is the so-called deformation gradient. The above equation describes a linear transformation of $\mathrm{d}\mathbf{r}$ onto $\mathrm{d}\mathbf{r}'$ by means of the tensor $\mathbf{F}$. A polar decomposition (multiplicative) gives

$$\mathbf{F} = \mathbf{I} + (\boldsymbol{\nabla} \otimes \mathbf{u})^T = \mathbf{Q}\mathbf{U} \tag{2.4}$$

where $\mathbf{U} = \mathbf{U}^T$ is a symmetric tensor and $\mathbf{Q}$ an orthogonal tensor. The tensor $\mathbf{U}$ describes a dilatation and the tensor $\mathbf{Q}$ a rotation in space.

Consequently, a general deformation of a 3-dimensional object in space may be decomposed in a Lagrangean representation as follows

1°) The neighborhood of a point P of the undeformed object, also called infinitesimal volume element, undergoes a dilatation by means of the tensor $\mathbf{U}$. We call this step the dilatation.

2°) Afterwards, the strained volume element undergoes a rotation by means of the tensor $\mathbf{Q}$. This rotation occur around an axis Δ going through point P. We call this step the rotation.

3°) Finally, the strained and rotated volume element undergoes a translation from its position P to the new position P$'$ by means of the displacement vector $\mathbf{u}$. We call this step the displacement.

As summary, we have

$$\boxed{\text{Deformation}} = \boxed{\text{Dilatation}} + \boxed{\text{Rotation}} + \boxed{\text{Displacement}} \tag{2.5}$$

This description is only valid for a single point and vary from point to point.

The general symmetric strain tensor $\widetilde{\boldsymbol{\mathcal{E}}}$ only contains the dilatation and is defined with the *Cauchy-Green* tensor $\mathbf{F}^T\mathbf{F} = \mathbf{U}^2$ as follows

$$\widetilde{\boldsymbol{\mathcal{E}}} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) = \frac{1}{2}(\mathbf{U}\mathbf{Q}^T\mathbf{Q}\mathbf{U} - \mathbf{I}) = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I}) \quad ; \quad \mathbf{Q}^T\mathbf{Q} = \mathbf{I} \quad , \quad \mathbf{U}^2 = \mathbf{U}\mathbf{U} \tag{2.6}$$

In order to get the components of the dilatation, we write

$$\begin{aligned} &\mathrm{d}\mathbf{r} = \mathbf{e}\mathrm{d}s \quad ; \quad \mathrm{d}\mathbf{r}_\perp = \mathbf{e}_\perp \mathrm{d}s_\perp \quad ; \quad \mathbf{e}\cdot\mathbf{e} = \mathbf{e}_\perp \cdot \mathbf{e}_\perp = 1 \quad ; \quad \mathbf{e}\cdot\mathbf{e}_\perp = 0 \\ &\mathrm{d}\mathbf{r}' = \mathbf{e}'\mathrm{d}s' \quad ; \quad \mathrm{d}\mathbf{r}'_\perp = \mathbf{e}'_\perp \mathrm{d}s'_\perp \quad ; \quad \mathbf{e}'\cdot\mathbf{e}' = \mathbf{e}'_\perp \cdot \mathbf{e}'_\perp = 1 \quad ; \quad -1 \leqslant \mathbf{e}'\cdot\mathbf{e}'_\perp \leqslant 1 \end{aligned} \tag{2.7}$$

From the usual theory of deformation, we have

$$\varepsilon = \frac{\mathrm{d}s' - \mathrm{d}s}{\mathrm{d}s} \quad ; \quad \varepsilon_\perp = \frac{\mathrm{d}s'_\perp - \mathrm{d}s_\perp}{\mathrm{d}s_\perp} \quad ; \quad \cos(\frac{\pi}{2} - \gamma) = \sin\gamma = \mathbf{e}'\cdot\mathbf{e}'_\perp$$

$$\left.\begin{aligned}\widetilde{\varepsilon} &= \mathbf{e}\cdot\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{e} = \frac{1}{2}\mathbf{e}\cdot(\mathbf{F}^T\mathbf{F}-\mathbf{I})\mathbf{e}\\ &= \frac{\mathrm{d}s'^2-\mathrm{d}s^2}{2\mathrm{d}s^2} = \varepsilon + \frac{1}{2}\varepsilon^2\end{aligned}\right\} \quad\Rightarrow\quad \varepsilon = \sqrt{1+2\widetilde{\varepsilon}}-1 = \sqrt{1+2\mathbf{e}\cdot\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{e}}-1 \tag{2.8}$$

$$\left.\begin{aligned}\widetilde{\varepsilon}_\perp &= \mathbf{e}_\perp\cdot\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{e}_\perp = \frac{1}{2}\mathbf{e}_\perp\cdot(\mathbf{F}^T\mathbf{F}-\mathbf{I})\mathbf{e}_\perp\\ &= \frac{\mathrm{d}s'^2_\perp-\mathrm{d}s^2_\perp}{2\mathrm{d}s^2_\perp} = \varepsilon_\perp + \frac{1}{2}\varepsilon_\perp^2\end{aligned}\right\} \quad\Rightarrow\quad \varepsilon_\perp = \sqrt{1+2\widetilde{\varepsilon}_\perp}-1 = \sqrt{1+2\mathbf{e}_\perp\cdot\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{e}_\perp}-1$$

$$\left.\begin{aligned}\frac{1}{2}\widetilde{\gamma} &= \mathbf{e}\cdot\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{e}_\perp = \frac{1}{2}\mathbf{e}\cdot(\mathbf{F}^T\mathbf{F}-\mathbf{I})\mathbf{e}_\perp\\ &= \frac{\mathrm{d}s'\mathrm{d}s'_\perp}{2\mathrm{d}s\mathrm{d}s_\perp}(\mathbf{e}'\cdot\mathbf{e}'_\perp) = \frac{1}{2}(1+\varepsilon)(1+\varepsilon_\perp)\sin\gamma\end{aligned}\right\} \quad\Rightarrow\quad \sin\gamma = \frac{2\mathbf{e}\cdot\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{e}_\perp}{\sqrt{(1+2\mathbf{e}\cdot\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{e})(1+2\mathbf{e}_\perp\cdot\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{e}_\perp)}}$$

where ε is the linear dilatation and γ the angular dilatation or shearing strain. The scalars $\widetilde{\varepsilon}, \widetilde{\varepsilon}_\perp$ and $\widetilde{\gamma}/2$ are the components of the strain tensor $\widetilde{\boldsymbol{\mathcal{E}}}$ corresponding to the tensors $\mathbf{e}\otimes\mathbf{e}, \mathbf{e}_\perp\otimes\mathbf{e}_\perp$ and $\mathbf{e}\otimes\mathbf{e}_\perp$, which are built on the unit vectors $\mathbf{e}$ and $\mathbf{e}_\perp$. Note that other components may be built on the third unit vector $\mathbf{e}_{b\perp} = \mathbf{e}\times\mathbf{e}_\perp = \mathbf{e}_\perp\boldsymbol{\mathcal{E}}\mathbf{e}$.

The deformation gradient may also be decomposed in an additive manner

$$\mathbf{F} = \mathbf{I} + (\boldsymbol{\nabla}\otimes\mathbf{u})^T = \mathbf{I} + \boldsymbol{\mathcal{E}} - \boldsymbol{\Omega} \tag{2.9}$$

with

$$\begin{aligned}\boldsymbol{\mathcal{E}} &= \boldsymbol{\mathcal{E}}^T = \frac{1}{2}[\boldsymbol{\nabla}\otimes\mathbf{u} + (\boldsymbol{\nabla}\otimes\mathbf{u})^T]\\ \boldsymbol{\Omega} &= -\boldsymbol{\Omega}^T = \frac{1}{2}[\boldsymbol{\nabla}\otimes\mathbf{u} - (\boldsymbol{\nabla}\otimes\mathbf{u})^T]\end{aligned} \tag{2.10}$$

where $\boldsymbol{\mathcal{E}}$ is a symmetric tensor and $\boldsymbol{\Omega}$ an antimetric tensor.

Special case of a small deformation

For $\mathbf{F}\simeq\mathbf{I}$, we have

$$\begin{aligned}\widetilde{\boldsymbol{\mathcal{E}}} &= \frac{1}{2}(\mathbf{F}^T\mathbf{F}-\mathbf{I}) = \frac{1}{2}[(\mathbf{I}+\boldsymbol{\mathcal{E}}+\boldsymbol{\Omega})(\mathbf{I}+\boldsymbol{\mathcal{E}}-\boldsymbol{\Omega})-\mathbf{I}]\\ &= \frac{1}{2}[\mathbf{I}+\boldsymbol{\mathcal{E}}-\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}}+\boldsymbol{\Omega}+\ldots-\mathbf{I}] \simeq \boldsymbol{\mathcal{E}}\end{aligned} \tag{2.11}$$

In a first approximation, the tensor $\boldsymbol{\mathcal{E}}$ describes the dilatation and the tensor $\boldsymbol{\Omega}$ the rotation of the volume element. Proof:

$$\begin{aligned}&\mathbf{U}\simeq\mathbf{I}+\mathbf{X} \quad ; \quad \mathbf{U}=\mathbf{U}^T\simeq\mathbf{I}+\mathbf{X}=\mathbf{I}+\mathbf{X}^T \quad\Rightarrow\quad \mathbf{X}=\mathbf{X}^T\\ &\mathbf{Q}\simeq\mathbf{I}+\mathbf{Y} \quad ; \quad \mathbf{I}=\mathbf{Q}^T\mathbf{Q}\simeq\mathbf{I}+\mathbf{Y}+\mathbf{Y}^T \quad\Rightarrow\quad \mathbf{Y}=-\mathbf{Y}^T\\ &\mathbf{F}=\mathbf{Q}\mathbf{U}=\mathbf{I}+\boldsymbol{\mathcal{E}}-\boldsymbol{\Omega}\simeq\mathbf{I}+\mathbf{X}+\mathbf{Y} \quad\Rightarrow\quad \mathbf{X}\simeq\boldsymbol{\mathcal{E}} \quad ; \quad \mathbf{Y}\simeq-\boldsymbol{\Omega}\\ &\Rightarrow\quad \mathbf{U}\simeq\mathbf{I}+\boldsymbol{\mathcal{E}} \quad ; \quad \mathbf{Q}\simeq\mathbf{I}-\boldsymbol{\Omega} \qquad \square\ \text{qed}\end{aligned} \tag{2.12}$$

2.1.2 Eulerian representation

The description of the object deformation in an Eulerian representation may be written as follows

$$\begin{aligned}
&\mathbf{r}' \quad \rightarrow \quad \mathbf{u} = \mathbf{u}(\mathbf{r}') \\
&\mathbf{r}' \quad \rightarrow \quad \mathbf{r} = \mathbf{r}(\mathbf{r}') = \mathbf{r}' - \mathbf{u}(\mathbf{r}') \\
&\mathrm{d}\mathbf{r} = \mathrm{d}\mathbf{r}'(\boldsymbol{\nabla}' \otimes \mathbf{r}) = \mathrm{d}\mathbf{r}' - \mathrm{d}\mathbf{u} = \mathrm{d}\mathbf{r}' - \mathrm{d}\mathbf{r}'(\boldsymbol{\nabla}' \otimes \mathbf{u}) = \mathbf{F}'\mathrm{d}\mathbf{r}' = \mathbf{F}'\mathbf{F}\mathrm{d}\mathbf{r} \\
&\mathrm{d}\mathbf{r}' = \mathbf{F}\mathrm{d}\mathbf{r} = \mathbf{F}\mathbf{F}'\mathrm{d}\mathbf{r}'
\end{aligned} \tag{2.13}$$

with

$$\mathbf{F}' = \mathbf{I} - (\boldsymbol{\nabla}' \otimes \mathbf{u})^T = \mathbf{F}^{-1} \quad ; \quad \mathbf{F}\mathbf{F}' = \mathbf{F}'\mathbf{F} = \mathbf{I} \tag{2.14}$$

where $\boldsymbol{\nabla}'$ represents the 3-dimensional derivative operator for the deformed configuration and $\mathbf{F}'$ the inverse of the deformation gradient. According to Lagrange, we have

$$\mathbf{F}' = \mathbf{F}^{-1} = \mathbf{U}^{-1}\mathbf{Q}^T \tag{2.15}$$

The polar decomposition according to Euler reads

$$\mathbf{F} = \mathbf{U}'\mathbf{Q} \qquad ; \qquad \mathbf{F}' = \mathbf{F}^{-1} = \mathbf{Q}^T\mathbf{U}'^{-1} \tag{2.16}$$

The symmetric tensor $\mathbf{U}'$ describes the dilatation of the volume element. In this case, we have

$$\boxed{\text{Deformation}} = \boxed{\text{Displacement}} + \boxed{\text{Rotation}} + \boxed{\text{Dilatation}} \tag{2.17}$$

Because both the undeformed and deformed object configurations are described by the same curvilinear coordinates, we have with $(\mathbf{F}^T)^{-1} = (\mathbf{F}^{-1})^T$

$$\begin{aligned}
&\mathrm{d}\mathbf{r} \cdot \boldsymbol{\nabla} = \mathrm{d}\mathbf{r}' \cdot \boldsymbol{\nabla}' = \mathrm{d}\mathbf{r} \cdot \mathbf{F}^T\boldsymbol{\nabla}' = \mathrm{d}\mathbf{r}' \cdot (\mathbf{F}^T)^{-1}\boldsymbol{\nabla} \qquad , \qquad \forall\, \mathrm{d}\mathbf{r}\,,\, \mathrm{d}\mathbf{r}' \\
&\Rightarrow \qquad \boldsymbol{\nabla} = \mathbf{F}^T\boldsymbol{\nabla}' \qquad ; \qquad \boldsymbol{\nabla}' = (\mathbf{F}^T)^{-1}\boldsymbol{\nabla}
\end{aligned} \tag{2.18}$$

2.2 Deformation of a curved surface in space

In the field of deformation analysis of opaque bodies by means of holographic interferometry, only the object surface can be recorded and not the interior of the body. This implies, that only an information on the object surface deformation may be collected with this optical technique. Without a constitutive law, no information can be obtained on the interior of a 3-dimensional opaque body by means of holographic interferometry. Therefore, we present here the relations needed to deal with the deformation of a 2-dimensional curved surface $\mathbb{A}^2$ in space.

In the case of a 2-dimensional curved surface deformation in space, we have

$$\theta^1, \theta^2 \qquad \rightarrow \qquad \begin{cases} \mathbf{r} = \mathbf{r}(\theta^1, \theta^2) \\ \mathbf{r}' = \mathbf{r}'(\theta^1, \theta^2) \end{cases} \tag{2.19}$$

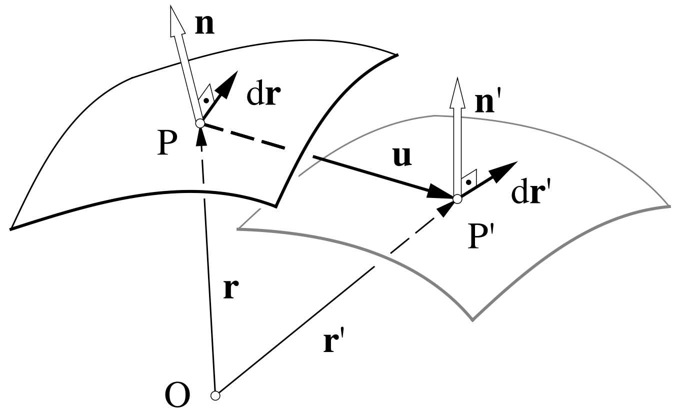

Fig. 2.2: Deformation of a curved surface in space

where $\mathbf{r}$ is the vector coordinate of a point P on the undeformed surface and $\mathbf{r}'$ the vector coordinate of a point P$'$ on the deformed surface.

2.2.1 Lagrangean representation

The description of the surface deformation in a Lagrangean representation may be written as follows

$$\begin{aligned} \mathbf{r} &\quad\rightarrow\quad \mathbf{u} = \mathbf{u}(\mathbf{r}) \\ \mathbf{r} &\quad\rightarrow\quad \mathbf{r}' = \mathbf{r}'(\mathbf{r}) = \mathbf{r} + \mathbf{u}(\mathbf{r}) \end{aligned} \tag{2.20}$$

where $\mathbf{u}$ describes the displacement of a point P in the new position P$'$. If no dislocations are present, the total differential $\mathrm{d}\mathbf{r}'$ reads

$$\mathrm{d}\mathbf{r}' = \mathbf{N}'\mathrm{d}\mathbf{r}' = \mathrm{d}\mathbf{r}(\boldsymbol{\nabla}_n \otimes \mathbf{r}') = \mathrm{d}\mathbf{r} + \mathrm{d}\mathbf{u} = \mathbf{N}\mathrm{d}\mathbf{r} + \mathrm{d}\mathbf{r}(\boldsymbol{\nabla}_n \otimes \mathbf{u}) = \mathbf{F}\mathbf{N}\mathrm{d}\mathbf{r} = \mathbf{F}_S\mathrm{d}\mathbf{r} \tag{2.21}$$

where $\mathbf{F}_S = \mathbf{N}'\mathbf{F}_S = \mathbf{F}\mathbf{N} = \mathbf{Q}\mathbf{U}\mathbf{N} = \mathbf{N} + (\boldsymbol{\nabla}_n \otimes \mathbf{u})^T$ is the so-called deformation gradient of the surface. The above equation describes a linear transformation of $\mathrm{d}\mathbf{r} = \mathbf{N}\mathrm{d}\mathbf{r}$ onto $\mathrm{d}\mathbf{r}' = \mathbf{N}'\mathrm{d}\mathbf{r}'$ by means of the tensor $\mathbf{F}_S$. A polar decomposition gives

$$\mathbf{F}_S = \mathbf{F}\mathbf{N} = \mathbf{N} + (\boldsymbol{\nabla}_n \otimes \mathbf{u})^T = \mathbf{Q}_S\mathbf{V} \qquad ; \qquad \mathbf{V} = \mathbf{V}^T = \mathbf{N}\mathbf{V}\mathbf{N} \tag{2.22}$$

where $\mathbf{V}$ is a 2-dimensional symmetric tensor and $\mathbf{Q}_S$ a 3-dimensional orthogonal tensor. The tensor $\mathbf{V}$ describes the bidimensional dilatation of an infinitesimal surface element and the tensor $\mathbf{Q}_S$ a rotation of this surface element in the 3-dimensional space. The unit normal $\mathbf{n}'$ of the deformed surface reads

$$\mathbf{n}' = \mathbf{Q}_S\mathbf{n} \tag{2.23}$$

Proof:

$$\begin{aligned} &\mathbf{F}_S = \mathbf{Q}_S\mathbf{V} = \mathbf{N}'\mathbf{F}_S = (\mathbf{I} - \mathbf{n}' \otimes \mathbf{n}')\mathbf{Q}_S\mathbf{V} = \mathbf{Q}_S\mathbf{V} - \mathbf{n}' \otimes \mathbf{n}'\mathbf{Q}_S\mathbf{V} \\ &\Rightarrow \quad \mathbf{n}'\mathbf{Q}_S\mathbf{V} = \mathbf{V}\mathbf{Q}_S^T\mathbf{n}' = 0 \qquad ; \qquad \mathbf{Q}_S^T\mathbf{n}' \neq 0 \end{aligned}$$

$$\Rightarrow \quad \mathbf{N}\mathbf{Q}_S^T\mathbf{n}' = 0 \quad ; \quad |\mathbf{Q}_S^T\mathbf{n}'| = 1 \tag{2.24}$$
$$\Rightarrow \quad \mathbf{Q}_S^T\mathbf{n}' = \mathbf{n} \quad \rightarrow \quad \mathbf{n}' = \mathbf{Q}_S\mathbf{n} \qquad \square \text{ qed}$$

Consequently, a general deformation of a 2-dimensional curved surface in space may be decomposed in a Lagrangean representation as follows

1°) The neighborhood of a point P on the undeformed curved surface, also called infinitesimal surface element, undergoes a bidimensional dilatation by means of the 2-dimensional tensor $\mathbf{V}$. We call this step the bidimensional dilatation because the surface element remains in the same tangential plane.

2°) Afterwards, the strained surface element undergoes a rotation by means of the tensor $\mathbf{Q}_S$. This rotation occur around an axis Δ_S going through point P. We call this step the rotation.

3°) Finally, the strained and rotated surface element undergoes a translation from its position P to the new position P′ by means of the displacement vector $\mathbf{u}$. We call this step the displacement.

As summary, we have

$$\boxed{\text{Deformation}} = \boxed{\text{Bidimensional dilatation}} + \boxed{\text{Rotation}} + \boxed{\text{Displacement}} \tag{2.25}$$

This description is only valid for a single point and vary from point to point on the surface.

Important note: The 3-dimensional rotation tensors $\mathbf{Q}_S$ and $\mathbf{Q}$ are in general not identical. This comes from the 3-dimensional symmetric tensor $\mathbf{U}$, which would not only acts as a bidimensional dilatation of the surface element but also as a rotation out of the tangential plane of the undeformed curved surface.

The general symmetric strain tensor of the surface $\widetilde{\gamma}$ only contains the bidimensional dilatation and is defined with the *Cauchy-Green* tensor $\mathbf{F}_S^T\mathbf{F}_S = \mathbf{V}^2$ of the surface as follows

$$\begin{aligned} \widetilde{\gamma} &= \mathbf{N}\widetilde{\gamma}\mathbf{N} = \frac{1}{2}(\mathbf{F}_S^T\mathbf{F}_S - \mathbf{N}) = \frac{1}{2}(\mathbf{V}\mathbf{Q}_S^T\mathbf{Q}_S\mathbf{V} - \mathbf{N}) = \frac{1}{2}(\mathbf{V}^2 - \mathbf{N}) \qquad & \mathbf{Q}_S^T\mathbf{Q}_S = \mathbf{I} \\ &= \frac{1}{2}(\mathbf{N}\mathbf{F}^T\mathbf{F}\mathbf{N} - \mathbf{N}) = \frac{1}{2}\mathbf{N}(\mathbf{F}^T\mathbf{F} - \mathbf{I})\mathbf{N} = \mathbf{N}\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{N} \qquad ; & \mathbf{V}^2 = \mathbf{V}\mathbf{V} \end{aligned} \tag{2.26}$$

In order to get the components of the bidimensional dilatation, we write

$$\begin{aligned} &\mathrm{d}\mathbf{r} = \mathbf{e}\mathrm{d}s \quad ; \quad \mathrm{d}\mathbf{r}_\perp = \mathbf{e}_\perp\mathrm{d}s_\perp \quad ; \quad \mathbf{e}\cdot\mathbf{e} = \mathbf{e}_\perp\cdot\mathbf{e}_\perp = 1 \quad ; \quad \mathbf{e}\cdot\mathbf{e}_\perp = 0 \\ &\mathrm{d}\mathbf{r}' = \mathbf{e}'\mathrm{d}s' \quad ; \quad \mathrm{d}\mathbf{r}'_\perp = \mathbf{e}'_\perp\mathrm{d}s'_\perp \quad ; \quad \mathbf{e}'\cdot\mathbf{e}' = \mathbf{e}'_\perp\cdot\mathbf{e}'_\perp = 1 \quad ; \quad -1 \leqslant \mathbf{e}'\cdot\mathbf{e}'_\perp \leqslant 1 \\ &\mathbf{e} = \mathbf{N}\mathbf{e} \quad ; \quad \mathbf{e}_\perp = \mathbf{N}\mathbf{e}_\perp \quad ; \quad \mathbf{e}' = \mathbf{N}'\mathbf{e}' \quad ; \quad \mathbf{e}'_\perp = \mathbf{N}'\mathbf{e}'_\perp \end{aligned} \tag{2.27}$$

From the usual theory of deformation, we have

$$\varepsilon = \frac{\mathrm{d}s' - \mathrm{d}s}{\mathrm{d}s} \quad ; \quad \varepsilon_\perp = \frac{\mathrm{d}s'_\perp - \mathrm{d}s_\perp}{\mathrm{d}s_\perp} \quad ; \quad \cos(\frac{\pi}{2} - \gamma) = \sin\gamma = \mathbf{e}'\cdot\mathbf{e}'_\perp$$

$$\left.\begin{aligned} \widetilde{\varepsilon} &= \mathbf{e}\cdot\widetilde{\gamma}\mathbf{e} = \frac{1}{2}\mathbf{e}\cdot(\mathbf{F}_S^T\mathbf{F}_S - \mathbf{N})\mathbf{e} \\ &= \frac{\mathrm{d}s'^2 - \mathrm{d}s^2}{2\mathrm{d}s^2} = \varepsilon + \frac{1}{2}\varepsilon^2 \end{aligned}\right\} \Rightarrow \quad \varepsilon = \sqrt{1 + 2\widetilde{\varepsilon}} - 1 = \sqrt{1 + 2\mathbf{e}\cdot\widetilde{\gamma}\mathbf{e}} - 1 \tag{2.28}$$

$$\left.\begin{aligned} \widetilde{\varepsilon}_\perp &= \mathbf{e}_\perp\cdot\widetilde{\gamma}\mathbf{e}_\perp = \frac{1}{2}\mathbf{e}_\perp\cdot(\mathbf{F}_S^T\mathbf{F}_S - \mathbf{N})\mathbf{e}_\perp \\ &= \frac{\mathrm{d}s'^2_\perp - \mathrm{d}s^2_\perp}{2\mathrm{d}s^2_\perp} = \varepsilon_\perp + \frac{1}{2}\varepsilon^2_\perp \end{aligned}\right\} \Rightarrow \quad \varepsilon_\perp = \sqrt{1 + 2\widetilde{\varepsilon}_\perp} - 1 = \sqrt{1 + 2\mathbf{e}_\perp\cdot\widetilde{\gamma}\mathbf{e}_\perp} - 1$$

$$\left.\begin{aligned}\frac{1}{2}\widetilde{\gamma} &= \mathbf{e}\cdot\widetilde{\boldsymbol{\gamma}}\mathbf{e}_\perp = \frac{1}{2}\mathbf{e}\cdot(\mathbf{F}_S^T\mathbf{F}_S-\mathbf{N})\mathbf{e}_\perp \\ &= \frac{\mathrm{d}s'\mathrm{d}s'_\perp}{2\mathrm{d}s\mathrm{d}s_\perp}(\mathbf{e}'\cdot\mathbf{e}'_\perp) = \frac{1}{2}(1+\varepsilon)(1+\varepsilon_\perp)\sin\gamma\end{aligned}\right\} \quad \Rightarrow \quad \sin\gamma = \frac{2\mathbf{e}\cdot\widetilde{\boldsymbol{\gamma}}\mathbf{e}_\perp}{\sqrt{(1+2\mathbf{e}\cdot\widetilde{\boldsymbol{\gamma}}\mathbf{e})(1+2\mathbf{e}_\perp\cdot\widetilde{\boldsymbol{\gamma}}\mathbf{e}_\perp)}}$$

where ε is the linear dilatation and γ the angular dilatation or the shearing strain. The scalars $\widetilde{\varepsilon}$, $\widetilde{\varepsilon}_\perp$ and $\widetilde{\gamma}/2$ are the components of the strain tensor $\widetilde{\boldsymbol{\gamma}}$ corresponding to the tensors $\mathbf{e}\otimes\mathbf{e}$, $\mathbf{e}_\perp\otimes\mathbf{e}_\perp$ and $\mathbf{e}\otimes\mathbf{e}_\perp$, which are built on the unit vectors $\mathbf{e}$ and $\mathbf{e}_\perp$ (often choosed in the tangential plane of the surface). Note that other components may be built on the third unit vector $\mathbf{e}_{\mathrm{b}\perp} = \mathbf{e}\times\mathbf{e}_\perp = \mathbf{e}_\perp\boldsymbol{\mathcal{E}}\mathbf{e}$ (often choosed equal to $\mathbf{n}$).

The deformation gradient of the surface may also be decomposed in an additive manner

$$\mathbf{F}_S = \mathbf{N} + (\boldsymbol{\nabla}_n\otimes\mathbf{u})^T = \mathbf{N} + (\boldsymbol{\mathcal{E}}-\boldsymbol{\Omega})\mathbf{N} \tag{2.29}$$

with

$$\begin{aligned}\boldsymbol{\mathcal{E}}\mathbf{N} &= \frac{1}{2}[(\boldsymbol{\nabla}\otimes\mathbf{u})\mathbf{N} + (\boldsymbol{\nabla}_n\otimes\mathbf{u})^T] \\ \boldsymbol{\Omega}\mathbf{N} &= \frac{1}{2}[(\boldsymbol{\nabla}\otimes\mathbf{u})\mathbf{N} - (\boldsymbol{\nabla}_n\otimes\mathbf{u})^T]\end{aligned} \tag{2.30}$$

We also may decompose the tensor $\boldsymbol{\nabla}_n\otimes\mathbf{u}$ as follows

$$\boldsymbol{\nabla}_n\otimes\mathbf{u} = (\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{I} = (\boldsymbol{\nabla}_n\otimes\mathbf{u})(\mathbf{N}+\mathbf{n}\otimes\mathbf{n}) = \underbrace{(\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{N}}_{\text{interior part}} + \underbrace{(\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{n}\otimes\mathbf{n}}_{\text{semi-exterior part}} \tag{2.31}$$

where $(\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{N} = \mathbf{N}(\boldsymbol{\mathcal{E}}+\boldsymbol{\Omega})\mathbf{N}$ and $(\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{n} = \mathbf{N}(\boldsymbol{\mathcal{E}}+\boldsymbol{\Omega})\mathbf{n}$.

Special case of a small deformation

For $\mathbf{F}_S \simeq \mathbf{N}$, we have

$$\begin{aligned}\widetilde{\boldsymbol{\gamma}} &= \frac{1}{2}(\mathbf{F}_S^T\mathbf{F}_S-\mathbf{N}) = \frac{1}{2}[(\mathbf{N}+\mathbf{N}\boldsymbol{\mathcal{E}}+\mathbf{N}\boldsymbol{\Omega})(\mathbf{N}+\boldsymbol{\mathcal{E}}\mathbf{N}-\boldsymbol{\Omega}\mathbf{N})-\mathbf{N}] \\ &= \frac{1}{2}[\mathbf{N}+\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N}-\mathbf{N}\boldsymbol{\Omega}\mathbf{N}+\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N}+\mathbf{N}\boldsymbol{\Omega}\mathbf{N}+\ldots-\mathbf{N}] \simeq \mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} = \boldsymbol{\gamma} \\ \mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} &= \boldsymbol{\gamma} = \frac{1}{2}[(\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{N}+\mathbf{N}(\boldsymbol{\nabla}_n\otimes\mathbf{u})^T] \simeq \widetilde{\boldsymbol{\gamma}} \\ \mathbf{N}\boldsymbol{\Omega}\mathbf{N} &= \frac{1}{2}[(\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{N}-\mathbf{N}(\boldsymbol{\nabla}_n\otimes\mathbf{u})^T] \simeq \mathit{\Omega}\mathbf{E} \\ (\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{n} &= \mathbf{N}(\boldsymbol{\mathcal{E}}+\boldsymbol{\Omega})\mathbf{n} = \mathbf{n}(\boldsymbol{\mathcal{E}}-\boldsymbol{\Omega})\mathbf{N} \simeq \boldsymbol{\omega} \\ \boldsymbol{\nabla}_n\otimes\mathbf{u} &= \mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N}+\mathbf{N}\boldsymbol{\Omega}\mathbf{N}+\mathbf{N}(\boldsymbol{\mathcal{E}}+\boldsymbol{\Omega})\mathbf{n}\otimes\mathbf{n} \simeq \widetilde{\boldsymbol{\gamma}}+\mathit{\Omega}\mathbf{E}+\boldsymbol{\omega}\otimes\mathbf{n} \\ \mathbf{F}_S &= \mathbf{N}+(\boldsymbol{\nabla}_n\otimes\mathbf{u})^T = \mathbf{N}+\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N}-\mathbf{N}\boldsymbol{\Omega}\mathbf{N}+\mathbf{n}\otimes\mathbf{n}(\boldsymbol{\mathcal{E}}-\boldsymbol{\Omega})\mathbf{N} \simeq \mathbf{N}+\widetilde{\boldsymbol{\gamma}}-\mathit{\Omega}\mathbf{E}+\mathbf{n}\otimes\boldsymbol{\omega}\end{aligned} \tag{2.32}$$

where, in a first approximation, the 2-dimensional symmetric tensor $\boldsymbol{\gamma} = \mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N}$ describes the bidimensional dilatation, the 2-dimensional antimetric tensor $\mathbf{N}\boldsymbol{\Omega}\mathbf{N}$ the in-plane rotation and the interior vector $\mathbf{N}(\boldsymbol{\mathcal{E}}+\boldsymbol{\Omega})\mathbf{n}$ the out-of-plane rotation of the surface element. Proof:

$$\mathbf{V} \simeq \mathbf{N}+\mathbf{X} \qquad ; \qquad \mathbf{X} = \mathbf{X}^T = \mathbf{N}\mathbf{X}\mathbf{N}$$

$$
\begin{aligned}
&\mathbf{Q}_S \simeq \mathbf{I} + \mathbf{Y} \quad ; \quad \mathbf{Y} = -\mathbf{Y}^T \\
&\mathbf{F}_S = \mathbf{Q}_S\mathbf{V} = \mathbf{N} + \mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} - \mathbf{N}\boldsymbol{\Omega}\mathbf{N} + \mathbf{n}\otimes\mathbf{n}(\boldsymbol{\mathcal{E}} - \boldsymbol{\Omega})\mathbf{N} \\
&\quad \simeq \mathbf{N} + \mathbf{X} + \mathbf{Y}\mathbf{N} = \mathbf{N} + \mathbf{N}\mathbf{X}\mathbf{N} + \mathbf{N}\mathbf{Y}\mathbf{N} + \mathbf{n}\otimes\mathbf{n}\mathbf{Y}\mathbf{N} \\
&\Rightarrow \quad \begin{cases} \mathbf{X} = \mathbf{N}\mathbf{X}\mathbf{N} = \mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} = \boldsymbol{\gamma} \quad ; \quad \mathbf{N}\mathbf{Y}\mathbf{N} = -\mathbf{N}\boldsymbol{\Omega}\mathbf{N} \\ \mathbf{n}\mathbf{Y}\mathbf{N} = -\mathbf{N}\mathbf{Y}\mathbf{n} = \mathbf{N}(\boldsymbol{\mathcal{E}} + \boldsymbol{\Omega})\mathbf{n} \end{cases} \\
&\mathbf{Y} = \mathbf{N}\mathbf{Y}\mathbf{N} + \mathbf{n}\otimes\mathbf{n}\mathbf{Y}\mathbf{N} + \mathbf{N}\mathbf{Y}\mathbf{n}\otimes\mathbf{n} + (\mathbf{n}\cdot\mathbf{Y}\mathbf{n})\mathbf{n}\otimes\mathbf{n} \\
&\quad = -\mathbf{N}\boldsymbol{\Omega}\mathbf{N} + \mathbf{n}\otimes\mathbf{n}(\boldsymbol{\mathcal{E}} - \boldsymbol{\Omega})\mathbf{N} - \mathbf{N}(\boldsymbol{\mathcal{E}} + \boldsymbol{\Omega})\mathbf{n}\otimes\mathbf{n} \\
&\mathbf{n}\cdot\mathbf{Y}\mathbf{n} = (\mathbf{Y}^T\mathbf{n})\cdot\mathbf{n} = -(\mathbf{Y}\mathbf{n})\cdot\mathbf{n} = -\mathbf{n}\cdot\mathbf{Y}\mathbf{n} = 0 \\
&\Rightarrow \quad \begin{cases} \mathbf{V} \simeq \mathbf{N} + \mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} = \mathbf{N} + \boldsymbol{\gamma} \\ \mathbf{Q}_S \simeq \mathbf{I} - \mathbf{N}\boldsymbol{\Omega}\mathbf{N} - \mathbf{N}(\boldsymbol{\mathcal{E}} + \boldsymbol{\Omega})\mathbf{n}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{n}(\boldsymbol{\mathcal{E}} - \boldsymbol{\Omega})\mathbf{N} \\ \quad \simeq \mathbf{I} - \Omega\mathbf{E} - \boldsymbol{\omega}\otimes\mathbf{n} + \mathbf{n}\otimes\boldsymbol{\omega} \end{cases} \qquad \square \text{ qed}
\end{aligned}
\tag{2.33}
$$

Consequently, the unit normal $\mathbf{n}'$ of the deformed curved surface can be approximated as follows

$$
\mathbf{n}' = \mathbf{Q}_S\mathbf{n} \simeq \mathbf{n} - \mathbf{N}(\boldsymbol{\mathcal{E}} + \boldsymbol{\Omega})\mathbf{n} \simeq \mathbf{n} - \boldsymbol{\omega} \tag{2.34}
$$

Note that both $\Omega\mathbf{E}$ and $\boldsymbol{\omega}$ will be exactly defined in the section dealing with nonlinear kinematics of deformation of curved surfaces.

2.2.2 Eulerian representation

The description of the surface deformation in an Eulerian representation may be written as follows

$$
\begin{aligned}
&\mathbf{r}' \quad \rightarrow \quad \mathbf{u} = \mathbf{u}(\mathbf{r}') \\
&\mathbf{r}' \quad \rightarrow \quad \mathbf{r} = \mathbf{r}(\mathbf{r}') = \mathbf{r}' - \mathbf{u}(\mathbf{r}') \\
&\mathrm{d}\mathbf{r} = \mathbf{N}\mathrm{d}\mathbf{r} = \mathrm{d}\mathbf{r}'(\boldsymbol{\nabla}_{n'}\otimes\mathbf{r}) = \mathrm{d}\mathbf{r}' - \mathrm{d}\mathbf{u} = \mathbf{N}'\mathrm{d}\mathbf{r}' - \mathrm{d}\mathbf{r}'(\boldsymbol{\nabla}_{n'}\otimes\mathbf{u}) \\
&\quad = \mathbf{F}'\mathbf{N}'\mathrm{d}\mathbf{r}' = \mathbf{F}'_S\mathrm{d}\mathbf{r}' = \mathbf{F}'_S\mathbf{F}_S\mathrm{d}\mathbf{r} \\
&\mathrm{d}\mathbf{r}' = \mathbf{N}'\mathrm{d}\mathbf{r}' = \mathbf{F}\mathbf{N}\mathrm{d}\mathbf{r} = \mathbf{F}_S\mathrm{d}\mathbf{r} = \mathbf{F}_S\mathbf{F}'_S\mathrm{d}\mathbf{r}'
\end{aligned}
\tag{2.35}
$$

with

$$
\mathbf{F}'_S = \mathbf{N}' - (\boldsymbol{\nabla}_{n'}\otimes\mathbf{u})^T \quad ; \quad \mathbf{F}'_S\mathbf{F}_S = \mathbf{N} \quad ; \quad \mathbf{F}_S\mathbf{F}'_S = \mathbf{N}' \tag{2.36}
$$

where $\boldsymbol{\nabla}_{n'}$ represents the 2-dimensional derivative operator on the deformed curved surface. Because both tensors $\mathbf{F}_S$ and $\mathbf{F}'_S$ contain projectors, the deformation gradient $\mathbf{F}'_S$ is not the inverse of $\mathbf{F}_S$. According to Lagrange, we have

$$
\mathbf{F}'_S = \mathbf{W}\mathbf{Q}_S^T \quad ; \quad \mathbf{V}\mathbf{W} = \mathbf{W}\mathbf{V} = \mathbf{N} \tag{2.37}
$$

where $\mathbf{W} = \mathbf{W}^T = \mathbf{N}\mathbf{W}\mathbf{N}$ describes the "inverse" bidimensional dilatation. The polar decomposition according to Euler reads

$$
\mathbf{F}_S = \mathbf{V}'\mathbf{Q}_S \quad ; \quad \mathbf{F}'_S = \mathbf{Q}_S^T\mathbf{W}' \quad ; \quad \mathbf{V}'\mathbf{W}' = \mathbf{W}'\mathbf{V}' = \mathbf{N}' \tag{2.38}
$$

where $\mathbf{V}' = \mathbf{V}'^T = \mathbf{N}'\mathbf{V}'\mathbf{N}'$ and $\mathbf{W}' = \mathbf{W}'^T = \mathbf{N}'\mathbf{W}'\mathbf{N}'$ respectively describe the bidimensional dilatation and the "inverse" bidimensional dilatation. Proof:

$$
\mathbf{n}' = \mathbf{Q}_S\mathbf{n} \quad \Rightarrow \quad \mathbf{n} = \mathbf{Q}_S^T\mathbf{n}' = \mathbf{n}'\mathbf{Q}_S
$$

$$
\begin{aligned}
\mathbf{F}'_S\mathbf{F}_S &= \mathbf{Q}_S^T\mathbf{W}'\mathbf{V}'\mathbf{Q}_S = \mathbf{Q}_S^T\mathbf{N}'\mathbf{Q}_S = \mathbf{Q}_S^T(\mathbf{I}-\mathbf{n}'\otimes\mathbf{n}')\mathbf{Q}_S \\
&= \mathbf{Q}_S^T\mathbf{Q}_S - \mathbf{Q}_S^T\mathbf{n}'\otimes\mathbf{n}'\mathbf{Q}_S = \mathbf{I}-\mathbf{n}\otimes\mathbf{n} = \mathbf{N} \\
\mathbf{F}_S\mathbf{F}'_S &= \mathbf{V}'\mathbf{Q}_S\mathbf{Q}_S^T\mathbf{W}' = \mathbf{V}'\mathbf{W}' = \mathbf{N}' \qquad \square \text{ qed}
\end{aligned} \tag{2.39}
$$

In this case, we have

$$
\boxed{\text{Deformation}} = \boxed{\text{Displacement}} + \boxed{\text{Rotation}} + \boxed{\text{Bidimensional dilatation}} \tag{2.40}
$$

Because both the undeformed and deformed surface configurations are described by the same curvilinear coordinates, we have

$$
\begin{aligned}
&\mathrm{d}\mathbf{r}\cdot\boldsymbol{\nabla}_n = \mathrm{d}\mathbf{r}'\cdot\boldsymbol{\nabla}_{n'} = \mathrm{d}\mathbf{r}\cdot\mathbf{N}\mathbf{F}^T\boldsymbol{\nabla}_{n'} = \mathrm{d}\mathbf{r}'\cdot\mathbf{N}'\mathbf{F}'^T\boldsymbol{\nabla}_n \\
&\qquad\qquad = \mathrm{d}\mathbf{r}\cdot\mathbf{F}_S^T\boldsymbol{\nabla}_{n'} = \mathrm{d}\mathbf{r}'\cdot\mathbf{F}_S'^T\boldsymbol{\nabla}_n \qquad , \qquad \forall\,\mathrm{d}\mathbf{r}\,,\,\mathrm{d}\mathbf{r}' \\
&\Rightarrow \qquad \boldsymbol{\nabla}_n = \mathbf{N}\mathbf{F}^T\boldsymbol{\nabla}_{n'} = \mathbf{F}_S^T\boldsymbol{\nabla}_{n'} \qquad ; \qquad \boldsymbol{\nabla}_{n'} = \mathbf{N}'\mathbf{F}'^T\boldsymbol{\nabla}_n = \mathbf{F}_S'^T\boldsymbol{\nabla}_n
\end{aligned} \tag{2.41}
$$

2.3 Nonlinear kinematics of deformation of a 3-dimensional object in space

In the case of large object deformation or in the case where the dilatation, rotation and displacement components have different orders of magnitude, we may for example encounter small strains together with moderate rotations and large displacements. In order to properly analyze such deformations, we must develop the tensors previously introduced in this chapter up to higher order terms. The purpose of this section is not to present an exhaustive theoretical background on this topic, but to introduce the nonlinear relations needed in the following sections.

2.3.1 Vector coordinates

The one-one mapping that associates the whole set of points $\{\mathrm{P}\}$ of the undeformed configuration to the set of points $\{\mathrm{P}'\}$ of the deformed configuration is described by a single set of curvilinear coordinates θ^i $(i = 1, 2, 3)$ called convected coordinates as follows

$$
\theta^1,\theta^2,\theta^3 \quad \longrightarrow \quad \begin{cases} \mathbf{r} = \mathbf{r}(\theta^1,\theta^2,\theta^3) \quad \longrightarrow \quad \mathbf{r}' = \mathbf{r}'(\mathbf{r}) = \mathbf{r}'[\mathbf{r}(\theta^1,\theta^2,\theta^3)] \\ \mathbf{r}' = \mathbf{r}'(\theta^1,\theta^2,\theta^3) \quad \longrightarrow \quad \mathbf{r} = \mathbf{r}(\mathbf{r}') = \mathbf{r}[\mathbf{r}'(\theta^1,\theta^2,\theta^3)] \end{cases} \tag{2.42}
$$

<u>Note</u>: We show here that it is possible to easily compute complicated calculations by only using the intrinsic notation of tensor calculus, without requiring the notation with indices or any covariant, contravariant or cartesian components.

2.3.2 Lagrangean representation of the deformation

$$
\begin{aligned}
&\mathbf{r} \quad \longrightarrow \quad \mathbf{u} = \mathbf{u}(\mathbf{r}) \\
&\mathbf{r} \quad \longrightarrow \quad \mathbf{r}' = \mathbf{r}'(\mathbf{r}) = \mathbf{r} + \mathbf{u}(\mathbf{r}) \\
&\mathrm{d}\mathbf{r}' = \mathrm{d}\mathbf{r} + \mathrm{d}\mathbf{u} = \mathrm{d}\mathbf{r} + \mathrm{d}\mathbf{r}(\boldsymbol{\nabla}\otimes\mathbf{u}) = [\mathbf{I} + (\boldsymbol{\nabla}\otimes\mathbf{u})^T]\mathrm{d}\mathbf{r} = \mathbf{F}\mathrm{d}\mathbf{r} \\
&\Longrightarrow \quad \mathbf{F} = \mathbf{I} + (\boldsymbol{\nabla}\otimes\mathbf{u})^T \quad : \qquad \text{deformation gradient}
\end{aligned} \tag{2.43}
$$

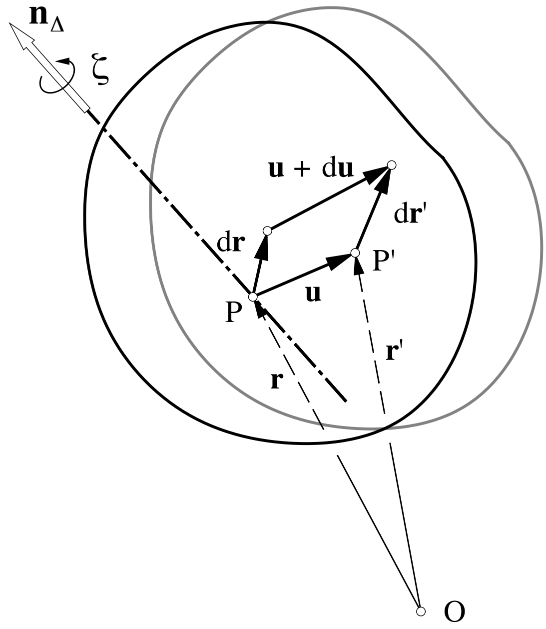

Fig. 2.3: Deformation of a 3-dimensional object in space

$$\mathrm{d}^2\mathbf{r}' = \mathrm{d}\mathbf{F}\mathrm{d}\mathbf{r} + \mathbf{F}\mathrm{d}^2\mathbf{r} = \mathbf{F}\mathrm{d}^2\mathbf{r} + \mathrm{d}\mathbf{r}(\boldsymbol{\nabla}\otimes\mathbf{F})\mathrm{d}\mathbf{r}$$

$$\Longrightarrow \quad \boldsymbol{\nabla}\otimes\mathbf{F} = \boldsymbol{\nabla}\otimes\boldsymbol{\nabla}\otimes\mathbf{u})^T \quad : \quad \text{derivative of the deformation gradient}$$

Polar decomposition:

Definition: $\mathbf{U}^2 = \mathbf{F}^T\mathbf{F}$ with $\mathbf{U} = \mathbf{U}^T$; $\mathbf{U}^{-1} = (\mathbf{U}^{-1})^T = (\mathbf{U}^T)^{-1}$

Definition: $\mathbf{Q} = \mathbf{F}\mathbf{U}^{-1}$ with $\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{I}$; $\mathbf{Q}^{-1} = \mathbf{Q}^T$
because $\mathbf{Q}^T\mathbf{Q} = \mathbf{U}^{-1}\mathbf{F}^T\mathbf{F}\mathbf{U}^{-1} = \mathbf{U}^{-1}\mathbf{U}\mathbf{U}\mathbf{U}^{-1} = \mathbf{I}$

$$\Longrightarrow \quad \mathbf{F} = \mathbf{I} + (\boldsymbol{\nabla}\otimes\mathbf{u})^T = \mathbf{Q}\mathbf{U} \quad : \quad \text{polar decomposition} \qquad (2.44)$$

with $\mathbf{F}^T\mathbf{F}$: *Cauchy-Green* tensor (symmetric, positive defined)
$\mathbf{U}$: symmetric tensor describing the dilatation
$\mathbf{Q}$: orthogonal tensor describing the rotation

A general deformation of a 3-dimensional object in space is described in the neighborhood of a point P by a dilatation followed by a rotation followed by a displacement of an infinitesimal volume element:

1°) The dilatation of the neighborhood of the point P of the undeformed configuration by means of the symmetric tensor $\mathbf{U}$.

2°) The "rigid body" rotation of the strained volume element around an axis Δ of direction $\mathbf{n}_\Delta$ going through point P by means of the orthogonal tensor $\mathbf{Q}$.

3°) The displacement of the strained and rotated volume element from its position P to the new position P′ of the deformed configuration by means of the displacement vector $\mathbf{u}$.

This description is only valid for a single point of the configuration and vary from point to point.

2.3.3 Definitions of the symmetric tensor $\mathcal{E}$ and of the antimetric tensor Ω

$$\begin{aligned} \boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^T &= \frac{1}{2}[\boldsymbol{\nabla} \otimes \mathbf{u} + (\boldsymbol{\nabla} \otimes \mathbf{u})^T] \\ \boldsymbol{\Omega} = -\boldsymbol{\Omega}^T &= \frac{1}{2}[\boldsymbol{\nabla} \otimes \mathbf{u} - (\boldsymbol{\nabla} \otimes \mathbf{u})^T] \end{aligned} \tag{2.45}$$

As previously mentioned, these tensors respectively approximate the dilatation and the rotation in the special case of small deformations, which means that only the first-order linear terms are relevant in their Taylor series development. For moderate deformation, we cannot use these above expressions to describe the dilatation and the rotation because the second-order nonlinear terms of both expressions contain coupled dilatation and rotation components.

2.3.4 Expressions of $\boldsymbol{\nabla} \otimes \mathbf{u}$ and $\mathbf{F}$ as function of $\mathcal{E}$ and Ω

$$\begin{aligned} &\boldsymbol{\nabla} \otimes \mathbf{u} = \boldsymbol{\Omega} + \boldsymbol{\mathcal{E}} \\ &\mathbf{F} = \mathbf{I} + (\boldsymbol{\nabla} \otimes \mathbf{u})^T = \mathbf{QU} = \mathbf{I} - \boldsymbol{\Omega} + \boldsymbol{\mathcal{E}} \end{aligned} \tag{2.46}$$

As we can see, to separate the dilatation from the rotation, it is necessary to deal with the polar decomposition of $\mathbf{F}$. The main purpose of this section is first to write the different tensors describing the deformation as function of the derivative of the displacement $\boldsymbol{\nabla} \otimes \mathbf{u}$ and as function of $\boldsymbol{\Omega}$ and $\boldsymbol{\mathcal{E}}$, and second to write their Taylor serie developments at least up to the second-order terms, which are built on quantities containing only “pure” dilatation and rotation components.

2.3.5 Definition of the symmetric strain tensor $\widetilde{\mathcal{E}}$

$$\begin{aligned} \widetilde{\boldsymbol{\mathcal{E}}} &= \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) = \frac{1}{2}(\mathbf{U}\mathbf{Q}^T\mathbf{Q}\mathbf{U} - \mathbf{I}) = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I}) \\ &= \boldsymbol{\mathcal{E}} - \frac{1}{2}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})(\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}}) \end{aligned} \tag{2.47}$$

This tensor obviously only contains the dilatation. In the particular case where the deformation gradient $\mathbf{F}$ is close to the identity $\mathbf{I}$, which means that $\mathbf{F} \simeq \mathbf{I}$ and $\mathbf{F}^T \simeq \mathbf{I}$, we have

$$\widetilde{\boldsymbol{\mathcal{E}}} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) = O(\varepsilon) \qquad ; \qquad 0 \leqslant |\varepsilon| \ll 1 \tag{2.48}$$

With the infinitesimal increments

$$\begin{aligned} &\mathrm{d}\mathbf{r} = \mathbf{e}\,\mathrm{d}s \\ &\mathrm{d}\mathbf{r}_\perp = \mathbf{e}_\perp \mathrm{d}s_\perp \\ &\mathrm{d}\mathbf{r}' = \mathbf{e}'\mathrm{d}s' = \mathbf{F}\mathrm{d}\mathbf{r} = \mathbf{F}\mathbf{e}\,\mathrm{d}s \\ &\mathrm{d}\mathbf{r}'_\perp = \mathbf{e}'_\perp \mathrm{d}s'_\perp = \mathbf{F}\mathrm{d}\mathbf{r}_\perp = \mathbf{F}\mathbf{e}_\perp \mathrm{d}s_\perp \end{aligned} \quad ; \quad \begin{aligned} &\mathbf{e} \cdot \mathbf{e} = 1 \\ &\mathbf{e}_\perp \cdot \mathbf{e}_\perp = 1 \\ &\mathbf{e}' \cdot \mathbf{e}' = 1 \\ &\mathbf{e}'_\perp \cdot \mathbf{e}'_\perp = 1 \end{aligned} \quad ; \quad \begin{aligned} &\mathbf{e} \cdot \mathbf{e}_\perp = 0 \\ &0 \leqslant |\mathbf{e}' \cdot \mathbf{e}'_\perp| \ll 1 \end{aligned} \tag{2.49}$$

the linear and angular dilatations read

$$\varepsilon = \frac{\mathrm{d}s' - \mathrm{d}s}{\mathrm{d}s} \quad ; \quad \varepsilon_\perp = \frac{\mathrm{d}s'_\perp - \mathrm{d}s_\perp}{\mathrm{d}s_\perp} \quad ; \quad \cos(\frac{\pi}{2} - \gamma) = \sin\gamma = \mathbf{e}' \cdot \mathbf{e}'_\perp$$

$$\left.\begin{aligned}\widetilde{\varepsilon} &= \mathbf{e}\cdot\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{e} = \frac{1}{2}\mathbf{e}\cdot(\mathbf{F}^T\mathbf{F}-\mathbf{I})\mathbf{e}\\ &= \frac{\mathrm{d}s'^2-\mathrm{d}s^2}{2\mathrm{d}s^2} = \varepsilon + \frac{1}{2}\varepsilon^2\end{aligned}\right\} \Rightarrow \quad \varepsilon = \sqrt{1+2\widetilde{\varepsilon}}-1 = \sqrt{1+2\mathbf{e}\cdot\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{e}}-1 \tag{2.50}$$

$$\left.\begin{aligned}\widetilde{\varepsilon}_\perp &= \mathbf{e}_\perp\cdot\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{e}_\perp = \frac{1}{2}\mathbf{e}_\perp\cdot(\mathbf{F}^T\mathbf{F}-\mathbf{I})\mathbf{e}_\perp\\ &= \frac{\mathrm{d}s'^2_\perp-\mathrm{d}s^2_\perp}{2\mathrm{d}s^2_\perp} = \varepsilon_\perp + \frac{1}{2}\varepsilon^2_\perp\end{aligned}\right\} \Rightarrow \quad \varepsilon_\perp = \sqrt{1+2\widetilde{\varepsilon}_\perp}-1 = \sqrt{1+2\mathbf{e}_\perp\cdot\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{e}_\perp}-1$$

$$\left.\begin{aligned}\frac{1}{2}\widetilde{\gamma} &= \mathbf{e}\cdot\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{e}_\perp = \frac{1}{2}\mathbf{e}\cdot(\mathbf{F}^T\mathbf{F}-\mathbf{I})\mathbf{e}_\perp\\ &= \frac{\mathrm{d}s'\mathrm{d}s'_\perp}{2\mathrm{d}s\mathrm{d}s_\perp}(\mathbf{e}'\cdot\mathbf{e}'_\perp) = \frac{1}{2}(1+\varepsilon)(1+\varepsilon_\perp)\sin\gamma\end{aligned}\right\} \Rightarrow \quad \sin\gamma = \frac{2\mathbf{e}\cdot\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{e}_\perp}{\sqrt{(1+2\mathbf{e}\cdot\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{e})(1+2\mathbf{e}_\perp\cdot\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{e}_\perp)}}$$

2.3.6 Developments of the tensors U and Q

Because of $\mathbf{F} = \mathbf{QU} \simeq \mathbf{I}$, both tensors $\mathbf{U}$ characterizing the dilatation and $\mathbf{Q}$ charaterizing the rotation are close to the identity $\mathbf{I}$ and can be developed as follows

$$\begin{aligned}\mathbf{U} &= \mathbf{I} + \eta\boldsymbol{\mathcal{E}}_1 + \frac{1}{2!}\eta^2\boldsymbol{\mathcal{E}}_2 + \frac{1}{3!}\eta^3\boldsymbol{\mathcal{E}}_3 + O(\eta^4) & & 0 \leqslant |\eta| \ll 1\\ \mathbf{Q} &= \mathbf{I} - \zeta\mathbf{E}_\Delta + \frac{1}{2!}\zeta^2\mathbf{E}^2_\Delta - \frac{1}{3!}\zeta^3\mathbf{E}^3_\Delta + O(\zeta^4) & \text{with} \quad & 0 \leqslant |\zeta| \ll 1\end{aligned} \tag{2.51}$$

where η and ζ are small independent parameters. For the tensors $\mathbf{U}$ and $\mathbf{U}^{-1}$, we have

$$\begin{aligned}&\mathbf{U} = \mathbf{U}^T \qquad ; \qquad \forall\, |\eta| \ll 1\\ &\Longrightarrow \quad \boldsymbol{\mathcal{E}}_1 = \boldsymbol{\mathcal{E}}_1^T \quad ; \quad \boldsymbol{\mathcal{E}}_2 = \boldsymbol{\mathcal{E}}_2^T \quad ; \quad \boldsymbol{\mathcal{E}}_3 = \boldsymbol{\mathcal{E}}_3^T\\ &\mathbf{U}^{-1}\mathbf{U} = \mathbf{U}\mathbf{U}^{-1} = \mathbf{I}\\ &\Longrightarrow \quad \mathbf{U}^{-1} = \mathbf{I} - \eta\boldsymbol{\mathcal{E}}_1 + \frac{1}{2}\eta^2(2\boldsymbol{\mathcal{E}}_1^2 - \boldsymbol{\mathcal{E}}_2) - \frac{1}{6}\eta^3(6\boldsymbol{\mathcal{E}}_1^3 - 3\boldsymbol{\mathcal{E}}_1\boldsymbol{\mathcal{E}}_2 - 3\boldsymbol{\mathcal{E}}_2\boldsymbol{\mathcal{E}}_1 + \boldsymbol{\mathcal{E}}_3) + O(\eta^4)\end{aligned} \tag{2.52}$$

For the tensors $\mathbf{Q}$ and $\mathbf{Q}^T$, we have

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{I} \qquad ; \qquad \forall\, |\zeta| \ll 1 \qquad \Longrightarrow \qquad \mathbf{E}_\Delta = -\mathbf{E}_\Delta^T$$

ζ : rotation angle (in radian) around the axis Δ going through point P

$\mathbf{E}_\Delta = \boldsymbol{\mathfrak{E}}\mathbf{n}_\Delta = \mathbf{n}_\Delta\boldsymbol{\mathfrak{E}}$: 2-dimensional second-rank permutation tensor

$\mathbf{n}_\Delta$: direction of the rotation axis Δ with $\mathbf{n}_\Delta\cdot\mathbf{n}_\Delta = 1$

$\mathbf{E}_\Delta\mathbf{E}_\Delta = \mathbf{E}^2_\Delta = -\mathbf{N}_\Delta$

$\mathbf{N}_\Delta = \mathbf{N}^T_\Delta = \mathbf{I} - \mathbf{n}_\Delta\otimes\mathbf{n}_\Delta$: normal projector (2.53)

$\mathbf{E}_\Delta \equiv \mathbf{N}_\Delta\mathbf{E}_\Delta \equiv \mathbf{E}_\Delta\mathbf{N}_\Delta \equiv \mathbf{N}_\Delta\mathbf{E}_\Delta\mathbf{N}_\Delta$

$\mathbf{E}^3_\Delta = \mathbf{E}^2_\Delta\mathbf{E}_\Delta = -\mathbf{N}_\Delta\mathbf{E}_\Delta = -\mathbf{E}_\Delta$

$\boldsymbol{\omega}_\Delta = \zeta\mathbf{n}_\Delta$: rotation vector (full describes the rotation of the volume element)

$$\zeta \mathbf{E}_\triangle = \zeta \boldsymbol{\mathcal{E}} \mathbf{n}_\triangle = \boldsymbol{\mathcal{E}} \boldsymbol{\omega}_\triangle = \boldsymbol{\omega}_\triangle \boldsymbol{\mathcal{E}}$$
$$\zeta^2 \mathbf{E}_\triangle^2 = (\boldsymbol{\mathcal{E}} \boldsymbol{\omega}_\triangle)(\boldsymbol{\mathcal{E}} \boldsymbol{\omega}_\triangle) = (\boldsymbol{\mathcal{E}} \boldsymbol{\omega}_\triangle)^2 = -\zeta^2 \mathbf{N}_\triangle = \boldsymbol{\omega}_\triangle \otimes \boldsymbol{\omega}_\triangle - (\boldsymbol{\omega}_\triangle \cdot \boldsymbol{\omega}_\triangle)\mathbf{I}$$

which leads to

$$\begin{aligned} \mathbf{Q} &= \mathbf{I} - \zeta \mathbf{E}_\triangle + \frac{1}{2}\zeta^2 \mathbf{E}_\triangle^2 - \frac{1}{6}\zeta^3 \mathbf{E}_\triangle^3 + O(\zeta^4) \\ &= \mathbf{I} - \zeta \mathbf{E}_\triangle - \frac{1}{2}\zeta^2 \mathbf{N}_\triangle + \frac{1}{6}\zeta^3 \mathbf{E}_\triangle + O(\zeta^4) \end{aligned} \tag{2.54a}$$

$$\begin{aligned} \mathbf{Q}^T &= \mathbf{I} + \zeta \mathbf{E}_\triangle + \frac{1}{2}\zeta^2 \mathbf{E}_\triangle^2 + \frac{1}{6}\zeta^3 \mathbf{E}_\triangle^3 + O(\zeta^4) \\ &= \mathbf{I} + \zeta \mathbf{E}_\triangle - \frac{1}{2}\zeta^2 \mathbf{N}_\triangle - \frac{1}{6}\zeta^3 \mathbf{E}_\triangle + O(\zeta^4) \end{aligned} \tag{2.54b}$$

Proof:
By introducing the two unit vectors $\mathbf{e}_\triangle \equiv \mathbf{N}_\triangle \mathbf{e}_\triangle$ and $\mathbf{e}_{\triangle\perp} = \mathbf{N}_\triangle \mathbf{e}_{\triangle\perp}$ such that

$$\mathbf{e}_\triangle \cdot \mathbf{e}_\triangle = \mathbf{e}_{\triangle\perp} \cdot \mathbf{e}_{\triangle\perp} = 1 \quad ; \quad \mathbf{e}_\triangle \cdot \mathbf{n}_\triangle = \mathbf{e}_{\triangle\perp} \cdot \mathbf{n}_\triangle = 0 \quad ; \quad \mathbf{e}_\triangle \perp \mathbf{e}_{\triangle\perp} = -\mathbf{E}_\triangle \mathbf{e}_\triangle \tag{2.55}$$

we have

$$\begin{aligned} \cos\zeta &= \mathbf{e}_\triangle \cdot \mathbf{Q}\mathbf{e}_\triangle = \mathbf{e}_{\triangle\perp} \cdot \mathbf{Q}\mathbf{e}_{\triangle\perp} \\ &= \mathbf{e}_\triangle \cdot \Big(\mathbf{I} - \frac{1}{1!}\zeta \underbrace{\mathbf{E}_\triangle}_{=\mathbf{E}_\triangle} + \frac{1}{2!}\zeta^2 \underbrace{\mathbf{E}_\triangle^2}_{=-\mathbf{N}_\triangle} - \frac{1}{3!}\zeta^3 \underbrace{\mathbf{E}_\triangle^3}_{=-\mathbf{E}_\triangle} + \frac{1}{4!}\zeta^4 \underbrace{\mathbf{E}_\triangle^4}_{=\mathbf{N}_\triangle} - \frac{1}{5!}\zeta^5 \underbrace{\mathbf{E}_\triangle^5}_{=\mathbf{E}_\triangle} + \frac{1}{6!}\zeta^6 \underbrace{\mathbf{E}_\triangle^6}_{=-\mathbf{N}_\triangle} - \dots \Big) \mathbf{e}_\triangle \\ &= \underbrace{\mathbf{e}_\triangle \cdot \mathbf{e}_\triangle}_{=1} - \frac{1}{1!}\zeta \underbrace{\mathbf{e}_\triangle \cdot \mathbf{E}_\triangle \mathbf{e}_\triangle}_{=0} - \frac{1}{2!}\zeta^2 \underbrace{\mathbf{e}_\triangle \cdot \mathbf{N}_\triangle \mathbf{e}_\triangle}_{=1} + \frac{1}{3!}\zeta^3 \underbrace{\mathbf{e}_\triangle \cdot \mathbf{E}_\triangle \mathbf{e}_\triangle}_{=0} + \frac{1}{4!}\zeta^4 \underbrace{\mathbf{e}_\triangle \cdot \mathbf{N}_\triangle \mathbf{e}_\triangle}_{=1} \\ &\quad - \frac{1}{5!}\zeta^5 \underbrace{\mathbf{e}_\triangle \cdot \mathbf{E}_\triangle \mathbf{e}_\triangle}_{=0} - \frac{1}{6!}\zeta^6 \underbrace{\mathbf{e}_\triangle \cdot \mathbf{N}_\triangle \mathbf{e}_\triangle}_{=1} - \dots \\ &= 1 - \frac{1}{2!}\zeta^2 + \frac{1}{4!}\zeta^4 - \frac{1}{6!}\zeta^6 + \dots \end{aligned} \tag{2.56a}$$

□ qed

Similarly, we have

$$\sin\zeta = \mathbf{e}_{\triangle\perp} \cdot \mathbf{Q}\mathbf{e}_\triangle = \mathbf{e}_\triangle \mathbf{E}_\triangle \cdot \mathbf{Q}\mathbf{e}_\triangle = \zeta - \frac{1}{3!}\zeta^3 + \frac{1}{5!}\zeta^5 - \frac{1}{7!}\zeta^7 + \dots \tag{2.56b}$$

Thus, the orthogonal rotation tensor $\mathbf{Q}$ can also be exactly written as follows

$$\mathbf{Q} \equiv \mathbf{N}_\triangle \cos\zeta - \mathbf{E}_\triangle \sin\zeta + \mathbf{n}_\triangle \otimes \mathbf{n}_\triangle \tag{2.57}$$

which can be verified with

$$\begin{aligned}\mathbf{Q}^T\mathbf{Q} &= (\mathbf{N}_\Delta \cos\zeta + \mathbf{E}_\Delta \sin\zeta + \mathbf{n}_\Delta \otimes \mathbf{n}_\Delta)(\mathbf{N}_\Delta \cos\zeta - \mathbf{E}_\Delta \sin\zeta + \mathbf{n}_\Delta \otimes \mathbf{n}_\Delta) \\ &= \mathbf{N}_\Delta \cos^2\zeta - \mathbf{E}_\Delta \cos\zeta \sin\zeta + \mathbf{E}_\Delta \sin\zeta\cos\zeta + \mathbf{N}_\Delta \sin^2\zeta + \mathbf{n}_\Delta \otimes \mathbf{n}_\Delta \\ &= \mathbf{N}_\Delta + \mathbf{n}_\Delta \otimes \mathbf{n}_\Delta = \mathbf{I} \qquad \square \text{ qed}\end{aligned} \tag{2.58}$$

2.3.7 Developments of $\mathbf{U}^2$, $\widetilde{\boldsymbol{\mathcal{E}}}$, $\mathbf{F}$, $\boldsymbol{\nabla}\otimes\mathbf{u}$, $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\Omega}$ as function of $\zeta\mathbf{E}_\Delta$, $\eta\boldsymbol{\mathcal{E}}_1$ and $\eta^2\boldsymbol{\mathcal{E}}_2$

By neglecting the third-order terms in $\zeta^3, \zeta^2\eta, \zeta\eta^2$ and η^3, we have

$$\begin{aligned}
&\mathbf{U}^2 = \mathbf{I} + 2\eta\boldsymbol{\mathcal{E}}_1 + \eta^2\boldsymbol{\mathcal{E}}_1^2 + \eta^2\boldsymbol{\mathcal{E}}_2 + O(\eta^3) = \mathbf{I} + O(\eta) \\
&\widetilde{\boldsymbol{\mathcal{E}}} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I}) = \eta\boldsymbol{\mathcal{E}}_1 + \frac{1}{2}\eta^2(\boldsymbol{\mathcal{E}}_1^2 + \boldsymbol{\mathcal{E}}_2) + O(\eta^3) = O(\eta) = O(\varepsilon) \quad \Longrightarrow \quad \eta = O(\varepsilon) \\
&\mathbf{F} = \mathbf{Q}\mathbf{U} = \mathbf{I} + (\boldsymbol{\nabla}\otimes\mathbf{u})^T \\
&\quad = \mathbf{I} - \zeta\mathbf{E}_\Delta + \eta\boldsymbol{\mathcal{E}}_1 + \frac{1}{2}\zeta^2\mathbf{E}_\Delta^2 - \zeta\eta\mathbf{E}_\Delta\boldsymbol{\mathcal{E}}_1 + \frac{1}{2}\eta^2\boldsymbol{\mathcal{E}}_2 + O(\zeta^3, \zeta^2\eta, \zeta\eta^2, \eta^3) = \mathbf{I} + O(\zeta, \eta) \\
&\boldsymbol{\nabla}\otimes\mathbf{u} = \mathbf{F}^T - \mathbf{I} = \boldsymbol{\Omega} + \boldsymbol{\mathcal{E}} = \mathbf{U}\mathbf{Q}^T - \mathbf{I} \\
&\qquad = \zeta\mathbf{E}_\Delta + \eta\boldsymbol{\mathcal{E}}_1 + \frac{1}{2}\zeta^2\mathbf{E}_\Delta^2 + \zeta\eta\boldsymbol{\mathcal{E}}_1\mathbf{E}_\Delta + \frac{1}{2}\eta^2\boldsymbol{\mathcal{E}}_2 + O(\zeta^3, \zeta^2\eta, \zeta\eta^2, \eta^3) = O(\zeta, \eta) \\
&\boldsymbol{\mathcal{E}} = \frac{1}{2}[\boldsymbol{\nabla}\otimes\mathbf{u} + (\boldsymbol{\nabla}\otimes\mathbf{u})^T] = \frac{1}{2}(\mathbf{Q}\mathbf{U} + \mathbf{U}\mathbf{Q}^T) - \mathbf{I} \\
&\quad = \eta\boldsymbol{\mathcal{E}}_1 + \frac{1}{2}\zeta^2\mathbf{E}_\Delta^2 - \frac{1}{2}\zeta\eta(\mathbf{E}_\Delta\boldsymbol{\mathcal{E}}_1 - \boldsymbol{\mathcal{E}}_1\mathbf{E}_\Delta) + \frac{1}{2}\eta^2\boldsymbol{\mathcal{E}}_2 + O(\zeta^4, \zeta^2\eta, \zeta\eta^2, \eta^3) = O(\zeta^2, \eta) \\
&\boldsymbol{\Omega} = \frac{1}{2}[\boldsymbol{\nabla}\otimes\mathbf{u} - (\boldsymbol{\nabla}\otimes\mathbf{u})^T] = -\frac{1}{2}(\mathbf{Q}\mathbf{U} - \mathbf{U}\mathbf{Q}^T) \\
&\quad = \zeta\mathbf{E}_\Delta + \frac{1}{2}\zeta\eta(\mathbf{E}_\Delta\boldsymbol{\mathcal{E}}_1 + \boldsymbol{\mathcal{E}}_1\mathbf{E}_\Delta) + O(\zeta^3, \zeta^2\eta, \zeta\eta^2) = O(\zeta)
\end{aligned} \tag{2.59}$$

2.3.8 Expressions of $\mathbf{U}$ and $\mathbf{U}^{-1}$ as function of $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\Omega}$

$$\begin{aligned}
&\mathbf{U}^2 = \mathbf{I} + 2\widetilde{\boldsymbol{\mathcal{E}}} = \mathbf{I} + 2\boldsymbol{\mathcal{E}} - (\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})(\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}}) \\
&\quad\;\; = \mathbf{I} + 2\eta\boldsymbol{\mathcal{E}}_1 + \eta^2\boldsymbol{\mathcal{E}}_1^2 + \eta^2\boldsymbol{\mathcal{E}}_2 + O(\eta^3) \\
&\eta\boldsymbol{\mathcal{E}}_1 = \boldsymbol{\mathcal{E}} - \frac{1}{2}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})(\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}}) - \frac{1}{2}\eta^2\boldsymbol{\mathcal{E}}_1^2 - \frac{1}{2}\eta^2\boldsymbol{\mathcal{E}}_2 + O(\eta^3) \\
&\frac{1}{2}\eta^2\boldsymbol{\mathcal{E}}_1^2 = \frac{1}{2}\boldsymbol{\mathcal{E}}^2 + O(\zeta^4, \zeta^2\eta, \zeta\eta^2, \eta^3) \\
&\Longrightarrow \quad \mathbf{U} = \mathbf{I} + \boldsymbol{\mathcal{E}} - \frac{1}{2}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})(\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}}) - \frac{1}{2}\boldsymbol{\mathcal{E}}^2 + O(\zeta^4, \zeta^2\eta, \zeta\eta^2, \eta^3) \\
&\qquad\quad\; \mathbf{U}^{-1} = \mathbf{I} - \boldsymbol{\mathcal{E}} + \frac{1}{2}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})(\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}}) + \frac{3}{2}\boldsymbol{\mathcal{E}}^2 + O(\zeta^4, \zeta^2\eta, \zeta\eta^2, \eta^3)
\end{aligned} \tag{2.60}$$

2.3.9 Expressions of Q and $\mathbf{Q}^T$ as function of $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\Omega}$

$$\mathbf{F} = \mathbf{I} - \boldsymbol{\Omega} + \boldsymbol{\mathcal{E}} = \mathbf{QU}$$

$$\begin{aligned}\mathbf{Q}^T\mathbf{F} = \mathbf{Q}^T\mathbf{QU} = \mathbf{U} &= \mathbf{I} + \boldsymbol{\mathcal{E}} - \frac{1}{2}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}}) - \frac{1}{2}\boldsymbol{\mathcal{E}}^2 + O(\zeta^4,\zeta^2\eta,\zeta\eta^2,\eta^3)\\ &= (\mathbf{I} + \zeta\mathbf{E}_\Delta + \frac{1}{2}\zeta^2\mathbf{E}_\Delta^2)(\mathbf{I} - \boldsymbol{\Omega} + \boldsymbol{\mathcal{E}}) + O(\zeta^3)\\ &= \mathbf{I} - (\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}}) + \zeta\mathbf{E}_\Delta - \zeta\mathbf{E}_\Delta(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}}) + \frac{1}{2}\zeta^2\mathbf{E}_\Delta^2 + O(\zeta^3,\zeta^2\eta)\end{aligned}$$

$$\zeta\mathbf{E}_\Delta = \boldsymbol{\Omega} - \frac{1}{2}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}}) - \frac{1}{2}\boldsymbol{\mathcal{E}}^2 + \zeta\mathbf{E}_\Delta(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}}) - \frac{1}{2}\zeta^2\mathbf{E}_\Delta^2 + O(\zeta^3,\zeta^2\eta,\zeta\eta^2,\eta^3)$$
$$\zeta\mathbf{E}_\Delta(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}}) = \boldsymbol{\Omega}(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}}) + O(\zeta^4,\zeta^2\eta,\zeta\eta^2)$$
$$\frac{1}{2}\zeta^2\mathbf{E}_\Delta^2 = \frac{1}{2}\boldsymbol{\Omega}^2 + O(\zeta^4,\zeta^2\eta)$$

$$\Longrightarrow\quad \begin{aligned}\mathbf{Q} &= \mathbf{I} - \boldsymbol{\Omega} + \frac{1}{2}(\boldsymbol{\Omega}^2 + \boldsymbol{\Omega}\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}\boldsymbol{\Omega}) + O(\zeta^3,\zeta^2\eta,\zeta\eta^2,\eta^3)\\ \mathbf{Q}^T &= \mathbf{I} + \boldsymbol{\Omega} + \frac{1}{2}(\boldsymbol{\Omega}^2 - \boldsymbol{\Omega}\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{E}}\boldsymbol{\Omega}) + O(\zeta^3,\zeta^2\eta,\zeta\eta^2,\eta^3)\end{aligned} \tag{2.61}$$

2.3.10 Expression of $\mathbf{F}^{-1}$ as function of $\boldsymbol{\nabla}'\otimes\mathbf{u}$, $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\Omega}$ (Lagrange)

$$\begin{aligned}&\mathbf{r}' \quad\longrightarrow\quad \mathbf{u} = \mathbf{u}(\mathbf{r}')\\ &\mathbf{r}' \quad\longrightarrow\quad \mathbf{r} = \mathbf{r}(\mathbf{r}') = \mathbf{r}' - \mathbf{u}(\mathbf{r}')\end{aligned}$$

$$\mathrm{d}\mathbf{r} = \mathrm{d}\mathbf{r}' - \mathrm{d}\mathbf{u} = \mathrm{d}\mathbf{r}' - \mathrm{d}\mathbf{r}'(\boldsymbol{\nabla}'\otimes\mathbf{u}) = [\mathbf{I} - (\boldsymbol{\nabla}'\otimes\mathbf{u})^T]\mathrm{d}\mathbf{r}' = \mathbf{F}^{-1}\mathrm{d}\mathbf{r}'$$

$$\mathbf{F}^{-1} = \mathbf{I} - (\boldsymbol{\nabla}'\otimes\mathbf{u})^T = (\mathbf{QU})^{-1} = \mathbf{U}^{-1}\mathbf{Q}^T \quad : \quad \text{polar decomposition}$$

$$\mathbf{F}^{-1}\mathbf{F} = \mathbf{F}\mathbf{F}^{-1} = \mathbf{F}^{-1}(\mathbf{I} - \boldsymbol{\Omega} + \boldsymbol{\mathcal{E}}) = \mathbf{I}$$

$$\Longrightarrow\quad \begin{aligned}\mathbf{F}^{-1} &= \mathbf{I} + (\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}}) + (\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})^2 + O(\zeta^3,\zeta^2\eta,\zeta\eta^2,\eta^3)\\ (\mathbf{F}^{-1})^T &= (\mathbf{F}^T)^{-1} = \mathbf{I} - (\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}}) + (\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})^2 + O(\zeta^3,\zeta^2\eta,\zeta\eta^2,\eta^3)\end{aligned} \tag{2.62}$$

2.3.11 Connection between the undeformed and deformed configurations

The connections between the base vectors and between the derivative operators corresponding to the undeformed and deformed configurations are

$$\begin{aligned}&\mathbf{g}_i = \mathbf{r}_{,i} = \frac{\partial\mathbf{r}}{\partial\theta^i} \;;\; \mathbf{g}^j\cdot\mathbf{g}_i = \delta^j{}_i && \mathbf{g}'_i = \mathbf{r}'_{,i} = \frac{\partial\mathbf{r}'}{\partial\theta^i} \;;\; \mathbf{g}'^j\cdot\mathbf{g}'_i = \delta^j{}_i\\ &\mathrm{d}\mathbf{r} = \frac{\partial\mathbf{r}}{\partial\theta^i}\mathrm{d}\theta^i = \mathbf{g}_i\mathrm{d}\theta^i \;;\; \boldsymbol{\nabla} = \mathbf{g}^k\frac{\partial}{\partial\theta^k} \quad ; && \mathrm{d}\mathbf{r}' = \frac{\partial\mathbf{r}'}{\partial\theta^i}\mathrm{d}\theta^i = \mathbf{g}'_i\mathrm{d}\theta^i \;;\; \boldsymbol{\nabla}' = \mathbf{g}'^k\frac{\partial}{\partial\theta^k}\end{aligned}$$

$$\begin{aligned}&\mathrm{d}\mathbf{r}\cdot\boldsymbol{\nabla} = \mathrm{d}\theta^i\mathbf{g}_i\cdot\mathbf{g}^k\frac{\partial}{\partial\theta^k} = \mathrm{d}\theta^i\frac{\partial}{\partial\theta^i} = \mathrm{d}\theta^i\mathbf{g}'_i\cdot\mathbf{g}'^k\frac{\partial}{\partial\theta^k} = \mathrm{d}\mathbf{r}'\cdot\boldsymbol{\nabla}'\\ &\mathrm{d}\mathbf{r}\cdot\boldsymbol{\nabla} = \mathrm{d}\mathbf{r}'\cdot(\mathbf{F}^{-1})^T\boldsymbol{\nabla} = \mathrm{d}\mathbf{r}'\cdot\boldsymbol{\nabla}' = \mathrm{d}\mathbf{r}\cdot\mathbf{F}^T\boldsymbol{\nabla}' \quad ; \quad \forall\,\mathrm{d}\mathbf{r},\mathrm{d}\mathbf{r}'\\ &\Longrightarrow\quad \boldsymbol{\nabla} = \mathbf{F}^T\boldsymbol{\nabla}' \quad ; \quad \boldsymbol{\nabla}' = (\mathbf{F}^{-1})^T\boldsymbol{\nabla} = (\mathbf{F}^T)^{-1}\boldsymbol{\nabla}\end{aligned} \tag{2.63}$$

$$\mathrm{d}\mathbf{r}' = \mathbf{g}_i'\mathrm{d}\theta^i = \mathbf{F}\mathrm{d}\mathbf{r} = \mathbf{F}\mathbf{g}_i\mathrm{d}\theta^i \quad ; \qquad \forall\, \mathrm{d}\theta^i$$

$$\Longrightarrow \quad \mathbf{g}_i' = \mathbf{F}\mathbf{g}_i$$

$$\boldsymbol{\nabla}' = \mathbf{g}'^k\frac{\partial}{\partial\theta^k} = (\mathbf{F}^{-1})^T\boldsymbol{\nabla} = (\mathbf{F}^{-1})^T\mathbf{g}^k\frac{\partial}{\partial\theta^k} \quad ; \qquad \forall\, \frac{\partial}{\partial\theta^k}$$

$$\Longrightarrow \quad \mathbf{g}'^j = (\mathbf{F}^{-1})^T\mathbf{g}^j$$

2.3.12 Expression of $\boldsymbol{\nabla}'$ as function of $\boldsymbol{\nabla}, \boldsymbol{\mathcal{E}}$ and $\boldsymbol{\Omega}$

$$\boldsymbol{\nabla}' = (\mathbf{F}^{-1})^T\boldsymbol{\nabla} = \boldsymbol{\nabla} - (\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\boldsymbol{\nabla} + (\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})^2\boldsymbol{\nabla} + O(\zeta^3,\zeta^2\eta,\zeta\eta^2,\eta^3)\boldsymbol{\nabla} \tag{2.64}$$

2.3.13 Expressions of the three invariants of an arbitrary 3-dimensional tensor $\mathbf{T}$

Let us now briefly recall the expressions of the three invariants I_1, I_2 et I_3 of $\mathbf{T}$

$$\begin{aligned}
I_1 &= \mathrm{tr}\,\mathbf{T} = \mathbf{T}\cdot\mathbf{I} && : \text{ Trace of } \mathbf{T}\\
I_2 &= \frac{1}{2}\mathbf{T}\cdot(\boldsymbol{\mathfrak{E}}\boldsymbol{\mathfrak{E}})^T\cdot\mathbf{T} && : \text{ Sum of the minor-determinants of } \mathbf{T}\\
I_3 &= \det\mathbf{T} = \frac{1}{6}\mathbf{T}\cdot(\boldsymbol{\mathfrak{E}}\mathbf{T}\boldsymbol{\mathfrak{E}})^T\cdot\mathbf{T} && : \text{ Determinant of } \mathbf{T}
\end{aligned} \tag{2.65}$$

because

$$\det(\mathbf{T}-\sigma\mathbf{I}) = \frac{1}{6}(\mathbf{T}-\sigma\mathbf{I})\cdot[\boldsymbol{\mathfrak{E}}(\mathbf{T}-\sigma\mathbf{I})\boldsymbol{\mathfrak{E}}]^T\cdot(\mathbf{T}-\sigma\mathbf{I}) = -\sigma^3 + I_1\sigma^2 - I_2\sigma + I_3 = 0 \tag{2.66}$$

with

$$\begin{aligned}
&(\boldsymbol{\mathfrak{E}}\boldsymbol{\mathfrak{E}})^T\cdot\mathbf{I} = \mathbf{I}\cdot(\boldsymbol{\mathfrak{E}}\boldsymbol{\mathfrak{E}})^T = -(\boldsymbol{\mathfrak{E}}\cdot\boldsymbol{\mathfrak{E}})^T = -\boldsymbol{\mathfrak{E}}\cdot\boldsymbol{\mathfrak{E}} = 2\mathbf{I}\\
&\mathbf{I}\cdot(\boldsymbol{\mathfrak{E}}\boldsymbol{\mathfrak{E}})^T\cdot\mathbf{I} = 2\mathbf{I}\cdot\mathbf{I} = 6 \quad ; \qquad \mathbf{I}\cdot\mathbf{I} = 3\\
&(\boldsymbol{\mathfrak{E}}\mathbf{T}\boldsymbol{\mathfrak{E}})^T\cdot\mathbf{I} = \mathbf{I}\cdot(\boldsymbol{\mathfrak{E}}\mathbf{T}\boldsymbol{\mathfrak{E}})^T = (\boldsymbol{\mathfrak{E}}\boldsymbol{\mathfrak{E}})^T\cdot\mathbf{T} = \mathbf{T}\cdot(\boldsymbol{\mathfrak{E}}\boldsymbol{\mathfrak{E}})^T\\
&\mathbf{I}\cdot(\boldsymbol{\mathfrak{E}}\mathbf{T}\boldsymbol{\mathfrak{E}})^T\cdot\mathbf{I} = \mathbf{T}\cdot(\boldsymbol{\mathfrak{E}}\boldsymbol{\mathfrak{E}})^T\cdot\mathbf{I} = \mathbf{I}\cdot(\boldsymbol{\mathfrak{E}}\boldsymbol{\mathfrak{E}})^T\cdot\mathbf{T} = 2\mathbf{T}\cdot\mathbf{I}
\end{aligned} \tag{2.67}$$

2.3.14 Summary

By neglecting the third-order terms in $\zeta^3, \zeta^2\eta, \zeta\eta^2$ and η^3, we have

$$\mathbf{F} = \mathbf{I} + (\boldsymbol{\nabla}\otimes\mathbf{u})^T = \mathbf{Q}\mathbf{U} = \mathbf{I} - \boldsymbol{\Omega} + \boldsymbol{\mathcal{E}} \tag{2.68}$$

$$\begin{aligned}
\mathbf{F}^{-1} &= \mathbf{I} - (\boldsymbol{\nabla}'\otimes\mathbf{u})^T = \mathbf{U}^{-1}\mathbf{Q}^T \cong \mathbf{I} + (\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}}) + (\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})^2\\
&\cong \mathbf{I} - (\boldsymbol{\nabla}\otimes\mathbf{u})^T + [(\boldsymbol{\nabla}\otimes\mathbf{u})^T]^2
\end{aligned} \tag{2.69}$$

$$\begin{aligned}\boldsymbol{\nabla}' &= (\mathbf{F}^{-1})^T\boldsymbol{\nabla} \cong \boldsymbol{\nabla} - (\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\boldsymbol{\nabla} + (\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})^2\boldsymbol{\nabla} \\ &\cong \boldsymbol{\nabla} - (\boldsymbol{\nabla}\otimes\mathbf{u})\boldsymbol{\nabla} + (\boldsymbol{\nabla}\otimes\mathbf{u})^2\boldsymbol{\nabla}\end{aligned} \tag{2.70}$$

$$\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^T = \frac{1}{2}[\boldsymbol{\nabla}\otimes\mathbf{u} + (\boldsymbol{\nabla}\otimes\mathbf{u})^T] = \frac{1}{2}(\mathbf{QU} + \mathbf{UQ}^T) - \mathbf{I} \tag{2.71}$$

$$\boldsymbol{\Omega} = -\boldsymbol{\Omega}^T = \frac{1}{2}[\boldsymbol{\nabla}\otimes\mathbf{u} - (\boldsymbol{\nabla}\otimes\mathbf{u})^T] = -\frac{1}{2}(\mathbf{QU} - \mathbf{UQ}^T) \tag{2.72}$$

$$\boldsymbol{\nabla}\otimes\mathbf{u} = \boldsymbol{\Omega} + \boldsymbol{\mathcal{E}} = \mathbf{UQ}^T - \mathbf{I} \tag{2.73}$$

$$(\boldsymbol{\nabla}\otimes\mathbf{u})^T = -(\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}}) = \mathbf{QU} - \mathbf{I} \tag{2.74}$$

$$\begin{aligned}\widetilde{\boldsymbol{\mathcal{E}}} &= \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I}) = \boldsymbol{\mathcal{E}} - \frac{1}{2}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}}) \\ &= \frac{1}{2}[\boldsymbol{\nabla}\otimes\mathbf{u} + (\boldsymbol{\nabla}\otimes\mathbf{u})^T] + \frac{1}{2}(\boldsymbol{\nabla}\otimes\mathbf{u})(\boldsymbol{\nabla}\otimes\mathbf{u})^T\end{aligned} \tag{2.75}$$

$$\begin{aligned}\mathbf{U} &\cong \mathbf{I} + \boldsymbol{\mathcal{E}} - \frac{1}{2}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}}) - \frac{1}{2}\boldsymbol{\mathcal{E}}^2 \\ &\cong \mathbf{I} + \frac{1}{2}[\boldsymbol{\nabla}\otimes\mathbf{u} + (\boldsymbol{\nabla}\otimes\mathbf{u})^T] - \frac{1}{8}(\boldsymbol{\nabla}\otimes\mathbf{u})^2 - \frac{1}{8}[(\boldsymbol{\nabla}\otimes\mathbf{u})^T]^2 \\ &\quad + \frac{1}{8}[3(\boldsymbol{\nabla}\otimes\mathbf{u})(\boldsymbol{\nabla}\otimes\mathbf{u})^T - (\boldsymbol{\nabla}\otimes\mathbf{u})^T(\boldsymbol{\nabla}\otimes\mathbf{u})]\end{aligned} \tag{2.76}$$

$$\begin{aligned}\mathbf{U}^{-1} &\cong \mathbf{I} - \boldsymbol{\mathcal{E}} + \frac{1}{2}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}}) + \frac{3}{2}\boldsymbol{\mathcal{E}}^2 \\ &\cong \mathbf{I} - \frac{1}{2}[\boldsymbol{\nabla}\otimes\mathbf{u} + (\boldsymbol{\nabla}\otimes\mathbf{u})^T] + \frac{3}{8}(\boldsymbol{\nabla}\otimes\mathbf{u})^2 + \frac{3}{8}[(\boldsymbol{\nabla}\otimes\mathbf{u})^T]^2 \\ &\quad - \frac{1}{8}[(\boldsymbol{\nabla}\otimes\mathbf{u})(\boldsymbol{\nabla}\otimes\mathbf{u})^T - 3(\boldsymbol{\nabla}\otimes\mathbf{u})^T(\boldsymbol{\nabla}\otimes\mathbf{u})]\end{aligned} \tag{2.77}$$

$$\begin{aligned}\mathbf{Q} &\cong \mathbf{I} - \boldsymbol{\Omega} + \frac{1}{2}(\boldsymbol{\Omega}^2 + \boldsymbol{\Omega}\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}\boldsymbol{\Omega}) \\ &\cong \mathbf{I} - \frac{1}{2}[\boldsymbol{\nabla}\otimes\mathbf{u} - (\boldsymbol{\nabla}\otimes\mathbf{u})^T] + \frac{3}{8}(\boldsymbol{\nabla}\otimes\mathbf{u})^2 - \frac{1}{8}[(\boldsymbol{\nabla}\otimes\mathbf{u})^T]^2 \\ &\quad - \frac{1}{8}[(\boldsymbol{\nabla}\otimes\mathbf{u})(\boldsymbol{\nabla}\otimes\mathbf{u})^T + (\boldsymbol{\nabla}\otimes\mathbf{u})^T(\boldsymbol{\nabla}\otimes\mathbf{u})]\end{aligned} \tag{2.78}$$

$$\begin{aligned}\mathbf{Q}^T &\cong \mathbf{I} + \boldsymbol{\Omega} + \frac{1}{2}(\boldsymbol{\Omega}^2 - \boldsymbol{\Omega}\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{E}}\boldsymbol{\Omega}) \\ &\cong \mathbf{I} + \frac{1}{2}[\boldsymbol{\nabla}\otimes\mathbf{u} - (\boldsymbol{\nabla}\otimes\mathbf{u})^T] - \frac{1}{8}(\boldsymbol{\nabla}\otimes\mathbf{u})^2 + \frac{3}{8}[(\boldsymbol{\nabla}\otimes\mathbf{u})^T]^2 \\ &\quad - \frac{1}{8}[(\boldsymbol{\nabla}\otimes\mathbf{u})(\boldsymbol{\nabla}\otimes\mathbf{u})^T + (\boldsymbol{\nabla}\otimes\mathbf{u})^T(\boldsymbol{\nabla}\otimes\mathbf{u})]\end{aligned} \tag{2.79}$$

2.3.15 Strain components of a 3-dimensional body in the 3-dimensional space written in a Cartesian system up to the second-order terms

Let us recall the general exact expression for the strain tensor of a 3-dimensional body

$$\widetilde{\boldsymbol{\mathcal{E}}} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) = \frac{1}{2}[(\boldsymbol{\nabla} \otimes \mathbf{u}) + (\boldsymbol{\nabla} \otimes \mathbf{u})^T] + \frac{1}{2}(\boldsymbol{\nabla} \otimes \mathbf{u})(\boldsymbol{\nabla} \otimes \mathbf{u})^T \tag{2.80}$$

where $\mathbf{F} = \mathbf{I} + (\boldsymbol{\nabla} \otimes \mathbf{u})^T$ is the deformation gradient of the body, $\mathbf{F}^T\mathbf{F}$ the related *Cauchy-Green* tensor, $\boldsymbol{\nabla} = \mathbf{g}^j \partial/\partial\theta^j$ the 3-dimensional derivative operator and $\mathbf{I} = \mathbf{g}_i \otimes \mathbf{g}^i$ the metric tensor in the 3-dimensional space (with $i, j = 1, 2, 3$).

Introducing a Cartesian system (P, x, y, z) in some point P of the body, we write

$$\widetilde{\boldsymbol{\mathcal{E}}} \mathrel{\hat{=}} \begin{bmatrix} \widetilde{\varepsilon}_x & \frac{1}{2}\widetilde{\gamma}_{xy} & \frac{1}{2}\widetilde{\gamma}_{xz} \\ \frac{1}{2}\widetilde{\gamma}_{xy} & \widetilde{\varepsilon}_y & \frac{1}{2}\widetilde{\gamma}_{yz} \\ \frac{1}{2}\widetilde{\gamma}_{xz} & \frac{1}{2}\widetilde{\gamma}_{yz} & \widetilde{\varepsilon}_z \end{bmatrix} \quad ; \quad \mathbf{I} \mathrel{\hat{=}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.81}$$

and

$$\boldsymbol{\nabla} \mathrel{\hat{=}} \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{Bmatrix} \quad ; \quad \mathbf{u} \mathrel{\hat{=}} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \quad ; \quad \mathbf{e} \mathrel{\hat{=}} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \quad ; \quad \mathbf{e}_\perp \mathrel{\hat{=}} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \quad ; \quad \mathbf{n} \mathrel{\hat{=}} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \tag{2.82}$$

where the sign ˆ draws attention to the fact that the base vectors are omitted in the matrix representation. In components, we have:

$$(\boldsymbol{\nabla} \otimes \mathbf{u}) \mathrel{\hat{=}} \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{Bmatrix} \otimes (\, u \quad v \quad w \,) = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} & \dfrac{\partial w}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} & \dfrac{\partial w}{\partial y} \\ \dfrac{\partial u}{\partial z} & \dfrac{\partial v}{\partial z} & \dfrac{\partial w}{\partial z} \end{bmatrix}$$

$$(\boldsymbol{\nabla} \otimes \mathbf{u})^T \mathrel{\hat{=}} \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{bmatrix} \tag{2.83}$$

$$(\nabla \otimes \mathbf{u})(\nabla \otimes \mathbf{u})^T \;\hat{=}$$

$$\hat{=} \begin{bmatrix} \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2 & \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} & \frac{\partial u}{\partial x}\frac{\partial u}{\partial z}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial z}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} & \left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2 & \frac{\partial u}{\partial y}\frac{\partial u}{\partial z}+\frac{\partial v}{\partial y}\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial x}\frac{\partial u}{\partial z}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial z}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial z} & \frac{\partial u}{\partial y}\frac{\partial u}{\partial z}+\frac{\partial v}{\partial y}\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\frac{\partial w}{\partial z} & \left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2 \end{bmatrix}$$

The components $\widetilde{\varepsilon}_x, \widetilde{\varepsilon}_y, \widetilde{\varepsilon}_z, \widetilde{\gamma}_{xy}/2, \widetilde{\gamma}_{xz}/2$ and $\widetilde{\gamma}_{yz}/2$ of the strain tensor $\widetilde{\boldsymbol{\mathcal{E}}}$ can now exactly be written as follows

$$\begin{aligned}
\widetilde{\varepsilon}_x &= \mathbf{e}\cdot\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{e} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \\
\widetilde{\varepsilon}_y &= \mathbf{e}_\perp\cdot\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{e}_\perp = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \\
\widetilde{\varepsilon}_z &= \mathbf{n}\cdot\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{n} = \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] \qquad (2.84) \\
\frac{1}{2}\widetilde{\gamma}_{xy} &= \mathbf{e}\cdot\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{e}_\perp = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \\
\frac{1}{2}\widetilde{\gamma}_{xz} &= \mathbf{e}\cdot\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{n} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\right) \\
\frac{1}{2}\widetilde{\gamma}_{yz} &= \mathbf{e}_\perp\cdot\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{n} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right)
\end{aligned}$$

To get the linear and angular dilatation ε_x, ε_y, ε_z, $\gamma_{xy}/2$, $\gamma_{xz}/2$ and $\gamma_{yz}/2$, we must take into account the following nonlinear expressions

$$\begin{aligned}
\varepsilon_x &= \sqrt{1+2\widetilde{\varepsilon}_x} - 1 & \qquad & \frac{1}{2}\gamma_{xy} = \frac{1}{2}\arcsin\left(\frac{\widetilde{\gamma}_{xy}}{\sqrt{(1+2\widetilde{\varepsilon}_x)(1+2\widetilde{\varepsilon}_y)}}\right) \\
\varepsilon_y &= \sqrt{1+2\widetilde{\varepsilon}_y} - 1 & ; \qquad & \frac{1}{2}\gamma_{xz} = \frac{1}{2}\arcsin\left(\frac{\widetilde{\gamma}_{xz}}{\sqrt{(1+2\widetilde{\varepsilon}_x)(1+2\widetilde{\varepsilon}_z)}}\right) \qquad (2.85) \\
\varepsilon_z &= \sqrt{1+2\widetilde{\varepsilon}_z} - 1 & \qquad & \frac{1}{2}\gamma_{yz} = \frac{1}{2}\arcsin\left(\frac{\widetilde{\gamma}_{yz}}{\sqrt{(1+2\widetilde{\varepsilon}_y)(1+2\widetilde{\varepsilon}_z)}}\right)
\end{aligned}$$

which read after development up to the second order terms

$$\varepsilon_x = \sqrt{1+2\widetilde{\varepsilon}_x} - 1 \simeq \widetilde{\varepsilon}_x - \frac{1}{2}\widetilde{\varepsilon}_x^2 + \ldots$$

$$
\begin{aligned}
\varepsilon_y &= \sqrt{1+2\widetilde{\varepsilon}_y} - 1 \simeq \widetilde{\varepsilon}_y - \frac{1}{2}\widetilde{\varepsilon}_y^2 + \ldots \\
\varepsilon_z &= \sqrt{1+2\widetilde{\varepsilon}_z} - 1 \simeq \widetilde{\varepsilon}_z - \frac{1}{2}\widetilde{\varepsilon}_z^2 + \ldots \\
\frac{1}{2}\gamma_{xy} &= \frac{1}{2}\arcsin\left(\frac{\widetilde{\gamma}_{xy}}{\sqrt{(1+2\widetilde{\varepsilon}_x)(1+2\widetilde{\varepsilon}_y)}}\right) \simeq \frac{1}{2}\widetilde{\gamma}_{xy} - \frac{1}{2}\widetilde{\gamma}_{xy}(\widetilde{\varepsilon}_x + \widetilde{\varepsilon}_y) + \ldots \\
\frac{1}{2}\gamma_{xz} &= \frac{1}{2}\arcsin\left(\frac{\widetilde{\gamma}_{xz}}{\sqrt{(1+2\widetilde{\varepsilon}_x)(1+2\widetilde{\varepsilon}_z)}}\right) \simeq \frac{1}{2}\widetilde{\gamma}_{xz} - \frac{1}{2}\widetilde{\gamma}_{xz}(\widetilde{\varepsilon}_x + \widetilde{\varepsilon}_z) + \ldots \\
\frac{1}{2}\gamma_{yz} &= \frac{1}{2}\arcsin\left(\frac{\widetilde{\gamma}_{yz}}{\sqrt{(1+2\widetilde{\varepsilon}_y)(1+2\widetilde{\varepsilon}_z)}}\right) \simeq \frac{1}{2}\widetilde{\gamma}_{yz} - \frac{1}{2}\widetilde{\gamma}_{yz}(\widetilde{\varepsilon}_y + \widetilde{\varepsilon}_z) + \ldots
\end{aligned}
\tag{2.86}
$$

Thus we get for this case

$$
\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \\
\varepsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \\
\varepsilon_z &= \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] \\
\frac{1}{2}\gamma_{xy} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \frac{1}{2}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} - \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \\
\frac{1}{2}\gamma_{xz} &= \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) - \frac{1}{2}\left(\frac{\partial u}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial z} - \frac{\partial v}{\partial x}\frac{\partial v}{\partial z}\right) \\
\frac{1}{2}\gamma_{yz} &= \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) - \frac{1}{2}\left(\frac{\partial v}{\partial y}\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial z} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial z}\right)
\end{aligned}
\tag{2.87}
$$

2.4 Nonlinear kinematics of deformation of a 2-dimensional curved surface in the 3-dimensional space

In the case of large deformation measurement of opaque bodies by means of holographic interferometry, or in the classical cases of deformation analysis of plates and shells, the dilatation, rotation and displacement components often have different orders of magnitude. Practically, one may encounter small strains together with moderate rotations and large displacements. In order to properly analyze such deformations, we must develop the tensors previously introduced in this chapter up to higher order terms. Because only the surface of an opaque object (not the interior of the body) can be recorded by means of holographic interferometry, it is necessary to introduce here the basic concepts of surface deformation.

2.4.1 Vector coordinates

The one-one mapping that associates the whole set of points $\{\mathrm{P}\}$ on the undeformed curved surface to the set of points $\{\mathrm{P}'\}$ on the deformed curved surface is described by a single set of curvilinear coordinates θ^α $(\alpha = 1, 2)$ called convected coordinates as follows

$$\theta^1, \theta^2 \quad \longrightarrow \quad \begin{cases} \mathbf{r} = \mathbf{r}(\theta^1, \theta^2) & \longrightarrow \quad \mathbf{r}' = \mathbf{r}'(\mathbf{r}) = \mathbf{r}'[\mathbf{r}(\theta^1, \theta^2)] \\ \mathbf{r}' = \mathbf{r}'(\theta^1, \theta^2) & \longrightarrow \quad \mathbf{r} = \mathbf{r}(\mathbf{r}') = \mathbf{r}[\mathbf{r}'(\theta^1, \theta^2)] \end{cases} \tag{2.88}$$

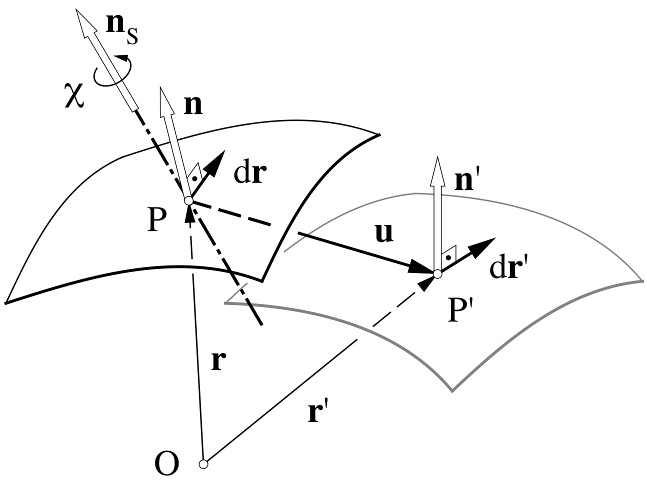

Fig. 2.4: Deformation of a 2-dimensional curved surface in space

2.4.2 Lagrangean representation of the deformation

$$\begin{aligned}
&\mathbf{r} \quad \longrightarrow \quad \mathbf{u} = \mathbf{u}(\mathbf{r}) \\
&\mathbf{r} \quad \longrightarrow \quad \mathbf{r}' = \mathbf{r}'(\mathbf{r}) = \mathbf{r} + \mathbf{u}(\mathbf{r}) \\
&\mathrm{d}\mathbf{r}' = \mathbf{N}'\mathrm{d}\mathbf{r}' = \mathrm{d}\mathbf{r} + \mathrm{d}\mathbf{u} = \mathbf{N}\mathrm{d}\mathbf{r} + \mathrm{d}\mathbf{r}(\boldsymbol{\nabla}_n \otimes \mathbf{u}) \\
&\qquad = [\mathbf{N} + (\boldsymbol{\nabla}_n \otimes \mathbf{u})^T]\mathrm{d}\mathbf{r} = \mathbf{F}\mathbf{N}\mathrm{d}\mathbf{r} = \mathbf{F}_S\mathrm{d}\mathbf{r} \\
&\mathbf{N}' = \mathbf{I} - \mathbf{n}' \otimes \mathbf{n}' \quad : \quad \text{normal projector} \quad , \quad \mathbf{n}' \perp \mathrm{d}\mathbf{r}' \quad , \quad (\mathbf{n}' \cdot \mathbf{n}' = 1) \\
&\Longrightarrow \quad \mathbf{F}_S \equiv \mathbf{F}\mathbf{N} \equiv \mathbf{N}'\mathbf{F}\mathbf{N} \equiv \mathbf{Q}\mathbf{U}\mathbf{N} \\
&\qquad\qquad \equiv \mathbf{N}'\mathbf{F}_S = \mathbf{N} + (\boldsymbol{\nabla}_n \otimes \mathbf{u})^T \quad : \quad \text{deformation gradient of the surface} \\
&\mathrm{d}^2\mathbf{r}' = \mathrm{d}\mathbf{F}_S\mathrm{d}\mathbf{r} + \mathbf{F}_S\mathrm{d}^2\mathbf{r} = \mathbf{F}_S\mathrm{d}^2\mathbf{r} + \mathrm{d}\mathbf{r}(\boldsymbol{\nabla}_n \otimes \mathbf{F}_S)\mathrm{d}\mathbf{r} \\
&\Longrightarrow \quad \boldsymbol{\nabla}_n \otimes \mathbf{F}_S = \boldsymbol{\nabla}_n \otimes \boldsymbol{\nabla}_n \otimes \mathbf{u})^T + \mathbf{B} \otimes \mathbf{n} + \mathbf{B} \otimes \mathbf{n})^T \quad : \quad \text{derivative of } \mathbf{F}_S
\end{aligned} \tag{2.89}$$

where $\mathbf{F}_S$ is a mixed semi-projection of the deformation gradient $\mathbf{F}$ of a 3-dimensional body.
Polar decomposition:

$$\mathbf{F}_S = \mathbf{N} + (\boldsymbol{\nabla}_n \otimes \mathbf{u})^T = \mathbf{Q}_S\mathbf{V} \quad : \quad \text{polar decomposition} \tag{2.90}$$

with $\mathbf{F}_S^T\mathbf{F}_S = \mathbf{V}^2 \quad ; \quad \mathbf{V} = \mathbf{V}^T = \mathbf{NVN} \quad ; \quad \mathbf{Q}_S^T\mathbf{Q}_S = \mathbf{Q}_S\mathbf{Q}_S^T = \mathbf{I} \quad ; \quad \mathbf{Q}_S^{-1} = \mathbf{Q}_S^T$

$\mathbf{F}_S^T\mathbf{F}_S$	:	*Cauchy-Green* tensor of the surface (2-dimensional, symmetric, pos. def.)
$\mathbf{V}$	:	2-dimensional symmetric tensor describing the dilatation of the surface
$\mathbf{Q}_S$	:	3-dimensional orthogonal tensor describing the rotation of the surface in space

Expression of the unit normal $\mathbf{n}'$ of the deformed surface:

$$\begin{aligned}
&\mathbf{F}_S = \mathbf{Q}_S\mathbf{V} = \mathbf{N}'\mathbf{F}_S = (\mathbf{I} - \mathbf{n}' \otimes \mathbf{n}')\mathbf{Q}_S\mathbf{V} = \mathbf{Q}_S\mathbf{V} - \mathbf{n}' \otimes \mathbf{n}'\mathbf{Q}_S\mathbf{V} \quad ; \quad \mathbf{n}' \neq 0 \\
&\Rightarrow \quad \mathbf{n}'\mathbf{Q}_S\mathbf{V} = 0 \quad ; \quad \forall\, \mathbf{Q}_S\mathbf{V} \neq 0 \quad ; \quad \forall\, \mathbf{Q}_S^T\mathbf{n}' \neq 0 \\
&\Rightarrow \quad \mathbf{n}'\mathbf{Q}_S\mathbf{N} = \mathbf{N}\mathbf{Q}_S^T\mathbf{n}' = 0 \quad \text{because} \quad \mathbf{V} = \mathbf{NVN} \\
&\Rightarrow \quad \mathbf{Q}_S^T\mathbf{n}' = \mathbf{n} \quad \Longleftrightarrow \quad \boxed{\mathbf{n}' = \mathbf{Q}_S\mathbf{n}}
\end{aligned} \tag{2.91}$$

Expression of the normal projector $\mathbf{N}'$ relative to the deformed surface:

$$\mathbf{N}' = \mathbf{I} - \mathbf{n}' \otimes \mathbf{n}' = \mathbf{Q}_S\mathbf{Q}_S^T - \mathbf{Q}_S\mathbf{n} \otimes \mathbf{n}\mathbf{Q}_S^T = \mathbf{Q}_S(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\mathbf{Q}_S^T = \mathbf{Q}_S\mathbf{N}\mathbf{Q}_S^T \tag{2.92}$$

A general deformation of a 2-dimensional curved surface in space is described in the neighborhood of a point P by a <u>bidimensional dilatation</u> in the tangential plane followed by a <u>rotation</u> followed by a <u>displacement</u> of an infinitesimal surface element:

1°) The bidimensional dilatation of the neighborhood of the point P on the undeformed curved surface by means of the 2-dimensional symmetric tensor $\mathbf{V} = \mathbf{NVN}$.

2°) The "rigid body" rotation of the strained surface element around an axis Δ_S of direction $\mathbf{n}_S$ going through point P by means of the 3-dimensional orthogonal tensor $\mathbf{Q}_S$.

3°) The displacement of the strained and rotated surface element from its position P on the undeformed surface to the new position P′ on the deformed surface by means of the displacement vector $\mathbf{u}$.

This description is only valid for a single point and vary from point to point on the surface. Let us also recall that the 3-dimensional rotation tensors $\mathbf{Q}_S$ and $\mathbf{Q}$ are in general not identical, that means $\mathbf{Q}_S \neq \mathbf{Q}$ and $\mathbf{n}_S \neq \mathbf{n}_\Delta$. This comes from the 3-dimensional symmetric tensor $\mathbf{U}$, which would not only acts as a bidimensional dilatation of the surface element but also as a rotation out of the tangential plane of the undeformed curved surface.

2.4.3 Expressions of $\boldsymbol{\nabla}_n \otimes \mathbf{u}$, $(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T$, $\mathbf{F}_S$ and $\mathbf{F}_S^T$ as function of $\mathbf{N}$, $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\Omega}$

$$\begin{aligned}
&\boldsymbol{\nabla}_n \otimes \mathbf{u} = \mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}}) \\
&(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T = -(\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}})\mathbf{N} \\
&\mathbf{F}_S = \mathbf{N} + (\boldsymbol{\nabla}_n \otimes \mathbf{u})^T = \mathbf{N} - (\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}})\mathbf{N} = (\mathbf{I} - \boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})\mathbf{N} \\
&\mathbf{F}_S^T = \mathbf{N} + \boldsymbol{\nabla}_n \otimes \mathbf{u} = \mathbf{N} + \mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}}) = \mathbf{N}(\mathbf{I} + \boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})
\end{aligned} \tag{2.93}$$

In holographic interferometry, we will see that the derivative of the displacement $\boldsymbol{\nabla}_n \otimes \mathbf{u}$, which contains the deformation, appears in the expression of the fringe vector $\widetilde{\mathbf{f}}_R$. This vector describes the fringe spacing and the fringe direction. Contrary to the measurement of small deformations where only the first-order linear terms are considered, moderate and large deformation measurements by means of holographic interferometry may only be done properly, if the quantities describing the dilatation and that describing the rotation respectively only contain dilatation and rotation components up to the higher order terms in their developments.

2.4.4 Definition of the 2-dimensional symmetric strain tensor $\widetilde{\gamma}$ of the surface

$$\begin{aligned}\widetilde{\gamma} &= \frac{1}{2}(\mathbf{F}_S^T\mathbf{F}_S - \mathbf{N}) = \frac{1}{2}(\mathbf{V}\mathbf{Q}_S^T\mathbf{Q}_S\mathbf{V} - \mathbf{N}) = \frac{1}{2}(\mathbf{V}^2 - \mathbf{N}) \\ &= \frac{1}{2}(\mathbf{N}\mathbf{F}^T\mathbf{F}\mathbf{N} - \mathbf{N}) = \frac{1}{2}\mathbf{N}(\mathbf{F}^T\mathbf{F} - \mathbf{I})\mathbf{N} = \frac{1}{2}\mathbf{N}(\mathbf{U}^2 - \mathbf{I})\mathbf{N} \\ &= \mathbf{N}\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{N} = \mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} - \frac{1}{2}\mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})(\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}})\mathbf{N}\end{aligned} \tag{2.94}$$

This tensor obviously only contains the dilatation. In the particular case where the surface deformation gradient $\mathbf{F}_S$ is close to the normal projector $\mathbf{N}$, which plays the role of the identity on the curved surface, we have with $\mathbf{F}_S = \mathbf{Q}_S\mathbf{V} \simeq \mathbf{N}$ and $\mathbf{F}_S^T \simeq \mathbf{N}$

$$\widetilde{\gamma} = \frac{1}{2}(\mathbf{F}_S^T\mathbf{F}_S - \mathbf{N}) = \mathbf{N}\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{N} = O(\varepsilon) \quad ; \qquad 0 \leqslant |\varepsilon| \ll 1 \tag{2.95}$$

With the infinitesimal increments

$$\begin{array}{llll} \mathrm{d}\mathbf{r} = \mathbf{e}\,\mathrm{d}s & \mathbf{N}\mathbf{e} \equiv \mathbf{e} & \mathbf{e}\cdot\mathbf{e} = 1 & \\ \mathrm{d}\mathbf{r}_\perp = \mathbf{e}_\perp \mathrm{d}s_\perp & \mathbf{N}\mathbf{e}_\perp \equiv \mathbf{e}_\perp & \mathbf{e}_\perp\cdot\mathbf{e}_\perp = 1 & \mathbf{e}\cdot\mathbf{e}_\perp = 0 \\ \mathrm{d}\mathbf{r}' = \mathbf{e}'\mathrm{d}s' = \mathbf{F}_S\mathrm{d}\mathbf{r} = \mathbf{F}_S\mathbf{e}\,\mathrm{d}s \quad ; & \mathbf{N}'\mathbf{e}' \equiv \mathbf{e}' \quad ; & \mathbf{e}'\cdot\mathbf{e}' = 1 \quad ; & 0 \leqslant |\mathbf{e}'\cdot\mathbf{e}'_\perp| \ll 1 \\ \mathrm{d}\mathbf{r}'_\perp = \mathbf{e}'_\perp\mathrm{d}s'_\perp = \mathbf{F}_S\mathrm{d}\mathbf{r}_\perp = \mathbf{F}_S\mathbf{e}_\perp\mathrm{d}s_\perp & \mathbf{N}'\mathbf{e}'_\perp \equiv \mathbf{e}'_\perp & \mathbf{e}'_\perp\cdot\mathbf{e}'_\perp = 1 & \end{array} \tag{2.96}$$

the linear and angular dilatations read

$$\begin{aligned} &\varepsilon = \frac{\mathrm{d}s' - \mathrm{d}s}{\mathrm{d}s} \quad ; \quad \varepsilon_\perp = \frac{\mathrm{d}s'_\perp - \mathrm{d}s_\perp}{\mathrm{d}s_\perp} \quad ; \quad \cos(\frac{\pi}{2} - \gamma) = \sin\gamma = \mathbf{e}'\cdot\mathbf{e}'_\perp \\ &\left.\begin{aligned}\widetilde{\varepsilon} &= \mathbf{e}\cdot\widetilde{\gamma}\mathbf{e} = \frac{1}{2}\mathbf{e}\cdot(\mathbf{F}_S^T\mathbf{F}_S - \mathbf{N})\mathbf{e} \\ &= \frac{\mathrm{d}s'^2 - \mathrm{d}s^2}{2\mathrm{d}s^2} = \varepsilon + \frac{1}{2}\varepsilon^2\end{aligned}\right\} \Rightarrow \quad \varepsilon = \sqrt{1 + 2\widetilde{\varepsilon}} - 1 = \sqrt{1 + 2\mathbf{e}\cdot\widetilde{\gamma}\mathbf{e}} - 1 \\ &\left.\begin{aligned}\widetilde{\varepsilon}_\perp &= \mathbf{e}_\perp\cdot\widetilde{\gamma}\mathbf{e}_\perp = \frac{1}{2}\mathbf{e}_\perp\cdot(\mathbf{F}_S^T\mathbf{F}_S - \mathbf{N})\mathbf{e}_\perp \\ &= \frac{\mathrm{d}s'^2_\perp - \mathrm{d}s^2_\perp}{2\mathrm{d}s^2_\perp} = \varepsilon_\perp + \frac{1}{2}\varepsilon^2_\perp\end{aligned}\right\} \Rightarrow \quad \varepsilon_\perp = \sqrt{1 + 2\widetilde{\varepsilon}_\perp} - 1 = \sqrt{1 + 2\mathbf{e}_\perp\cdot\widetilde{\gamma}\mathbf{e}_\perp} - 1 \\ &\left.\begin{aligned}\frac{1}{2}\widetilde{\gamma} &= \mathbf{e}\cdot\widetilde{\gamma}\mathbf{e}_\perp = \frac{1}{2}\mathbf{e}\cdot(\mathbf{F}_S^T\mathbf{F}_S - \mathbf{N})\mathbf{e}_\perp \\ &= \frac{\mathrm{d}s'\mathrm{d}s'_\perp}{2\mathrm{d}s\mathrm{d}s_\perp}(\mathbf{e}'\cdot\mathbf{e}'_\perp) = \frac{1}{2}(1+\varepsilon)(1+\varepsilon_\perp)\sin\gamma\end{aligned}\right\} \Rightarrow \quad \sin\gamma = \frac{2\mathbf{e}\cdot\widetilde{\gamma}\mathbf{e}_\perp}{\sqrt{(1 + 2\mathbf{e}\cdot\widetilde{\gamma}\mathbf{e})(1 + 2\mathbf{e}_\perp\cdot\widetilde{\gamma}\mathbf{e}_\perp)}}\end{aligned} \tag{2.97}$$

2.4.5 Developments of the tensors $\mathbf{V}$ and $\mathbf{Q}_S$

Because of $\mathbf{F}_S = \mathbf{Q}_S\mathbf{V} \simeq \mathbf{N}$, the tensor $\mathbf{V}$ characterizing the dilatation is close to the normal projector $\mathbf{N}$ and the tensor $\mathbf{Q}_S$ characterizing the rotation is close to the identity $\mathbf{I}$. For the tensor $\mathbf{V}$, we write

$$\begin{aligned}
&\mathbf{V} = \mathbf{N} + \mathbf{V}_1 \qquad \text{with} \quad \mathbf{V}_1 = \mathbf{V}_1^T = \mathbf{N}\mathbf{V}_1\mathbf{N}\\
&\mathbf{V}^2 = \mathbf{N} + 2\widetilde{\gamma} = \mathbf{N} + 2\mathbf{V}_1 + \mathbf{V}_1^2\\
&\mathbf{V}_1 = \widetilde{\gamma} - \frac{1}{2}\mathbf{V}_1^2 = O(\varepsilon) = O(\eta)\\
&\mathbf{V}_1^2 = \widetilde{\gamma}^2 + O(\eta^3)\\
&\Longrightarrow \quad \mathbf{V} = \mathbf{N} + \widetilde{\gamma} - \frac{1}{2}\widetilde{\gamma}^2 + O(\eta^3)
\end{aligned} \tag{2.98}$$

For the tensors $\mathbf{Q}_S$ and $\mathbf{Q}_S^T$, we have

$$\left.\begin{aligned}
\mathbf{Q}_S &= \mathbf{I} - \chi\mathbf{E}_S + \frac{1}{2}\chi^2\mathbf{E}_S^2 + O(\chi^3)\\
\mathbf{Q}_S^T &= \mathbf{I} + \chi\mathbf{E}_S + \frac{1}{2}\chi^2\mathbf{E}_S^2 + O(\chi^3)
\end{aligned}\right. \qquad \text{with} \qquad 0 \leqslant |\chi| \ll 1 \tag{2.99}$$

where χ is a small parameter describing the rotation angle in radian. We have

$$\begin{aligned}
&\mathbf{Q}_S^T\mathbf{Q}_S = \mathbf{Q}_S\mathbf{Q}_S^T = \mathbf{I} \quad ; \quad \forall\, |\chi| \ll 1 \qquad \Longrightarrow \qquad \mathbf{E}_S = -\mathbf{E}_S^T\\
&\chi \quad : \qquad \text{rotation angle (in radian) around the axis } \Delta_S \text{ going through point P}\\
&\mathbf{E}_S = \boldsymbol{\mathcal{E}}\mathbf{n}_S = \mathbf{n}_S\boldsymbol{\mathcal{E}} \quad : \qquad \text{2-dimensional second-rank permutation tensor}\\
&\mathbf{n}_S \quad : \qquad \text{direction of the rotation axis } \Delta_S \text{ with } \mathbf{n}_S\cdot\mathbf{n}_S = 1\\
&\mathbf{N}_S = \mathbf{N}_S^T = -\mathbf{E}_S^2 = \mathbf{I} - \mathbf{n}_S\otimes\mathbf{n}_S \quad : \qquad \text{normal projector}\\
&\mathbf{E}_S \equiv \mathbf{N}_S\mathbf{E}_S \equiv \mathbf{E}_S\mathbf{N}_S \equiv \mathbf{N}_S\mathbf{E}_S\mathbf{N}_S\\
&\boldsymbol{\omega}_S = \chi\mathbf{n}_S \quad : \qquad \text{rotation vector (full describes the rotation of the surface element)}\\
&\chi\mathbf{E}_S = \chi\boldsymbol{\mathcal{E}}\mathbf{n}_S = \boldsymbol{\mathcal{E}}\boldsymbol{\omega}_S = \boldsymbol{\omega}_S\boldsymbol{\mathcal{E}}
\end{aligned} \tag{2.100}$$

2.4.6 Decomposition of the rotation vector ω_S in interior and exterior parts

$$\boxed{\boldsymbol{\omega}_S \equiv \chi\mathbf{n}_S \equiv \Omega\mathbf{n} + \mathbf{E}\,\boldsymbol{\omega}} \qquad \text{with} \quad \boldsymbol{\omega} \equiv \mathbf{N}\,\boldsymbol{\omega} \quad \text{by definition} \tag{2.101}$$

where $\Omega\mathbf{n}$ and $\mathbf{E}\,\boldsymbol{\omega}$ respectively describe the exterior and the interior parts of $\boldsymbol{\omega}_S$ relatively to the tangential plane of the surface element.

2.4.7 Decomposition of the tensors $\mathbf{Q}_S$ and $\mathbf{Q}_S^T$ in interior, semi-exterior and exterior parts

$$\begin{aligned}
\chi\mathbf{E}_S = \boldsymbol{\mathcal{E}}\boldsymbol{\omega}_S &= [\mathbf{E}\otimes\mathbf{n} - \mathbf{E}\otimes\mathbf{n})^T + \mathbf{n}\otimes\mathbf{E}](\Omega\mathbf{n} + \mathbf{E}\,\boldsymbol{\omega})\\
&= \Omega\mathbf{E} + \boldsymbol{\omega}\otimes\mathbf{n} - \mathbf{n}\otimes\boldsymbol{\omega}
\end{aligned}$$

$$\chi^2 \mathbf{E}_S^2 = (\boldsymbol{\mathcal{E}}\boldsymbol{\omega}_S)(\boldsymbol{\mathcal{E}}\boldsymbol{\omega}_S) = (\varOmega\mathbf{E} + \boldsymbol{\omega}\otimes\mathbf{n} - \mathbf{n}\otimes\boldsymbol{\omega})(\varOmega\mathbf{E} + \boldsymbol{\omega}\otimes\mathbf{n} - \mathbf{n}\otimes\boldsymbol{\omega})$$
$$= -\varOmega^2\mathbf{N} + \varOmega\mathbf{E}\boldsymbol{\omega}\otimes\mathbf{n} - \boldsymbol{\omega}\otimes\boldsymbol{\omega} + \mathbf{n}\otimes\varOmega\mathbf{E}\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\boldsymbol{\omega})\mathbf{n}\otimes\mathbf{n}$$

$$\Longrightarrow \quad \begin{cases} \mathbf{Q}_S = \mathbf{I} - \varOmega\mathbf{E} - \boldsymbol{\omega}\otimes\mathbf{n} + \mathbf{n}\otimes\boldsymbol{\omega} + \frac{1}{2}\varOmega\mathbf{E}\boldsymbol{\omega}\otimes\mathbf{n} + \frac{1}{2}\mathbf{n}\otimes\varOmega\mathbf{E}\boldsymbol{\omega} \\ \qquad - \frac{1}{2}[\varOmega^2\mathbf{N} + \boldsymbol{\omega}\otimes\boldsymbol{\omega} + (\boldsymbol{\omega}\cdot\boldsymbol{\omega})\mathbf{n}\otimes\mathbf{n}] + O(\chi^3) \end{cases} \tag{2.102}$$

$$\Longrightarrow \quad \begin{cases} \mathbf{Q}_S^T = \mathbf{I} + \varOmega\mathbf{E} + \boldsymbol{\omega}\otimes\mathbf{n} - \mathbf{n}\otimes\boldsymbol{\omega} + \frac{1}{2}\varOmega\mathbf{E}\boldsymbol{\omega}\otimes\mathbf{n} + \frac{1}{2}\mathbf{n}\otimes\varOmega\mathbf{E}\boldsymbol{\omega} \\ \qquad - \frac{1}{2}[\varOmega^2\mathbf{N} + \boldsymbol{\omega}\otimes\boldsymbol{\omega} + (\boldsymbol{\omega}\cdot\boldsymbol{\omega})\mathbf{n}\otimes\mathbf{n}] + O(\chi^3) \end{cases}$$

For the orders of magnitude, we have

$$\left.\begin{array}{l} \mathbf{V} = \mathbf{N} + O(\eta) \quad ; \quad \mathbf{Q}_S = \mathbf{I} + O(\chi) \\ \mathbf{U} = \mathbf{I} + O(\eta) \quad ; \quad \mathbf{Q} = \mathbf{I} + O(\zeta) \\ \boldsymbol{\nabla}_n\otimes\mathbf{u} = \mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}}) = \mathbf{N}(\mathbf{F}^T - \mathbf{I}) = \mathbf{N}(\mathbf{U}\mathbf{Q}^T - \mathbf{I}) = O(\zeta,\eta) \\ \qquad = \mathbf{F}_S^T - \mathbf{N} = \mathbf{V}\mathbf{Q}_S^T - \mathbf{N} = O(\chi,\eta) \end{array}\right\} \Longrightarrow \quad \chi = O(\zeta,\eta) \tag{2.103}$$

2.4.8 Developments of $\boldsymbol{\nabla}_n\otimes\mathbf{u}$, $\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N}$ and $\mathbf{N}\boldsymbol{\Omega}\mathbf{N}$ as function of $\chi\mathbf{E}_S$ and $\widetilde{\gamma}$

$$\begin{aligned} \boldsymbol{\nabla}_n\otimes\mathbf{u} &= \mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}}) = \mathbf{F}_S^T - \mathbf{N} = \mathbf{V}\mathbf{Q}_S^T - \mathbf{N} \\ &= \chi\mathbf{N}\mathbf{E}_S + \widetilde{\gamma} + \frac{1}{2}\chi^2\mathbf{N}\mathbf{E}_S^2 + \chi\widetilde{\gamma}\mathbf{E}_S - \frac{1}{2}\widetilde{\gamma}^2 + O(\chi^3,\chi^2\eta,\chi\eta^2,\eta^3) = O(\chi,\eta) \\ \mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} &= \frac{1}{2}[(\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{N} + \mathbf{N}(\boldsymbol{\nabla}_n\otimes\mathbf{u})^T] \\ &= \widetilde{\gamma} + \frac{1}{2}\chi^2\mathbf{N}\mathbf{E}_S^2\mathbf{N} - \frac{1}{2}\chi(\mathbf{N}\mathbf{E}_S\widetilde{\gamma} - \widetilde{\gamma}\mathbf{E}_S\mathbf{N}) - \frac{1}{2}\widetilde{\gamma}^2 + O(\chi^4,\chi^2\eta,\chi\eta^2,\eta^3) = O(\chi^2,\eta) \\ \mathbf{N}\boldsymbol{\Omega}\mathbf{N} &= \frac{1}{2}[(\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{N} - \mathbf{N}(\boldsymbol{\nabla}_n\otimes\mathbf{u})^T] \\ &= \chi\mathbf{N}\mathbf{E}_S\mathbf{N} + \frac{1}{2}\chi(\mathbf{N}\mathbf{E}_S\widetilde{\gamma} + \widetilde{\gamma}\mathbf{E}_S\mathbf{N}) + O(\chi^3,\chi^2\eta,\chi\eta^2) = O(\chi) \end{aligned} \tag{2.104}$$

2.4.9 Expression of $\mathbf{V}$ as function of $\mathbf{N}$, $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\Omega}$

$$\begin{aligned} \widetilde{\gamma}^2 &= \mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} - \frac{1}{2}\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})(\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}})\mathbf{N} - \frac{1}{2}\mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})(\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}})\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} \\ &\quad + \frac{1}{4}\mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})(\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}})\mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})(\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}})\mathbf{N} \\ &= \mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} + O(\chi^4,\chi^2\eta,\chi\eta^2,\eta^3) \end{aligned} \tag{2.105}$$

$$\Longrightarrow \quad \mathbf{V} = \mathbf{N} + \mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} - \frac{1}{2}\mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})(\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}})\mathbf{N} - \frac{1}{2}\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} + O(\chi^4,\chi^2\eta,\chi\eta^2,\eta^3)$$

2.4.10 Expression of $\mathbf{Q}_S$ as function of $\mathbf{N}, \boldsymbol{\mathcal{E}}$ and $\boldsymbol{\Omega}$

$$\begin{aligned} &\mathbf{F}_S^T = \mathbf{V}\mathbf{Q}_S^T = \mathbf{N}(\mathbf{I}+\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}}) \\ &\mathbf{F}_S^T\mathbf{Q}_S = \mathbf{V}\mathbf{Q}_S^T\mathbf{Q}_S = \mathbf{V} = \mathbf{N}(\mathbf{I}+\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{Q}_S \end{aligned} \tag{2.106}$$

with the two essential conditions:

$$\begin{aligned} &(a) \quad \mathbf{V}\mathbf{n} = \mathbf{N}(\mathbf{I}+\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{Q}_S\mathbf{n} = 0 \\ &(b) \quad \mathbf{V}\mathbf{N} = \mathbf{N}(\mathbf{I}+\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{Q}_S\mathbf{N} = \mathbf{V} \end{aligned} \tag{2.107}$$

Before solving (a) and (b), we write

$$\begin{aligned}
&\mathbf{n}\cdot\mathbf{E}_S\mathbf{n} = (\mathbf{E}_S\mathbf{n})\cdot\mathbf{n} = \mathbf{n}\mathbf{E}_S^T\cdot\mathbf{n} = -\mathbf{n}\cdot\mathbf{E}_S\mathbf{n} = 0 \qquad \rightarrow \quad \begin{cases}\text{a scalar equal to}\\ \text{its opposite is zero}\end{cases} \\
&\mathbf{E}_S\mathbf{n} = \mathbf{I}\mathbf{E}_S\mathbf{n} = (\mathbf{N}+\mathbf{n}\otimes\mathbf{n})\mathbf{E}_S\mathbf{n} = \mathbf{N}\mathbf{E}_S\mathbf{n} + \mathbf{n}(\mathbf{n}\cdot\mathbf{E}_S\mathbf{n}) \\
&\Rightarrow \quad \mathbf{E}_S\mathbf{n} \equiv \mathbf{N}\mathbf{E}_S\mathbf{n} \\
&\mathbf{N}\mathbf{E}_S^2\mathbf{n} = \mathbf{N}\mathbf{E}_S\mathbf{I}\mathbf{E}_S\mathbf{n} = \mathbf{N}\mathbf{E}_S(\mathbf{N}+\mathbf{n}\otimes\mathbf{n})\mathbf{E}_S\mathbf{n} = \mathbf{N}\mathbf{E}_S\mathbf{N}\mathbf{E}_S\mathbf{n} + \mathbf{N}\mathbf{E}_S\mathbf{n}(\mathbf{n}\cdot\mathbf{E}_S\mathbf{n}) \\
&\Rightarrow \quad \mathbf{N}\mathbf{E}_S^2\mathbf{n} \equiv \mathbf{N}\mathbf{E}_S\mathbf{N}\mathbf{E}_S\mathbf{n} \\
&\mathbf{E}_S\mathbf{N} = \mathbf{I}\mathbf{E}_S\mathbf{N} = (\mathbf{N}+\mathbf{n}\otimes\mathbf{n})\mathbf{E}_S\mathbf{N} \\
&\Rightarrow \quad \mathbf{E}_S\mathbf{N} \equiv \mathbf{N}\mathbf{E}_S\mathbf{N} + \mathbf{n}\otimes\mathbf{n}\mathbf{E}_S\mathbf{N} \\
&\mathbf{N}\mathbf{E}_S^2\mathbf{N} = \mathbf{N}\mathbf{E}_S\mathbf{I}\mathbf{E}_S\mathbf{N} = \mathbf{N}\mathbf{E}_S(\mathbf{N}+\mathbf{n}\otimes\mathbf{n})\mathbf{E}_S\mathbf{N} \\
&\Rightarrow \quad \mathbf{N}\mathbf{E}_S^2\mathbf{N} \equiv \mathbf{N}\mathbf{E}_S\mathbf{N}\mathbf{E}_S\mathbf{N} + \mathbf{N}\mathbf{E}_S\mathbf{n}\otimes\mathbf{n}\mathbf{E}_S\mathbf{N} \\
&\mathbf{E}_S = \mathbf{I}\mathbf{E}_S\mathbf{I} = (\mathbf{N}+\mathbf{n}\otimes\mathbf{n})\mathbf{E}_S(\mathbf{N}+\mathbf{n}\otimes\mathbf{n}) \\
&\quad = \mathbf{N}\mathbf{E}_S\mathbf{N} + \mathbf{N}\mathbf{E}_S\mathbf{n}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{n}\mathbf{E}_S\mathbf{N} + (\mathbf{n}\cdot\mathbf{E}_S\mathbf{n})\mathbf{n}\otimes\mathbf{n} \\
&\quad \Rightarrow \quad \mathbf{E}_S \equiv \mathbf{N}\mathbf{E}_S\mathbf{N} + \mathbf{N}\mathbf{E}_S\mathbf{n}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{n}\mathbf{E}_S\mathbf{N}
\end{aligned} \tag{2.108}$$

Solving (a)

$$\begin{aligned}
0 = \mathbf{V}\mathbf{n} = \mathbf{N}(\mathbf{I}+\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{Q}_S\mathbf{n} &= [\mathbf{N}+\mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})]\left(\mathbf{I} - \chi\mathbf{E}_S + \frac{1}{2}\chi^2\mathbf{E}_S^2\right)\mathbf{n} + O(\chi^3) \\
&= -\chi\mathbf{N}\mathbf{E}_S\mathbf{n} + \mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n} - \chi\mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{E}_S\mathbf{n} \\
&\quad + \frac{1}{2}\chi^2\mathbf{N}\mathbf{E}_S^2\mathbf{n} + O(\chi^3,\chi^2\eta)
\end{aligned} \tag{2.109}$$

$$\Rightarrow \quad \chi\mathbf{N}\mathbf{E}_S\mathbf{n} = \mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n} - \chi\mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{N}\mathbf{E}_S\mathbf{n} + \frac{1}{2}\chi^2\mathbf{N}\mathbf{E}_S\mathbf{N}\mathbf{E}_S\mathbf{n} + O(\chi^3,\chi^2\eta)$$

$$\Rightarrow \quad \chi\mathbf{n}\mathbf{E}_S\mathbf{N} = \mathbf{n}(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N} + \chi\mathbf{n}\mathbf{E}_S\mathbf{N}(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N} - \frac{1}{2}\chi^2\mathbf{n}\mathbf{E}_S\mathbf{N}\mathbf{E}_S\mathbf{N} + O(\chi^3,\chi^2\eta)$$

Solving (b)

$$\mathbf{V} = \mathbf{N}(\mathbf{I}+\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{Q}_S\mathbf{N} = \mathbf{N}+\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} - \frac{1}{2}\mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N} - \frac{1}{2}\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} + O(\chi^4,\chi^2\eta,\chi\eta^2,\eta^3)$$
$$= [\mathbf{N}+\mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})]\left(\mathbf{I}-\chi\mathbf{E}_S+\frac{1}{2}\chi^2\mathbf{E}_S^2\right)\mathbf{N}+O(\chi^3)$$
$$= \mathbf{N}-\chi\mathbf{N}\mathbf{E}_S\mathbf{N}+\mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{N}-\chi\mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{E}_S\mathbf{N}+\frac{1}{2}\chi^2\mathbf{N}\mathbf{E}_S^2\mathbf{N} + O(\chi^3,\chi^2\eta)$$

$$\Rightarrow \begin{cases} \chi\mathbf{N}\mathbf{E}_S\mathbf{N} = \mathbf{N}\boldsymbol{\Omega}\mathbf{N}+\frac{1}{2}\mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N}+\frac{1}{2}\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N}-\chi\mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{N}\mathbf{E}_S\mathbf{N} \\ \qquad -\chi\mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n}\otimes\mathbf{n}\mathbf{E}_S\mathbf{N}+\frac{1}{2}\chi^2\mathbf{N}\mathbf{E}_S\mathbf{N}\mathbf{E}_S\mathbf{N}+\frac{1}{2}\chi^2\mathbf{N}\mathbf{E}_S\mathbf{n}\otimes\mathbf{n}\mathbf{E}_S\mathbf{N} \\ \qquad + O(\chi^3,\chi^2\eta,\chi\eta^2,\eta^3) \end{cases} \tag{2.110}$$

The second-order terms can be explicitly written with the following three linear approximations

$$\begin{aligned} \chi\mathbf{N}\mathbf{E}_S\mathbf{n} &= \mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n}+O(\chi^2,\chi\eta) \\ \chi\mathbf{n}\mathbf{E}_S\mathbf{N} &= \mathbf{n}(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N}+O(\chi^2,\chi\eta) \\ \chi\mathbf{N}\mathbf{E}_S\mathbf{N} &= \mathbf{N}\boldsymbol{\Omega}\mathbf{N}+O(\chi^2,\chi\eta,\eta^2) \end{aligned} \tag{2.111}$$

We get

$$\begin{aligned} \chi\mathbf{N}\mathbf{E}_S\mathbf{n} &= \mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n}-\mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n}+\frac{1}{2}\mathbf{N}\boldsymbol{\Omega}\mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n}+O(\chi^3,\chi^2\eta,\chi\eta^2,\eta^3) \\ \chi\mathbf{n}\mathbf{E}_S\mathbf{N} &= \mathbf{n}(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N}+\mathbf{n}(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N}(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N}-\frac{1}{2}\mathbf{n}(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N}\boldsymbol{\Omega}\mathbf{N}+O(\chi^3,\chi^2\eta,\chi\eta^2,\eta^3) \\ \chi\mathbf{N}\mathbf{E}_S\mathbf{N} &= \mathbf{N}\boldsymbol{\Omega}\mathbf{N}+\frac{1}{2}\mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N}-\mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{N}\boldsymbol{\Omega}\mathbf{N}+\frac{1}{2}\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} \\ &\quad +\frac{1}{2}\mathbf{N}\boldsymbol{\Omega}\mathbf{N}\boldsymbol{\Omega}\mathbf{N}-\frac{1}{2}\mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n}\otimes\mathbf{n}(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N}+O(\chi^3,\chi^2\eta,\chi\eta^2,\eta^3) \end{aligned} \tag{2.112}$$

The antimetric tensor $\chi\mathbf{E}_S$ and symmetric tensor $\chi^2\mathbf{E}_S^2$ read

$$\begin{aligned} \chi\mathbf{E}_S &\equiv \chi\mathbf{N}\mathbf{E}_S\mathbf{N}+\chi\mathbf{N}\mathbf{E}_S\mathbf{n}\otimes\mathbf{n}+\chi\mathbf{n}\otimes\mathbf{n}\mathbf{E}_S\mathbf{N} \\ &= \mathbf{N}\boldsymbol{\Omega}\mathbf{N}+\mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n}\otimes\mathbf{n}+\mathbf{n}\otimes\mathbf{n}(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N}-\frac{1}{2}(\mathbf{N}\boldsymbol{\Omega}\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N}+\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N}\boldsymbol{\Omega}\mathbf{N}) \\ &\quad -\mathbf{N}\left(\frac{1}{2}\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}}\right)\mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n}\otimes\mathbf{n}+\mathbf{n}\otimes\mathbf{n}(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N}\left(\frac{1}{2}\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}}\right)\mathbf{N} \\ &\quad +O(\chi^3,\chi^2\eta,\chi\eta^2,\eta^3) \end{aligned} \tag{2.113}$$

$$\chi^2\mathbf{E}_S^2 = \mathbf{N\Omega N\Omega N} + \mathbf{N\Omega N}(\mathbf{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{n}(\mathbf{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N\Omega N}$$
$$+\mathbf{N}(\mathbf{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n}\otimes\mathbf{n}(\mathbf{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N} + [\mathbf{n}\cdot(\mathbf{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N}(\mathbf{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n}]\,\mathbf{n}\otimes\mathbf{n} \tag{2.114}$$
$$+O(\chi^3,\chi^2\eta,\chi\eta^2,\eta^3)$$

The orthogonal tensor $\mathbf{Q}_S$ can now be written as follows

$$\mathbf{Q}_S = \mathbf{I} - \chi\mathbf{E}_S + \frac{1}{2}\chi^2\mathbf{E}_S^2 + O(\chi^3)$$
$$= \mathbf{I} - \mathbf{N\Omega N} - \mathbf{N}(\mathbf{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n}\otimes\mathbf{n} - \mathbf{n}\otimes\mathbf{n}(\mathbf{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N} + \frac{1}{2}(\mathbf{N\Omega N\Omega N} + \mathbf{N\Omega N}\boldsymbol{\mathcal{E}}\mathbf{N} + \mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N\Omega N})$$
$$+\mathbf{N}(\mathbf{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{N}(\mathbf{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{n}(\mathbf{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} + \frac{1}{2}\mathbf{N}(\mathbf{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n}\otimes\mathbf{n}(\mathbf{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N}$$
$$-\frac{1}{2}[\mathbf{N}(\mathbf{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n}]^2\,\mathbf{n}\otimes\mathbf{n} + O(\chi^3,\chi^2\eta,\chi\eta^2,\eta^3) \tag{2.115}$$

2.4.11 Expressions of $\boldsymbol{\omega}$, $\mathit{\Omega}\mathbf{E}$ and $\mathit{\Omega}$ as function of $\mathbf{N}, \boldsymbol{\mathcal{E}}, \mathbf{\Omega}$ and $\boldsymbol{\nabla}_n\otimes\mathbf{u}$

From the relation

$$\chi\mathbf{E}_S \equiv \mathit{\Omega}\mathbf{E} + \boldsymbol{\omega}\otimes\mathbf{n} - \mathbf{n}\otimes\boldsymbol{\omega}$$
$$= \mathbf{N\Omega N} + \mathbf{N}(\mathbf{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{n}(\mathbf{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N} - \frac{1}{2}(\mathbf{N\Omega N}\boldsymbol{\mathcal{E}}\mathbf{N} + \mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N\Omega N}) \tag{2.116}$$
$$-\mathbf{N}\left(\frac{1}{2}\mathbf{\Omega}+\boldsymbol{\mathcal{E}}\right)\mathbf{N}(\mathbf{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{n}(\mathbf{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N}\left(\frac{1}{2}\mathbf{\Omega}-\boldsymbol{\mathcal{E}}\right)\mathbf{N} + O(\chi^3,\chi^2\eta,\chi\eta^2,\eta^3)$$

we get

$$\boldsymbol{\omega} = \chi\mathbf{E}_S\mathbf{n} = \chi\mathbf{N}\mathbf{E}_S\mathbf{n}$$
$$= \mathbf{N}(\mathbf{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n} - \mathbf{N}\left(\frac{1}{2}\mathbf{\Omega}+\boldsymbol{\mathcal{E}}\right)\mathbf{N}(\mathbf{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n} + O(\chi^3,\chi^2\eta,\chi\eta^2,\eta^3)$$
$$= (\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{n} - \left[\frac{3}{4}(\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{N} + \frac{1}{4}\mathbf{N}(\boldsymbol{\nabla}_n\otimes\mathbf{u})^T\right](\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{n} + O(\chi^3,\chi^2\eta,\chi\eta^2,\eta^3) \tag{2.117}$$

$$\mathit{\Omega}\mathbf{E} = \chi\mathbf{N}\mathbf{E}_S\mathbf{N}$$
$$= \mathbf{N\Omega N} - \frac{1}{2}(\mathbf{N\Omega N}\boldsymbol{\mathcal{E}}\mathbf{N} + \mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N\Omega N}) + O(\chi^3,\chi^2\eta,\chi\eta^2,\eta^3)$$
$$= \frac{1}{2}[(\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{N} - \mathbf{N}(\boldsymbol{\nabla}_n\otimes\mathbf{u})^T] - \frac{1}{4}[(\boldsymbol{\nabla}_n\otimes\mathbf{u})(\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{N} - \mathbf{N}(\boldsymbol{\nabla}_n\otimes\mathbf{u})^T(\boldsymbol{\nabla}_n\otimes\mathbf{u})^T]$$
$$+O(\chi^3,\chi^2\eta,\chi\eta^2,\eta^3) \tag{2.118}$$

$$
\begin{aligned}
\mathit{\Omega} &= -\frac{1}{2}\chi \mathbf{E}_S\cdot\mathbf{E} = -\frac{1}{2}\chi\mathbf{E}\cdot\mathbf{E}_S = -\frac{1}{2}\mathit{\Omega}\mathbf{E}\cdot\mathbf{E} \\
&= -\frac{1}{2}\mathbf{N\Omega N}\cdot\mathbf{E} + \frac{1}{4}(\mathbf{N\Omega N}\boldsymbol{\mathcal{E}}\mathbf{N} + \mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N\Omega N})\cdot\mathbf{E} + O(\chi^3,\chi^2\eta,\chi\eta^2,\eta^3) \\
&= -\frac{1}{4}[(\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{N} - \mathbf{N}(\boldsymbol{\nabla}_n\otimes\mathbf{u})^T]\cdot\mathbf{E} + \frac{1}{8}[(\boldsymbol{\nabla}_n\otimes\mathbf{u})(\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{N} - \mathbf{N}(\boldsymbol{\nabla}_n\otimes\mathbf{u})^T(\boldsymbol{\nabla}_n\otimes\mathbf{u})^T]\cdot\mathbf{E} \\
&\quad + O(\chi^3,\chi^2\eta,\chi\eta^2,\eta^3)
\end{aligned} \tag{2.119}
$$

2.4.12 Expression of $\boldsymbol{\nabla}_n\otimes\mathbf{u}$ as function of $\widetilde{\boldsymbol{\gamma}}$, $\boldsymbol{\omega}$ and $\mathit{\Omega}$

$$
\begin{aligned}
\boldsymbol{\nabla}_n\otimes\mathbf{u} &= \mathbf{F}_S^T - \mathbf{N} = \mathbf{V}\mathbf{Q}_S^T - \mathbf{N} \\
&= \Big(\mathbf{N} + \widetilde{\boldsymbol{\gamma}} - \frac{1}{2}\widetilde{\boldsymbol{\gamma}}^2\Big)\Big(\mathbf{I} + \mathit{\Omega}\mathbf{E} + \boldsymbol{\omega}\otimes\mathbf{n} - \mathbf{n}\otimes\boldsymbol{\omega} + \frac{1}{2}\mathit{\Omega}\mathbf{E}\boldsymbol{\omega}\otimes\mathbf{n} + \frac{1}{2}\mathbf{n}\otimes\mathit{\Omega}\mathbf{E}\boldsymbol{\omega} \\
&\qquad - \frac{1}{2}[\mathit{\Omega}^2\mathbf{N} + \boldsymbol{\omega}\otimes\boldsymbol{\omega} + (\boldsymbol{\omega}\cdot\boldsymbol{\omega})\mathbf{n}\otimes\mathbf{n}]\Big) - \mathbf{N} + O(\chi^3,\eta^3) \\
&= \widetilde{\boldsymbol{\gamma}} + \mathit{\Omega}\mathbf{E} + \boldsymbol{\omega}\otimes\mathbf{n} - \frac{1}{2}(\widetilde{\boldsymbol{\gamma}}^2 + \mathit{\Omega}^2\mathbf{N} + \boldsymbol{\omega}\otimes\boldsymbol{\omega}) + \mathit{\Omega}\widetilde{\boldsymbol{\gamma}}\mathbf{E} + \widetilde{\boldsymbol{\gamma}}\boldsymbol{\omega}\otimes\mathbf{n} + \frac{1}{2}\mathit{\Omega}\mathbf{E}\boldsymbol{\omega}\otimes\mathbf{n} \\
&\quad + O(\chi^3,\chi^2\eta,\chi\eta^2,\eta^3)
\end{aligned} \tag{2.120}
$$

2.4.13 Expression of the unit normal $\mathbf{n}'$ of the deformed curved surface as function of $\boldsymbol{\omega}$ and $\mathit{\Omega}$

$$
\mathbf{n}' = \mathbf{Q}_S\mathbf{n} = \mathbf{n} - \boldsymbol{\omega} + \frac{1}{2}\mathit{\Omega}\mathbf{E}\boldsymbol{\omega} - \frac{1}{2}(\boldsymbol{\omega}\cdot\boldsymbol{\omega})\mathbf{n} + O(\chi^3) \tag{2.121}
$$

2.4.14 Expression of the normal projector $\mathbf{N}'$ as function of $\boldsymbol{\omega}$ and $\mathit{\Omega}$

$$
\begin{aligned}
\mathbf{N}' &= \mathbf{Q}_S\mathbf{N}\mathbf{Q}_S^T = \mathbf{I} - \mathbf{n}'\otimes\mathbf{n}' \qquad (2.122) \\
&= \mathbf{N} + \mathbf{n}\otimes\boldsymbol{\omega} + \boldsymbol{\omega}\otimes\mathbf{n} - \frac{1}{2}\mathit{\Omega}\mathbf{E}\boldsymbol{\omega}\otimes\mathbf{n} - \frac{1}{2}\mathbf{n}\otimes\mathit{\Omega}\mathbf{E}\boldsymbol{\omega} - \boldsymbol{\omega}\otimes\boldsymbol{\omega} + (\boldsymbol{\omega}\cdot\boldsymbol{\omega})\mathbf{n}\otimes\mathbf{n} + O(\chi^3)
\end{aligned}
$$

2.4.15 Expression of $\mathbf{F}'_S$ as function of $\boldsymbol{\nabla}'_n\otimes\mathbf{u}, \mathbf{N}, \boldsymbol{\mathcal{E}}$ and $\boldsymbol{\Omega}$ (Lagrange)

$$
\begin{aligned}
&\mathbf{r}' \longrightarrow \mathbf{u} = \mathbf{u}(\mathbf{r}') \\
&\mathbf{r}' \longrightarrow \mathbf{r} = \mathbf{r}(\mathbf{r}') = \mathbf{r}' - \mathbf{u}(\mathbf{r}') \\
&\mathrm{d}\mathbf{r} = \mathbf{N}\mathrm{d}\mathbf{r} = \mathrm{d}\mathbf{r}' - \mathrm{d}\mathbf{u} = \mathbf{N}'\mathrm{d}\mathbf{r}' - \mathrm{d}\mathbf{r}'(\boldsymbol{\nabla}'_n\otimes\mathbf{u}) \\
&\quad = [\mathbf{N}' - (\boldsymbol{\nabla}'_n\otimes\mathbf{u})^T]\mathrm{d}\mathbf{r}' = \mathbf{F}^{-1}\mathbf{N}'\mathrm{d}\mathbf{r}' = \mathbf{F}'_S\mathrm{d}\mathbf{r}' \\
&\Longrightarrow \left\{\begin{aligned} &\mathbf{F}'_S \equiv \mathbf{F}^{-1}\mathbf{N}' \equiv \mathbf{N}\mathbf{F}^{-1}\mathbf{N}' \equiv \mathbf{U}^{-1}\mathbf{Q}^T\mathbf{N}' \\ &\quad \equiv \mathbf{N}\mathbf{F}'_S = \mathbf{N}' - (\boldsymbol{\nabla}'_n\otimes\mathbf{u})^T = \mathbf{W}\mathbf{Q}_S^T \quad : \quad \text{polar decomposition}\end{aligned}\right.
\end{aligned} \tag{2.123}
$$

where $\mathbf{W} = \mathbf{W}^T = \mathbf{NWN}$. It follows that

$$\begin{aligned}
&\mathrm{d}\mathbf{r}' = \mathbf{F}_S \mathrm{d}\mathbf{r} = \mathbf{F}_S \mathbf{F}'_S \mathrm{d}\mathbf{r}' = \mathbf{N}' \mathrm{d}\mathbf{r}' \quad ; \qquad \forall\, \mathrm{d}\mathbf{r}' \equiv \mathbf{N}' \mathrm{d}\mathbf{r}' \\
&\Rightarrow \quad \mathbf{F}_S \mathbf{F}'_S = \mathbf{Q}_S \mathbf{V} \mathbf{W} \mathbf{Q}_S^T = \mathbf{N}' \quad \Longrightarrow \quad \mathbf{VW} = \mathbf{N} \\
&\mathrm{d}\mathbf{r} = \mathbf{F}'_S \mathrm{d}\mathbf{r}' = \mathbf{F}'_S \mathbf{F}_S \mathrm{d}\mathbf{r} = \mathbf{N} \mathrm{d}\mathbf{r} \quad ; \qquad \forall\, \mathrm{d}\mathbf{r} \equiv \mathbf{N} \mathrm{d}\mathbf{r} \\
&\Rightarrow \quad \mathbf{F}'_S \mathbf{F}_S = \mathbf{W} \mathbf{Q}_S^T \mathbf{Q}_S \mathbf{V} = \mathbf{WV} = \mathbf{N}
\end{aligned} \tag{2.124}$$

For the tensor $\mathbf{W}$, we write

$$\begin{aligned}
&\mathbf{W} = \mathbf{N} + \mathbf{W}_1 \qquad \text{with} \quad \mathbf{W}_1 = \mathbf{W}_1^T = \mathbf{N}\mathbf{W}_1\mathbf{N} \\
&\mathbf{N} = \mathbf{WV} = (\mathbf{N} + \mathbf{W}_1)[\mathbf{N} + \widetilde{\gamma} - \frac{1}{2}\widetilde{\gamma}^2 + O(\eta^3)] \\
&\mathbf{W}_1 = -\widetilde{\gamma} - \mathbf{W}_1 \widetilde{\gamma} + \frac{1}{2}\widetilde{\gamma}^2 + O(\eta^3) = O(\varepsilon) = O(\eta) \\
&\mathbf{W}_1 \widetilde{\gamma} = -\widetilde{\gamma}^2 + O(\eta^3) \\
&\Rightarrow \quad \begin{cases} \mathbf{W} = \mathbf{N} - \widetilde{\gamma} + \dfrac{3}{2}\widetilde{\gamma}^2 + O(\eta^3) \\ \quad = \mathbf{N} - \mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} + \dfrac{1}{2}\mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})(\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}})\mathbf{N} + \dfrac{3}{2}\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} + O(\chi^4, \chi^2\eta, \chi\eta^2, \eta^3) \end{cases}
\end{aligned} \tag{2.125}$$

The tensor $\mathbf{F}'_S$ can now be written as follows

$$\begin{aligned}
\mathbf{F}'_S = \mathbf{W}\mathbf{Q}_S^T &= \mathbf{N} - \widetilde{\gamma} + \varOmega\mathbf{E} + \boldsymbol{\omega} \otimes \mathbf{n} + \frac{1}{2}(3\widetilde{\gamma}^2 - \varOmega^2\mathbf{N} - \boldsymbol{\omega} \otimes \boldsymbol{\omega}) \\
&\quad - \varOmega\widetilde{\gamma}\mathbf{E} - \widetilde{\gamma}\,\boldsymbol{\omega} \otimes \mathbf{n} + \frac{1}{2}\varOmega\mathbf{E}\,\boldsymbol{\omega} \otimes \mathbf{n} + O(\chi^3, \chi^2\eta, \chi\eta^2, \eta^3) \\
&= \mathbf{N} + \mathbf{N}(\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}})\mathbf{N} + \mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})\mathbf{n} \otimes \mathbf{n} + \mathbf{N}(\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}})\mathbf{N}(\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}})\mathbf{N} \\
&\quad + \mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})\mathbf{n} \otimes \mathbf{n}(\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}})\mathbf{N} - 2\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})\mathbf{n} \otimes \mathbf{n} + O(\chi^3, \chi^2\eta, \chi\eta^2, \eta^3)
\end{aligned} \tag{2.126}$$

2.4.16 Connexion between the undeformed and deformed configurations

The connections between the base vectors and between the derivative operators corresponding to the undeformed and deformed curved surface are

$$\begin{aligned}
&\mathbf{a}_\alpha = \mathbf{r}_{,\alpha} = \frac{\partial \mathbf{r}}{\partial \theta^\alpha} \quad ; \quad \mathbf{a}^\beta \cdot \mathbf{a}_\alpha = \delta^\beta{}_\alpha \qquad ; \qquad \mathbf{a}'_\alpha = \mathbf{r}'_{,\alpha} = \frac{\partial \mathbf{r}'}{\partial \theta^\alpha} \quad ; \quad \mathbf{a}'^\beta \cdot \mathbf{a}'_\alpha = \delta^\beta{}_\alpha \\
&\mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \theta^\alpha}\mathrm{d}\theta^\alpha = \mathbf{a}_\alpha \mathrm{d}\theta^\alpha \quad ; \quad \boldsymbol{\nabla}_n = \mathbf{a}^\beta \frac{\partial}{\partial \theta^\beta} \qquad \mathrm{d}\mathbf{r}' = \frac{\partial \mathbf{r}'}{\partial \theta^\alpha}\mathrm{d}\theta^\alpha = \mathbf{a}'_\alpha \mathrm{d}\theta^\alpha \quad ; \quad \boldsymbol{\nabla}'_n = \mathbf{a}'^\beta \frac{\partial}{\partial \theta^\beta} \\
&\mathrm{d}\mathbf{r} \cdot \boldsymbol{\nabla}_n = \mathrm{d}\theta^\alpha \mathbf{a}_\alpha \cdot \mathbf{a}^\beta \frac{\partial}{\partial \theta^\beta} = \mathrm{d}\theta^\alpha \frac{\partial}{\partial \theta^\alpha} = \mathrm{d}\theta^\alpha \mathbf{a}'_\alpha \cdot \mathbf{a}'^\beta \frac{\partial}{\partial \theta^\beta} = \mathrm{d}\mathbf{r}' \cdot \boldsymbol{\nabla}'_n \\
&\mathrm{d}\mathbf{r} \cdot \boldsymbol{\nabla}_n = \mathrm{d}\mathbf{r}' \cdot \mathbf{F}'^T_S \boldsymbol{\nabla}_n = \mathrm{d}\mathbf{r}' \cdot \boldsymbol{\nabla}'_n = \mathrm{d}\mathbf{r} \cdot \mathbf{F}^T_S \boldsymbol{\nabla}'_n \quad ; \qquad \forall\, \mathrm{d}\mathbf{r} \equiv \mathbf{N}\mathrm{d}\mathbf{r},\ \mathrm{d}\mathbf{r}' \equiv \mathbf{N}'\mathrm{d}\mathbf{r}' \\
&\Longrightarrow \quad \boldsymbol{\nabla}_n = \mathbf{F}^T_S \boldsymbol{\nabla}'_n \quad ; \qquad \boldsymbol{\nabla}'_n = \mathbf{F}'^T_S \boldsymbol{\nabla}_n
\end{aligned}$$

$$\begin{aligned}
&\mathrm{d}\mathbf{r}' = \mathbf{a}'_\alpha \mathrm{d}\theta^\alpha = \mathbf{F}_S \mathrm{d}\mathbf{r} = \mathbf{F}_S \mathbf{a}_\alpha \mathrm{d}\theta^\alpha \quad ; \qquad \forall\, \mathrm{d}\theta^\alpha \\
&\Longrightarrow \quad \mathbf{a}'_\alpha = \mathbf{F}_S \mathbf{a}_\alpha \\
&\boldsymbol{\nabla}'_n = \mathbf{a}'^\beta \frac{\partial}{\partial\theta^\beta} = \mathbf{F}'^T_S \boldsymbol{\nabla}_n = \mathbf{F}'^T_S \mathbf{a}^\beta \frac{\partial}{\partial\theta^\beta} \quad ; \qquad \forall\, \frac{\partial}{\partial\theta^\beta} \\
&\Longrightarrow \quad \mathbf{a}'^\beta = \mathbf{F}'^T_S \mathbf{a}^\beta \qquad ; \qquad \mathbf{n}' = \mathbf{Q}_S \mathbf{n}
\end{aligned} \tag{2.127}$$

2.4.17 Expression of $\boldsymbol{\nabla}'_n$ as function of $\boldsymbol{\nabla}_n$, $\widetilde{\gamma}$, $\boldsymbol{\omega}$ and Ω

$$\begin{aligned}
\boldsymbol{\nabla}'_n = \mathbf{F}'^T_S \boldsymbol{\nabla}_n = \boldsymbol{\nabla}_n - (\widetilde{\gamma} + \Omega\mathbf{E} - \mathbf{n}\otimes\boldsymbol{\omega})\boldsymbol{\nabla}_n + \frac{1}{2}(3\widetilde{\gamma}^2 - \Omega^2\mathbf{N} - \boldsymbol{\omega}\otimes\boldsymbol{\omega} \\
+ 2\Omega\mathbf{E}\widetilde{\gamma} - 2\mathbf{n}\otimes\widetilde{\gamma}\,\boldsymbol{\omega} + \mathbf{n}\otimes\Omega\mathbf{E}\,\boldsymbol{\omega})\boldsymbol{\nabla}_n + O(\chi^3, \chi^2\eta, \chi\eta^2, \eta^3)\boldsymbol{\nabla}_n
\end{aligned} \tag{2.128}$$

2.4.18 Expressions of the three invariants of an arbitrary 2-dimensional tensor $\mathbf{T}$

Let us now briefly recall the expressions of the three invariants I_1, I_2 and I_3 of a 2-dimensional tensor $\mathbf{T} = \mathbf{NTN}$

$$\begin{aligned}
&I_1 = \mathrm{tr}\,\mathbf{T} = 2H_T = \mathbf{T}\cdot\mathbf{N} && : \quad \text{Trace of } \mathbf{T} \\
&I_2 = \det(\mathbf{T} + \mathbf{n}\otimes\mathbf{n}) = K_T = -\frac{1}{2}\mathbf{T}\cdot\mathbf{E}\mathbf{T}^T\mathbf{E} && : \quad \text{(Minor-)determinant of } \mathbf{T} \\
&I_3 = \det\mathbf{T} = 0 && : \quad \text{Determinant of } \mathbf{T}
\end{aligned} \tag{2.129}$$

2.4.19 Determination of the curvature tensor $\mathbf{B}$ of the undeformed surface

The curvature κ and the radius of curvature R relative to some direction $\mathbf{e}$ in the tangential plane of the surface are defined as follows

$$\kappa = \frac{1}{R} = \frac{\mathbf{n}\cdot\mathrm{d}^2\mathbf{r}}{\mathrm{d}s^2} \qquad ; \qquad \mathrm{d}\mathbf{r} \equiv \mathbf{N}\mathrm{d}\mathbf{r} = \mathbf{e}\mathrm{d}s \qquad ; \qquad \mathbf{e}\cdot\mathbf{e} = 1 \tag{2.130}$$

We have

$$\begin{aligned}
&\mathbf{n}\cdot\mathrm{d}\mathbf{r} \equiv 0 \quad ; \qquad \mathrm{d}\mathbf{r} \perp \mathbf{n} \\
&\mathrm{d}(\mathbf{n}\cdot\mathrm{d}\mathbf{r}) = \mathrm{d}\mathbf{n}\cdot\mathrm{d}\mathbf{r} + \mathbf{n}\cdot\mathrm{d}^2\mathbf{r} = 0 \\
&\mathbf{n}\cdot\mathrm{d}^2\mathbf{r} = -\mathrm{d}\mathbf{n}\cdot\mathrm{d}\mathbf{r} = -\mathrm{d}\mathbf{r}\cdot(\boldsymbol{\nabla}_n\otimes\mathbf{n})\mathrm{d}\mathbf{r} = -\mathbf{e}\cdot(\boldsymbol{\nabla}_n\otimes\mathbf{n})\mathbf{e}\mathrm{d}s^2 \\
&\Rightarrow \quad \kappa = \frac{1}{R} = \mathbf{e}\cdot\mathbf{B}\mathbf{e} \qquad \text{with} \qquad \mathbf{B} = \mathbf{B}^T = \mathbf{NBN} = -\boldsymbol{\nabla}_n\otimes\mathbf{n}
\end{aligned} \tag{2.131}$$

$$\begin{aligned}
&\kappa_1 = \frac{1}{R_1} \quad ; \qquad \kappa_2 = \frac{1}{R_2} && : \quad \text{Eigenvalues of } \mathbf{B} \\
&H_B = \frac{1}{2}\,\mathrm{tr}\,\mathbf{B} = \frac{1}{2}\mathbf{B}\cdot\mathbf{N} = \frac{1}{2}(\kappa_1+\kappa_2) = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) && : \quad \text{Mean curvature} \\
&K_B = \det(\mathbf{B} + \mathbf{n}\otimes\mathbf{n}) = -\frac{1}{2}\mathbf{B}\cdot\mathbf{EBE} = \kappa_1\kappa_2 = \frac{1}{R_1 R_2} && : \quad \text{Gaussian curvature}
\end{aligned}$$

To take into account the order of magnitude of $\mathbf{B}$, we introduce a radius of comparison, i.e. a curvature of comparison, and write by definition

$$\begin{aligned} \kappa_n^2 &= \frac{1}{R_n^2} = \frac{1}{2}(\kappa_1^2+\kappa_2^2) = \frac{1}{2}\left(\frac{1}{R_1^2}+\frac{1}{R_2^2}\right) = \frac{1}{2}\mathbf{B}\cdot\mathbf{B} \\ &= \frac{1}{2}\,\mathrm{tr}^2\,\mathbf{B} - \det(\mathbf{B}+\mathbf{n}\otimes\mathbf{n}) = \frac{1}{2}[(\mathbf{B}\cdot\mathbf{N})^2 + \mathbf{B}\cdot\mathbf{EBE}] \end{aligned} \tag{2.132}$$

$$\mathbf{B} = O(\kappa_n) = O\left(\frac{1}{R_n}\right) \qquad \text{with} \qquad \begin{array}{lll} R_n & : & \text{Radius of comparison} \\ \kappa_n & : & \text{Curvature of comparison} \end{array}$$

2.4.20 Definition of the tensor of curvature change $\boldsymbol{\kappa}$

$$\boldsymbol{\kappa} = \mathbf{N}\boldsymbol{\kappa}\mathbf{N} = -(\boldsymbol{\nabla}_n \otimes \boldsymbol{\omega})\mathbf{N} \qquad \text{with} \qquad \boldsymbol{\kappa} \neq \boldsymbol{\kappa}^T \tag{2.133}$$

2.4.21 Definition of $\boldsymbol{\kappa}_S$

The tensor $\boldsymbol{\kappa}_S$ is built on the derivative of the rotation vector $\boldsymbol{\omega}_S$ as follows

$$\begin{aligned} \boldsymbol{\kappa}_S &= -\boldsymbol{\nabla}_n \otimes \boldsymbol{\omega}_S = -\boldsymbol{\nabla}_n \otimes (\Omega\mathbf{n} + \mathbf{E}\,\boldsymbol{\omega}) = O(\kappa_\omega) \\ &= -\boldsymbol{\nabla}_n\Omega \otimes \mathbf{n} - \Omega\boldsymbol{\nabla}_n \otimes \mathbf{n} - (\boldsymbol{\nabla}_n \otimes \mathbf{E})\,\boldsymbol{\omega} + (\boldsymbol{\nabla}_n \otimes \boldsymbol{\omega})\mathbf{E} \\ &= \Omega\mathbf{B} - \boldsymbol{\kappa}\mathbf{E} - (\mathbf{BE}\,\boldsymbol{\omega} + \boldsymbol{\nabla}_n\Omega) \otimes \mathbf{n} \end{aligned} \tag{2.134}$$

With the comparison parameter κ_ω, the orders of magnitude are

$$\begin{aligned} \boldsymbol{\kappa}_S &= O(\kappa_\omega) = O(\chi\kappa_n) \\ \boldsymbol{\kappa} &= O(\kappa_\omega) = O(\chi\kappa_n) \\ \boldsymbol{\nabla}_n\Omega &= O(\kappa_\omega) = O(\chi\kappa_n) \end{aligned} \tag{2.135}$$

2.4.22 Determination of the curvature tensor $\mathbf{B}'$ of the deformed surface

Before writing $\mathbf{B}'$ explicitly, let us first take a look at some derivatives

$$\begin{aligned} \boldsymbol{\nabla}_n \otimes \boldsymbol{\omega} &= \boldsymbol{\nabla}_n \otimes (\mathbf{N}\,\boldsymbol{\omega}) = (\boldsymbol{\nabla}_n \otimes \mathbf{N})\,\boldsymbol{\omega} + (\boldsymbol{\nabla}_n \otimes \boldsymbol{\omega})\mathbf{N} \\ &= [\mathbf{B}\otimes\mathbf{n} + \mathbf{B}\otimes\mathbf{n})^T]\,\boldsymbol{\omega} + (\boldsymbol{\nabla}_n \otimes \boldsymbol{\omega})\mathbf{N} = \mathbf{B}\,\boldsymbol{\omega}\otimes\mathbf{n} - \boldsymbol{\kappa} \end{aligned}$$

$$\begin{aligned} \boldsymbol{\nabla}_n \otimes (\Omega\mathbf{E}\,\boldsymbol{\omega}) &= \boldsymbol{\nabla}_n\Omega \otimes \mathbf{E}\,\boldsymbol{\omega} + \Omega(\boldsymbol{\nabla}_n \otimes \mathbf{E})\,\boldsymbol{\omega} - \Omega(\boldsymbol{\nabla}_n \otimes \boldsymbol{\omega})\mathbf{E} \\ &= \boldsymbol{\nabla}_n\Omega \otimes \mathbf{E}\,\boldsymbol{\omega} + \Omega\mathbf{BE}\,\boldsymbol{\omega}\otimes\mathbf{n} + \Omega\,\boldsymbol{\kappa}\mathbf{E} \end{aligned}$$

$$\begin{aligned} \boldsymbol{\nabla}_n \otimes [(\boldsymbol{\omega}\cdot\boldsymbol{\omega})\mathbf{n}] &= \boldsymbol{\nabla}_n(\boldsymbol{\omega}\cdot\boldsymbol{\omega}) \otimes \mathbf{n} + (\boldsymbol{\omega}\cdot\boldsymbol{\omega})\boldsymbol{\nabla}_n \otimes \mathbf{n} = 2(\boldsymbol{\nabla}_n \otimes \boldsymbol{\omega})\,\boldsymbol{\omega}\otimes\mathbf{n} - (\boldsymbol{\omega}\cdot\boldsymbol{\omega})\mathbf{B} \\ &= -2\,\boldsymbol{\kappa}\,\boldsymbol{\omega}\otimes\mathbf{n} - (\boldsymbol{\omega}\cdot\boldsymbol{\omega})\mathbf{B} \end{aligned}$$

With $\mathbf{n}' = n_3'\mathbf{g}'^3$ and $n_3'\Gamma'^3_{\alpha\beta} = \mathbf{n}'\cdot\mathbf{a}'_{\alpha,\beta}$, we have

$$\mathbf{B}' = \mathbf{B}'^T = \mathbf{N}'\mathbf{B}'\mathbf{N}' = -\mathbf{F}'^T_S\,\boldsymbol{\nabla}_n \otimes (\mathbf{Q}_S\mathbf{n}) = -\boldsymbol{\nabla}'_n \otimes \mathbf{n}' = n_3'\Gamma'^3_{\alpha\beta}\,\mathbf{a}'^\alpha \otimes \mathbf{a}'^\beta$$

$$= \left[-\nabla_n + (\widetilde{\gamma} + \mathit{\Omega}\mathbf{E} - \mathbf{n}\otimes\boldsymbol{\omega} - \frac{3}{2}\widetilde{\gamma}^2 + \frac{1}{2}\mathit{\Omega}^2\mathbf{N} + \frac{1}{2}\boldsymbol{\omega}\otimes\boldsymbol{\omega} - \mathit{\Omega}\mathbf{E}\widetilde{\gamma} + \mathbf{n}\otimes\widetilde{\gamma}\boldsymbol{\omega} - \frac{1}{2}\mathbf{n}\otimes\mathit{\Omega}\mathbf{E}\boldsymbol{\omega})\nabla_n\right] \otimes \left[\mathbf{n} - \boldsymbol{\omega} + \frac{1}{2}\mathit{\Omega}\mathbf{E}\boldsymbol{\omega} - \frac{1}{2}(\boldsymbol{\omega}\cdot\boldsymbol{\omega})\mathbf{n}\right]$$

$$= \left[-\mathbf{N} + \widetilde{\gamma} + \mathit{\Omega}\mathbf{E} - \mathbf{n}\otimes\boldsymbol{\omega} - \frac{3}{2}\widetilde{\gamma}^2 + \frac{1}{2}\mathit{\Omega}^2\mathbf{N} + \frac{1}{2}\boldsymbol{\omega}\otimes\boldsymbol{\omega} - \mathit{\Omega}\mathbf{E}\widetilde{\gamma} + \mathbf{n}\otimes\widetilde{\gamma}\boldsymbol{\omega} - \frac{1}{2}\mathbf{n}\otimes\mathit{\Omega}\mathbf{E}\boldsymbol{\omega}\right] \left[-\mathbf{B} + \boldsymbol{\kappa} + \frac{1}{2}\nabla_n\mathit{\Omega}\otimes\mathbf{E}\boldsymbol{\omega} + \frac{1}{2}\mathit{\Omega}\boldsymbol{\kappa}\mathbf{E} + \frac{1}{2}(\boldsymbol{\omega}\cdot\boldsymbol{\omega})\mathbf{B} + \left(-\mathbf{B} + \boldsymbol{\kappa} + \frac{1}{2}\mathit{\Omega}\mathbf{B}\mathbf{E}\right)\boldsymbol{\omega}\otimes\mathbf{n}\right]$$

$$= \mathbf{B} - \boldsymbol{\kappa} - \widetilde{\gamma}\mathbf{B} - \mathit{\Omega}\mathbf{E}\mathbf{B} + \mathbf{n}\otimes\mathbf{B}\boldsymbol{\omega} + \mathbf{B}\boldsymbol{\omega}\otimes\mathbf{n} + \frac{3}{2}\widetilde{\gamma}^2\mathbf{B} - \frac{1}{2}\mathit{\Omega}^2\mathbf{B} - \frac{1}{2}(\boldsymbol{\omega}\cdot\boldsymbol{\omega})\mathbf{B}$$
$$- \frac{1}{2}\boldsymbol{\omega}\otimes\mathbf{B}\boldsymbol{\omega} + \mathit{\Omega}\mathbf{E}\widetilde{\gamma}\mathbf{B} + \widetilde{\gamma}\boldsymbol{\kappa} + \mathit{\Omega}\mathbf{E}\boldsymbol{\kappa} - \frac{1}{2}\mathit{\Omega}\boldsymbol{\kappa}\mathbf{E} - \frac{1}{2}\nabla_n\mathit{\Omega}\otimes\mathbf{E}\boldsymbol{\omega} - \mathbf{n}\otimes\mathbf{B}\widetilde{\gamma}\boldsymbol{\omega}$$
$$- \widetilde{\gamma}\mathbf{B}\boldsymbol{\omega}\otimes\mathbf{n} + \frac{1}{2}\mathbf{n}\otimes\mathit{\Omega}\mathbf{B}\mathbf{E}\boldsymbol{\omega} - \frac{1}{2}\mathit{\Omega}\mathbf{B}\mathbf{E}\boldsymbol{\omega}\otimes\mathbf{n} - \mathit{\Omega}\mathbf{E}\mathbf{B}\boldsymbol{\omega}\otimes\mathbf{n} - \mathbf{n}\otimes\boldsymbol{\omega}\boldsymbol{\kappa} - \boldsymbol{\kappa}\boldsymbol{\omega}\otimes\mathbf{n}$$
$$+ (\boldsymbol{\omega}\cdot\mathbf{B}\boldsymbol{\omega})\mathbf{n}\otimes\mathbf{n} + O(\chi^3, \chi^2\eta, \chi\eta^2)\kappa_n$$

$$= \mathbf{B} - \frac{1}{2}[\boldsymbol{\kappa} + \boldsymbol{\kappa}^T + \mathbf{B}\widetilde{\gamma} + \widetilde{\gamma}\mathbf{B} - \mathit{\Omega}(\mathbf{B}\mathbf{E} - \mathbf{E}\mathbf{B})] + \mathbf{n}\otimes\mathbf{B}\boldsymbol{\omega} + \mathbf{B}\boldsymbol{\omega}\otimes\mathbf{n}$$
$$+ \frac{3}{4}(\mathbf{B}\widetilde{\gamma}^2 + \widetilde{\gamma}^2\mathbf{B}) - \frac{1}{2}[\mathit{\Omega}^2 + (\boldsymbol{\omega}\cdot\boldsymbol{\omega})]\mathbf{B} - \frac{1}{4}(\boldsymbol{\omega}\otimes\mathbf{B}\boldsymbol{\omega} + \mathbf{B}\boldsymbol{\omega}\otimes\boldsymbol{\omega})$$
$$- \frac{1}{2}\mathit{\Omega}(\mathbf{B}\widetilde{\gamma}\mathbf{E} - \mathbf{E}\widetilde{\gamma}\mathbf{B}) + \frac{1}{2}(\widetilde{\gamma}\boldsymbol{\kappa} + \boldsymbol{\kappa}^T\widetilde{\gamma}) + \frac{1}{2}\mathit{\Omega}(\mathbf{E}\boldsymbol{\kappa} - \boldsymbol{\kappa}^T\mathbf{E}) + \frac{1}{4}\mathit{\Omega}(\mathbf{E}\boldsymbol{\kappa}^T - \boldsymbol{\kappa}\mathbf{E})$$
$$- \frac{1}{4}(\nabla_n\mathit{\Omega}\otimes\mathbf{E}\boldsymbol{\omega} + \mathbf{E}\boldsymbol{\omega}\otimes\nabla_n\mathit{\Omega}) - \frac{1}{2}[\mathbf{n}\otimes(\mathbf{B}\widetilde{\gamma}\boldsymbol{\omega} + \widetilde{\gamma}\mathbf{B}\boldsymbol{\omega}) + (\mathbf{B}\widetilde{\gamma}\boldsymbol{\omega} + \widetilde{\gamma}\mathbf{B}\boldsymbol{\omega})\otimes\mathbf{n}]$$
$$- \frac{1}{2}(\mathbf{n}\otimes\mathit{\Omega}\mathbf{E}\mathbf{B}\boldsymbol{\omega} + \mathit{\Omega}\mathbf{E}\mathbf{B}\boldsymbol{\omega}\otimes\mathbf{n}) - \frac{1}{2}[\mathbf{n}\otimes\boldsymbol{\omega}(\boldsymbol{\kappa} + \boldsymbol{\kappa}^T) + (\boldsymbol{\kappa} + \boldsymbol{\kappa}^T)\boldsymbol{\omega}\otimes\mathbf{n}]$$
$$+ (\boldsymbol{\omega}\cdot\mathbf{B}\boldsymbol{\omega})\mathbf{n}\otimes\mathbf{n} + O(\chi^3, \chi^2\eta, \chi\eta^2)\kappa_n \tag{2.136}$$

2.4.23 Determination of the normal curvature change

The exact relation for the difference of the normal curvatures between both deformed and undeformed curved surfaces, i.e for the normal curvature change reads

$$\kappa' - \kappa = \frac{1}{R'} - \frac{1}{R} = \mathbf{e}'\cdot\mathbf{B}'\mathbf{e}' - \mathbf{e}\cdot\mathbf{B}\mathbf{e} = \frac{\mathbf{e}\cdot\mathbf{F}_S^T\mathbf{B}'\mathbf{F}_S\mathbf{e}}{1 + 2\mathbf{e}\cdot\widetilde{\gamma}\mathbf{e}} - \mathbf{e}\cdot\mathbf{B}\mathbf{e} \tag{2.137}$$

2.4.24 Summary

By neglecting the third-order terms in χ^3, $\chi^2\eta$, $\chi\eta^2$ and η^3, we have

$$\mathbf{F}_S = \mathbf{N} + (\nabla_n\otimes\mathbf{u})^T = \mathbf{Q}_S\mathbf{V} = \mathbf{F}\mathbf{N} = \mathbf{Q}\mathbf{U}\mathbf{N} = \mathbf{N} - (\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}})\mathbf{N}$$
$$\cong \mathbf{N} + \widetilde{\gamma} - \mathit{\Omega}\mathbf{E} + \mathbf{n}\otimes\boldsymbol{\omega} - \frac{1}{2}(\widetilde{\gamma}^2 + \mathit{\Omega}^2\mathbf{N} + \boldsymbol{\omega}\otimes\boldsymbol{\omega}) \tag{2.138}$$
$$- \mathit{\Omega}\mathbf{E}\widetilde{\gamma} + \mathbf{n}\otimes\widetilde{\gamma}\boldsymbol{\omega} + \frac{1}{2}\mathbf{n}\otimes\mathit{\Omega}\mathbf{E}\boldsymbol{\omega}$$

$$\begin{aligned}\boldsymbol{\nabla}_n \otimes \mathbf{u} &= \mathbf{F}_S^T - \mathbf{N} = \mathbf{V}\mathbf{Q}_S^T - \mathbf{N} = \mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}}) \\ &\cong \widetilde{\gamma} + \Omega\mathbf{E} + \boldsymbol{\omega}\otimes\mathbf{n} - \frac{1}{2}(\widetilde{\gamma}^2 + \Omega^2\mathbf{N} + \boldsymbol{\omega}\otimes\boldsymbol{\omega}) \\ &\quad + \Omega\widetilde{\gamma}\mathbf{E} + \widetilde{\gamma}\,\boldsymbol{\omega}\otimes\mathbf{n} + \frac{1}{2}\Omega\mathbf{E}\,\boldsymbol{\omega}\otimes\mathbf{n}\end{aligned} \tag{2.139}$$

$$\begin{aligned}\mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})\mathbf{n} &= (\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{n} = \mathbf{n}(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T = -\mathbf{n}(\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}})\mathbf{N} \\ &\cong \boldsymbol{\omega} + \widetilde{\gamma}\,\boldsymbol{\omega} + \frac{1}{2}\Omega\mathbf{E}\,\boldsymbol{\omega}\end{aligned} \tag{2.140}$$

$$\begin{aligned}\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} &= \frac{1}{2}[(\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{N} + \mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T] \\ &\cong \widetilde{\gamma} - \frac{1}{2}(\widetilde{\gamma}^2 + \Omega^2\mathbf{N} + \boldsymbol{\omega}\otimes\boldsymbol{\omega}) + \frac{1}{2}\Omega(\widetilde{\gamma}\mathbf{E} - \mathbf{E}\widetilde{\gamma})\end{aligned} \tag{2.141}$$

$$\begin{aligned}\mathbf{N}\boldsymbol{\Omega}\mathbf{N} &= \frac{1}{2}[(\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{N} - \mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T] \\ &\cong \Omega\mathbf{E} + \frac{1}{2}\Omega(\widetilde{\gamma}\mathbf{E} + \mathbf{E}\widetilde{\gamma})\end{aligned} \tag{2.142}$$

$$\begin{aligned}\widetilde{\gamma} &= \frac{1}{2}(\mathbf{F}_S^T\mathbf{F}_S - \mathbf{N}) = \frac{1}{2}(\mathbf{V}^2 - \mathbf{N}) = \mathbf{N}\widetilde{\boldsymbol{\mathcal{E}}}\mathbf{N} = \mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} - \frac{1}{2}\mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})(\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}})\mathbf{N} \\ &= \frac{1}{2}[(\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{N} + \mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T] + \frac{1}{2}(\boldsymbol{\nabla}_n \otimes \mathbf{u})(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T\end{aligned} \tag{2.143}$$

$$\begin{aligned}\boldsymbol{\omega} &\cong \mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})\mathbf{n} - \mathbf{N}\left(\frac{1}{2}\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}}\right)\mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})\mathbf{n} \\ &\cong (\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{n} - \left[\frac{3}{4}(\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{N} + \frac{1}{4}\mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T\right](\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{n}\end{aligned} \tag{2.144}$$

$$\begin{aligned}\Omega\mathbf{E} &\cong \mathbf{N}\boldsymbol{\Omega}\mathbf{N} - \frac{1}{2}(\mathbf{N}\boldsymbol{\Omega}\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} + \mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N}\boldsymbol{\Omega}\mathbf{N}) \\ &\cong \frac{1}{2}[(\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{N} - \mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T] - \frac{1}{4}\big((\boldsymbol{\nabla}_n \otimes \mathbf{u})^2\mathbf{N} - \mathbf{N}[(\boldsymbol{\nabla}_n \otimes \mathbf{u})^2]^T\big)\end{aligned} \tag{2.145}$$

$$\begin{aligned}\Omega &\cong -\frac{1}{2}\mathbf{N}\boldsymbol{\Omega}\mathbf{N}\cdot\mathbf{E} + \frac{1}{4}(\mathbf{N}\boldsymbol{\Omega}\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} + \mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N}\boldsymbol{\Omega}\mathbf{N})\cdot\mathbf{E} \\ &\cong -\frac{1}{4}[(\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{N} - \mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T]\cdot\mathbf{E} + \frac{1}{8}\big((\boldsymbol{\nabla}_n \otimes \mathbf{u})^2\mathbf{N} - \mathbf{N}[(\boldsymbol{\nabla}_n \otimes \mathbf{u})^2]^T\big)\cdot\mathbf{E}\end{aligned} \tag{2.146}$$

$$\Omega\mathbf{E}\,\boldsymbol{\omega} \cong \mathbf{N}\boldsymbol{\Omega}\mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})\mathbf{n} = \frac{1}{2}[(\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{N} - \mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T](\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{n} \tag{2.147}$$

$$\boldsymbol{\omega}\cdot\boldsymbol{\omega} = \boldsymbol{\omega}^2 \cong [\mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})\mathbf{n}]^2 = [(\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{n}]^2 \tag{2.148}$$

$$
\begin{aligned}
\mathbf{V} &\cong \mathbf{N} + \widetilde{\gamma} - \frac{1}{2}\widetilde{\gamma}^2 \\
&\cong \mathbf{N} + \mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} - \frac{1}{2}\mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})(\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}})\mathbf{N} - \frac{1}{2}\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} \\
&\cong \mathbf{N} + \frac{1}{2}[(\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{N} + \mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T] + \frac{1}{2}(\boldsymbol{\nabla}_n \otimes \mathbf{u})(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T \\
&\quad - \frac{1}{8}[(\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{N} + \mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T]^2
\end{aligned}
\tag{2.149}
$$

$$
\begin{aligned}
\mathbf{W} &\cong \mathbf{N} - \widetilde{\gamma} + \frac{3}{2}\widetilde{\gamma}^2 \\
&\cong \mathbf{N} - \mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} + \frac{1}{2}\mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})(\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}})\mathbf{N} + \frac{3}{2}\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} \\
&\cong \mathbf{N} - \frac{1}{2}[(\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{N} + \mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T] - \frac{1}{2}(\boldsymbol{\nabla}_n \otimes \mathbf{u})(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T \\
&\quad + \frac{3}{8}[(\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{N} + \mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T]^2
\end{aligned}
\tag{2.150}
$$

$$
\begin{aligned}
\mathbf{Q}_S &\cong \mathbf{I} - \mathit{\Omega}\mathbf{E} - \boldsymbol{\omega} \otimes \mathbf{n} + \mathbf{n} \otimes \boldsymbol{\omega} + \frac{1}{2}\mathit{\Omega}\mathbf{E}\boldsymbol{\omega} \otimes \mathbf{n} + \frac{1}{2}\mathbf{n} \otimes \mathit{\Omega}\mathbf{E}\boldsymbol{\omega} \\
&\qquad - \frac{1}{2}[\mathit{\Omega}^2\mathbf{N} + \boldsymbol{\omega} \otimes \boldsymbol{\omega} + (\boldsymbol{\omega} \cdot \boldsymbol{\omega})\mathbf{n} \otimes \mathbf{n}] \\
&\cong \mathbf{I} - \mathbf{N}\boldsymbol{\Omega}\mathbf{N} - \mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})\mathbf{n} \otimes \mathbf{n} - \mathbf{n} \otimes \mathbf{n}(\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}})\mathbf{N} \\
&\qquad + \frac{1}{2}(\mathbf{N}\boldsymbol{\Omega}\mathbf{N}\boldsymbol{\Omega}\mathbf{N} + \mathbf{N}\boldsymbol{\Omega}\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} + \mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N}\boldsymbol{\Omega}\mathbf{N}) + \mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})\mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})\mathbf{n} \otimes \mathbf{n} \\
&\qquad + \mathbf{n} \otimes \mathbf{n}(\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}})\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N} + \frac{1}{2}\mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})\mathbf{n} \otimes \mathbf{n}(\boldsymbol{\Omega} - \boldsymbol{\mathcal{E}})\mathbf{N} - \frac{1}{2}[\mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})\mathbf{n}]^2\, \mathbf{n} \otimes \mathbf{n} \\
&\cong \mathbf{I} - \frac{1}{2}[(\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{N} - \mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T] - (\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{n} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{n}(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T \\
&\qquad + \frac{1}{2}(\boldsymbol{\nabla}_n \otimes \mathbf{u})^2\mathbf{N} - \frac{1}{8}[(\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{N} + \mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T]^2 + (\boldsymbol{\nabla}_n \otimes \mathbf{u})^2\mathbf{n} \otimes \mathbf{n} \\
&\qquad - \frac{1}{2}\mathbf{n} \otimes \mathbf{n}(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T[(\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{N} + \mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T] - \frac{1}{2}(\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{n} \otimes \mathbf{n}(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T \\
&\qquad - \frac{1}{2}[(\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{n}]^2\, \mathbf{n} \otimes \mathbf{n}
\end{aligned}
\tag{2.151}
$$

$$
\begin{aligned}
\mathbf{n}' = \mathbf{Q}_S\mathbf{n} &\cong \mathbf{n} - \boldsymbol{\omega} + \frac{1}{2}\mathit{\Omega}\mathbf{E}\boldsymbol{\omega} - \frac{1}{2}(\boldsymbol{\omega} \cdot \boldsymbol{\omega})\mathbf{n} \\
&\cong \mathbf{n} - \mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})\mathbf{n} + \mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})\mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})\mathbf{n} - \frac{1}{2}[\mathbf{N}(\boldsymbol{\Omega} + \boldsymbol{\mathcal{E}})\mathbf{n}]^2\, \mathbf{n} \\
&\cong \mathbf{n} - (\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{n} + (\boldsymbol{\nabla}_n \otimes \mathbf{u})^2\mathbf{n} - \frac{1}{2}[(\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{n}]^2\, \mathbf{n}
\end{aligned}
\tag{2.152}
$$

$$
\begin{aligned}
\mathbf{N}' &= \mathbf{Q}_S\mathbf{N}\mathbf{Q}_S^T = \mathbf{I} - \mathbf{n}'\otimes\mathbf{n}' \\
&\cong \mathbf{N} + \mathbf{n}\otimes\boldsymbol{\omega} + \boldsymbol{\omega}\otimes\mathbf{n} - \frac{1}{2}\Omega\mathbf{E}\boldsymbol{\omega}\otimes\mathbf{n} - \frac{1}{2}\mathbf{n}\otimes\Omega\mathbf{E}\boldsymbol{\omega} - \boldsymbol{\omega}\otimes\boldsymbol{\omega} + (\boldsymbol{\omega}\cdot\boldsymbol{\omega})\mathbf{n}\otimes\mathbf{n} \\
&\cong \mathbf{N} + \mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n}\otimes\mathbf{n} - \mathbf{n}\otimes\mathbf{n}(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N} - \mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n}\otimes\mathbf{n} \\
&\qquad - \mathbf{n}\otimes\mathbf{n}(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N}(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N} + \mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n}\otimes\mathbf{n}(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N} + [\mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n}]^2\mathbf{n}\otimes\mathbf{n} \\
&\cong \mathbf{N} + (\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{n}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{n}(\boldsymbol{\nabla}_n\otimes\mathbf{u})^T - (\boldsymbol{\nabla}_n\otimes\mathbf{u})^2\mathbf{n}\otimes\mathbf{n} - \mathbf{n}\otimes\mathbf{n}[(\boldsymbol{\nabla}_n\otimes\mathbf{u})^2]^T \\
&\qquad - (\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{n}\otimes\mathbf{n}(\boldsymbol{\nabla}_n\otimes\mathbf{u})^T + [(\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{n}]^2\mathbf{n}\otimes\mathbf{n}
\end{aligned}
\tag{2.153}
$$

$$
\begin{aligned}
\mathbf{F}_S' &= \mathbf{N}' - (\boldsymbol{\nabla}_n'\otimes\mathbf{u})^T = \mathbf{W}\mathbf{Q}_S^T = \mathbf{F}^{-1}\mathbf{N}' = \mathbf{U}^{-1}\mathbf{Q}^T\mathbf{N}' \\
&\cong \mathbf{N} - \widetilde{\gamma} + \Omega\mathbf{E} + \boldsymbol{\omega}\otimes\mathbf{n} + \frac{1}{2}(3\widetilde{\gamma}^2 - \Omega^2\mathbf{N} - \boldsymbol{\omega}\otimes\boldsymbol{\omega}) - \Omega\widetilde{\gamma}\mathbf{E} \\
&\qquad - \widetilde{\gamma}\boldsymbol{\omega}\otimes\mathbf{n} + \frac{1}{2}\Omega\mathbf{E}\boldsymbol{\omega}\otimes\mathbf{n} \\
&\cong \mathbf{N} + \mathbf{N}(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N} + \mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n}\otimes\mathbf{n} + \mathbf{N}(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N}(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N} \\
&\qquad + \mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n}\otimes\mathbf{n}(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N} - 2\mathbf{N}\boldsymbol{\mathcal{E}}\mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n}\otimes\mathbf{n} \\
&\cong \mathbf{N} - \mathbf{N}(\boldsymbol{\nabla}_n\otimes\mathbf{u})^T + (\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{n}\otimes\mathbf{n} + \mathbf{N}[(\boldsymbol{\nabla}_n\otimes\mathbf{u})^2]^T \\
&\qquad - (\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{n}\otimes\mathbf{n}(\boldsymbol{\nabla}_n\otimes\mathbf{u})^T - [(\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{N} + \mathbf{N}(\boldsymbol{\nabla}_n\otimes\mathbf{u})^T](\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{n}\otimes\mathbf{n}
\end{aligned}
\tag{2.154}
$$

$$
\begin{aligned}
\boldsymbol{\nabla}_n' &= \mathbf{F}_S'^T\boldsymbol{\nabla}_n \\
&\cong \boldsymbol{\nabla}_n - (\widetilde{\gamma} + \Omega\mathbf{E} - \mathbf{n}\otimes\boldsymbol{\omega})\boldsymbol{\nabla}_n \\
&\qquad + \frac{1}{2}(3\widetilde{\gamma}^2 - \Omega^2\mathbf{N} - \boldsymbol{\omega}\otimes\boldsymbol{\omega} + 2\Omega\mathbf{E}\widetilde{\gamma} - 2\mathbf{n}\otimes\widetilde{\gamma}\boldsymbol{\omega} + \mathbf{n}\otimes\Omega\mathbf{E}\boldsymbol{\omega})\boldsymbol{\nabla}_n \\
&\cong \boldsymbol{\nabla}_n - \mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\boldsymbol{\nabla}_n - \mathbf{n}[\mathbf{n}\cdot(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})\boldsymbol{\nabla}_n] + \mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\boldsymbol{\nabla}_n \\
&\qquad + \mathbf{N}(\boldsymbol{\Omega}+\boldsymbol{\mathcal{E}})\mathbf{n}[\mathbf{n}\cdot(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})\boldsymbol{\nabla}_n] + 2\mathbf{n}[\mathbf{n}\cdot(\boldsymbol{\Omega}-\boldsymbol{\mathcal{E}})\mathbf{N}\boldsymbol{\mathcal{E}}\boldsymbol{\nabla}_n] \\
&\cong \boldsymbol{\nabla}_n - (\boldsymbol{\nabla}_n\otimes\mathbf{u})\boldsymbol{\nabla}_n + \mathbf{n}[\mathbf{n}\cdot(\boldsymbol{\nabla}_n\otimes\mathbf{u})^T\boldsymbol{\nabla}_n] \\
&\qquad + (\boldsymbol{\nabla}_n\otimes\mathbf{u})^2\boldsymbol{\nabla}_n - (\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{n}[\mathbf{n}\cdot(\boldsymbol{\nabla}_n\otimes\mathbf{u})^T\boldsymbol{\nabla}_n] \\
&\qquad - \mathbf{n}\Big(\mathbf{n}\cdot(\boldsymbol{\nabla}_n\otimes\mathbf{u})^T[(\boldsymbol{\nabla}_n\otimes\mathbf{u})\mathbf{N} + \mathbf{N}(\boldsymbol{\nabla}_n\otimes\mathbf{u})^T]\boldsymbol{\nabla}_n\Big)
\end{aligned}
\tag{2.155}
$$

$$
\begin{aligned}
\mathbf{B}' = \mathbf{B}'^T = \mathbf{N}'\mathbf{B}'\mathbf{N}' &= -\mathbf{F}_S'^T \boldsymbol{\nabla}_n \otimes (\mathbf{Q}_S \mathbf{n}) = -\boldsymbol{\nabla}_n' \otimes \mathbf{n}' = n_3' \Gamma_{\alpha\beta}'^3 \, \mathbf{a}'^\alpha \otimes \mathbf{a}'^\beta \\
&\cong \mathbf{B} - \frac{1}{2}[\boldsymbol{\kappa} + \boldsymbol{\kappa}^T + \mathbf{B}\widetilde{\gamma} + \widetilde{\gamma}\mathbf{B} - \Omega(\mathbf{BE} - \mathbf{EB})] + \mathbf{n} \otimes \mathbf{B}\boldsymbol{\omega} + \mathbf{B}\boldsymbol{\omega} \otimes \mathbf{n} \\
&\quad + \frac{3}{4}(\mathbf{B}\widetilde{\gamma}^2 + \widetilde{\gamma}^2\mathbf{B}) - \frac{1}{2}[\Omega^2 + (\boldsymbol{\omega} \cdot \boldsymbol{\omega})]\mathbf{B} - \frac{1}{4}(\boldsymbol{\omega} \otimes \mathbf{B}\boldsymbol{\omega} + \mathbf{B}\boldsymbol{\omega} \otimes \boldsymbol{\omega}) \\
&\quad - \frac{1}{2}\Omega(\mathbf{B}\widetilde{\gamma}\mathbf{E} - \mathbf{E}\widetilde{\gamma}\mathbf{B}) + \frac{1}{2}(\widetilde{\gamma}\boldsymbol{\kappa} + \boldsymbol{\kappa}^T\widetilde{\gamma}) + \frac{1}{2}\Omega(\mathbf{E}\boldsymbol{\kappa} - \boldsymbol{\kappa}^T\mathbf{E}) \\
&\quad + \frac{1}{4}\Omega(\mathbf{E}\boldsymbol{\kappa}^T - \boldsymbol{\kappa}\mathbf{E}) - \frac{1}{4}(\boldsymbol{\nabla}_n\Omega \otimes \mathbf{E}\boldsymbol{\omega} + \mathbf{E}\boldsymbol{\omega} \otimes \boldsymbol{\nabla}_n\Omega) \\
&\quad - \frac{1}{2}[\mathbf{n} \otimes (\mathbf{B}\widetilde{\gamma}\boldsymbol{\omega} + \widetilde{\gamma}\mathbf{B}\boldsymbol{\omega}) + (\mathbf{B}\widetilde{\gamma}\boldsymbol{\omega} + \widetilde{\gamma}\mathbf{B}\boldsymbol{\omega}) \otimes \mathbf{n}] \\
&\quad - \frac{1}{2}(\mathbf{n} \otimes \Omega\mathbf{EB}\boldsymbol{\omega} + \Omega\mathbf{EB}\boldsymbol{\omega} \otimes \mathbf{n}) \\
&\quad - \frac{1}{2}[\mathbf{n} \otimes \boldsymbol{\omega}(\boldsymbol{\kappa} + \boldsymbol{\kappa}^T) + (\boldsymbol{\kappa} + \boldsymbol{\kappa}^T)\boldsymbol{\omega} \otimes \mathbf{n}] + (\boldsymbol{\omega} \cdot \mathbf{B}\boldsymbol{\omega})\mathbf{n} \otimes \mathbf{n}
\end{aligned}
\tag{2.156}
$$

2.4.25 Strain components of a 2-dimensional curved surface in the 3-dimensional space written in a Cartesian system up to the second-order terms

Let us recall the general expression for the in-plane strain tensor of a 2-dimensional curved surface

$$
\widetilde{\gamma} = \frac{1}{2}(\mathbf{F}_S^T\mathbf{F}_S - \mathbf{N}) = \frac{1}{2}[(\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{N} + \mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T] + \frac{1}{2}(\boldsymbol{\nabla}_n \otimes \mathbf{u})(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T \tag{2.157}
$$

where $\mathbf{F}_S = \mathbf{N} + (\boldsymbol{\nabla}_n \otimes \mathbf{u})^T$ is the deformation gradient of the surface, $\mathbf{F}_S^T\mathbf{F}_S$ the related *Cauchy-Green* tensor, $\boldsymbol{\nabla}_n = \mathbf{a}^\beta \partial/\partial\theta^\beta$ the 2-dimensional derivative operator and $\mathbf{N} = \mathbf{a}_\alpha \otimes \mathbf{a}^\alpha = \mathbf{I} - \mathbf{n} \otimes \mathbf{n}$ the metric tensor of the surface (with $\alpha, \beta = 1, 2$). The 3-dimensional displacement $\mathbf{u}$, which generally points out of the surface, can be decomposed into interior and exterior parts as follows

$$
\mathbf{u} = \mathbf{Iu} = (\mathbf{N} + \mathbf{n} \otimes \mathbf{n})\mathbf{u} = \mathbf{Nu} + (\mathbf{n} \cdot \mathbf{u})\mathbf{n} = \mathbf{v} + w\mathbf{n} \tag{2.158}
$$

where $\mathbf{v} = \mathbf{Nu}$ and $w = \mathbf{n} \cdot \mathbf{u}$. The decomposition of the derivative $\boldsymbol{\nabla}_n \otimes \mathbf{u}$ and its transpose read

$$
\begin{aligned}
\boldsymbol{\nabla}_n \otimes \mathbf{u} &= (\boldsymbol{\nabla}_n \otimes \mathbf{v})\mathbf{N} - w\mathbf{B} + (\mathbf{Bv} + \boldsymbol{\nabla}_n w) \otimes \mathbf{n} \\
(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T &= \mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{v})^T - w\mathbf{B} + \mathbf{n} \otimes (\mathbf{Bv} + \boldsymbol{\nabla}_n w)
\end{aligned}
\tag{2.159}
$$

where

$$
\boldsymbol{\nabla}_n \otimes \mathbf{v} = (\boldsymbol{\nabla}_n \otimes \mathbf{v})\mathbf{N} + \mathbf{Bv} \otimes \mathbf{n} \tag{2.160}
$$

which gives

$$
\begin{aligned}
(\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{N} &= (\boldsymbol{\nabla}_n \otimes \mathbf{v})\mathbf{N} - w\mathbf{B} \\
\mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T &= \mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{v})^T - w\mathbf{B}
\end{aligned}
\tag{2.161}
$$

and

$$
\begin{aligned}
(\boldsymbol{\nabla}_n \otimes \mathbf{u})(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T = {} & (\boldsymbol{\nabla}_n \otimes \mathbf{v})\mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{v})^T - w(\boldsymbol{\nabla}_n \otimes \mathbf{v})\mathbf{B} - w\mathbf{B}(\boldsymbol{\nabla}_n \otimes \mathbf{v})^T \\
& + w^2\mathbf{B}^2 + (\mathbf{Bv} + \boldsymbol{\nabla}_n w) \otimes (\mathbf{Bv} + \boldsymbol{\nabla}_n w)
\end{aligned}
\tag{2.162}
$$

Thus, we get explicitly

$$\widetilde{\gamma} = \frac{1}{2}[(\boldsymbol{\nabla}_n \otimes \mathbf{v})\mathbf{N} + \mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{v})^T] - w\mathbf{B} + \frac{1}{2}w^2\mathbf{B}^2 + \frac{1}{2}(\boldsymbol{\nabla}_n \otimes \mathbf{v})\mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{v})^T \\ - \frac{1}{2}w[(\boldsymbol{\nabla}_n \otimes \mathbf{v})\mathbf{B} + \mathbf{B}(\boldsymbol{\nabla}_n \otimes \mathbf{v})^T] + \frac{1}{2}(\mathbf{B}\mathbf{v} + \boldsymbol{\nabla}_n w) \otimes (\mathbf{B}\mathbf{v} + \boldsymbol{\nabla}_n w) \tag{2.163}$$

In order to write the tensor $\widetilde{\gamma}$ in a matrix representation, we introduce in a point P on the curved surface a Cartesian base with <u>constant</u> orthogonal unit base vectors $\mathbf{e}, \mathbf{e}_\perp$ and $\mathbf{e}_3$ in space. In point P, the direction $\mathbf{e}_3$ is perpendicular to the surface and gives the direction of the z-axis. The unit base vectors $\mathbf{e}$ and $\mathbf{e}_\perp$ are tangential to the surface and perpendicular to each other ($\mathbf{e} \perp \mathbf{e}_\perp$), respectively giving the directions of the x and y-axis. Note that in another point $\bar{\mathrm{P}}$, $\mathbf{e}_3$ may not be normal and $\mathbf{e}$ and $\mathbf{e}_\perp$ not tangential to the surface, which means that $\mathbf{e}_3 = \mathbf{n}$ is only valid in point P. Recalling that the sign $\hat{}$ means that the base vectors are omitted in the matrix representation, we write

$$\mathbf{e} \mathrel{\hat{=}} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \quad ; \quad \mathbf{e}_\perp \mathrel{\hat{=}} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \quad ; \quad \mathbf{e}_3 = \mathbf{e} \times \mathbf{e}_\perp = \mathbf{e}_\perp \boldsymbol{\mathcal{E}} \mathbf{e} \mathrel{\hat{=}} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \tag{2.164}$$

The identity tensor reads

$$\mathbf{I} = \mathbf{g}^i \otimes \mathbf{g}_i = \mathbf{g}_i \otimes \mathbf{g}^i = \mathbf{e} \otimes \mathbf{e} + \mathbf{e}_\perp \otimes \mathbf{e}_\perp + \mathbf{e}_3 \otimes \mathbf{e}_3 \mathrel{\hat{=}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.165}$$

The metric tensor of the surface (normal projector) reads

$$\mathbf{N} = \mathbf{a}^\alpha \otimes \mathbf{a}_\alpha = \mathbf{a}_\alpha \otimes \mathbf{a}^\alpha = \mathbf{I} - \mathbf{n} \otimes \mathbf{n} \tag{2.166}$$

which means in point P

$$\mathbf{N} = \mathbf{I} - \mathbf{e}_3 \otimes \mathbf{e}_3 = \mathbf{e} \otimes \mathbf{e} + \mathbf{e}_\perp \otimes \mathbf{e}_\perp \mathrel{\hat{=}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.167}$$

Both base vectors $\mathbf{a}_\alpha$ and $\mathbf{a}^\beta$ can now be decomposed and written relatively to the constant Cartesian base $(\mathbf{e}, \mathbf{e}_\perp, \mathbf{e}_3)$ as follows

$$\mathbf{a}_\alpha = \mathbf{a}_\alpha \mathbf{I} = \mathbf{a}_\alpha(\mathbf{e} \otimes \mathbf{e} + \mathbf{e}_\perp \otimes \mathbf{e}_\perp + \mathbf{e}_3 \otimes \mathbf{e}_3) = (\mathbf{a}_\alpha \cdot \mathbf{e})\mathbf{e} + (\mathbf{a}_\alpha \cdot \mathbf{e}_\perp)\mathbf{e}_\perp + (\mathbf{a}_\alpha \cdot \mathbf{e}_3)\mathbf{e}_3 \\ \mathbf{a}^\beta = \mathbf{a}^\beta \mathbf{I} = \mathbf{a}^\beta(\mathbf{e} \otimes \mathbf{e} + \mathbf{e}_\perp \otimes \mathbf{e}_\perp + \mathbf{e}_3 \otimes \mathbf{e}_3) = (\mathbf{a}^\beta \cdot \mathbf{e})\mathbf{e} + (\mathbf{a}^\beta \cdot \mathbf{e}_\perp)\mathbf{e}_\perp + (\mathbf{a}^\beta \cdot \mathbf{e}_3)\mathbf{e}_3 \tag{2.168}$$

In point P, we have $\mathbf{a}_\alpha \cdot \mathbf{e}_3 = \mathbf{a}^\beta \cdot \mathbf{e}_3 = 0$ and we get

$$\mathbf{a}_\alpha = \mathbf{a}_\alpha \mathbf{N} = \mathbf{a}_\alpha(\mathbf{e} \otimes \mathbf{e} + \mathbf{e}_\perp \otimes \mathbf{e}_\perp) = (\mathbf{a}_\alpha \cdot \mathbf{e})\mathbf{e} + (\mathbf{a}_\alpha \cdot \mathbf{e}_\perp)\mathbf{e}_\perp \\ \mathbf{a}^\beta = \mathbf{a}^\beta \mathbf{N} = \mathbf{a}^\beta(\mathbf{e} \otimes \mathbf{e} + \mathbf{e}_\perp \otimes \mathbf{e}_\perp) = (\mathbf{a}^\beta \cdot \mathbf{e})\mathbf{e} + (\mathbf{a}^\beta \cdot \mathbf{e}_\perp)\mathbf{e}_\perp \tag{2.169}$$

The 2-dimensional antimetric permutation tensor of the surface reads

$$\mathbf{E} = E_{\alpha\beta}\mathbf{a}^{\alpha} \otimes \mathbf{a}^{\beta} = \boldsymbol{\mathcal{E}}\mathbf{n} \tag{2.170}$$

where $E_{\alpha\beta} = +\sqrt{a}$ if $\alpha\beta = 12$, $E_{\alpha\beta} = -\sqrt{a}$ if $\alpha\beta = 21$, $E_{\alpha\beta} = 0$ if $\alpha = \beta$ and $a = \det a_{\alpha\beta}$. In point P, we have

$$\mathbf{E} = \boldsymbol{\mathcal{E}}\mathbf{e}_3 = \mathbf{e} \otimes \mathbf{e}_{\perp} - \mathbf{e}_{\perp} \otimes \mathbf{e} \mathrel{\hat{=}} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.171}$$

Let us now decompose and write the interior part $\mathbf{v} = \mathbf{N}\mathbf{u} = v_{\alpha}\mathbf{a}^{\alpha}$ of the displacement $\mathbf{u} = \mathbf{v} + w\mathbf{n}$ in the Cartesian system as follows

$$\begin{aligned} \mathbf{v} &= v_{\alpha}\mathbf{a}^{\alpha} = v_{\alpha}\mathbf{a}^{\alpha}\mathbf{I} = v_{\alpha}\mathbf{a}^{\alpha}(\mathbf{e} \otimes \mathbf{e} + \mathbf{e}_{\perp} \otimes \mathbf{e}_{\perp} + \mathbf{e}_3 \otimes \mathbf{e}_3) \\ &= v_{\alpha}(\mathbf{a}^{\alpha} \cdot \mathbf{e})\mathbf{e} + v_{\alpha}(\mathbf{a}^{\alpha} \cdot \mathbf{e}_{\perp})\mathbf{e}_{\perp} + v_{\alpha}(\mathbf{a}^{\alpha} \cdot \mathbf{e}_3)\mathbf{e}_3 \\ &= u\mathbf{e} + v\mathbf{e}_{\perp} + v_z\mathbf{e}_3 \end{aligned} \quad ; \quad \mathbf{v} \mathrel{\hat{=}} \begin{Bmatrix} u \\ v \\ v_z \end{Bmatrix} \tag{2.172}$$

where

$$u = v_x = (\mathbf{a}^{\alpha} \cdot \mathbf{e})v_{\alpha} \quad ; \quad v = v_y = (\mathbf{a}^{\alpha} \cdot \mathbf{e}_{\perp})v_{\alpha} \quad ; \quad v_z = (\mathbf{a}^{\alpha} \cdot \mathbf{e}_3)v_{\alpha} \tag{2.173}$$

are the components of $\mathbf{v}$ relatively to the x, y and z-axes by definition. In point P, the interior and exterior parts of $\mathbf{u}$ read

$$\mathbf{v} = u\mathbf{e} + v\mathbf{e}_{\perp} \mathrel{\hat{=}} \begin{Bmatrix} u \\ v \\ 0 \end{Bmatrix} \quad ; \quad w\mathbf{n} \mathrel{\hat{=}} \begin{Bmatrix} 0 \\ 0 \\ w \end{Bmatrix} \tag{2.174}$$

Written in the Cartesian system, the 2-dimensional derivative operator reads

$$\boldsymbol{\nabla}_n = \mathbf{a}^{\beta}\frac{\partial}{\partial\theta^{\beta}} = \mathbf{a}^{\beta}\mathbf{I}\frac{\partial}{\partial\theta^{\beta}} = [(\mathbf{a}^{\beta} \cdot \mathbf{e})\mathbf{e} + (\mathbf{a}^{\beta} \cdot \mathbf{e}_{\perp})\mathbf{e}_{\perp} + (\mathbf{a}^{\beta} \cdot \mathbf{e}_3)\mathbf{e}_3]\frac{\partial}{\partial\theta^{\beta}} \tag{2.175}$$

In point P, we have

$$\boldsymbol{\nabla}_n = \mathbf{a}^{\beta}\frac{\partial}{\partial\theta^{\beta}} = \mathbf{a}^{\beta}\mathbf{N}\frac{\partial}{\partial\theta^{\beta}} = [(\mathbf{a}^{\beta} \cdot \mathbf{e})\mathbf{e} + (\mathbf{a}^{\beta} \cdot \mathbf{e}_{\perp})\mathbf{e}_{\perp}]\frac{\partial}{\partial\theta^{\beta}} = \mathbf{e}\frac{\partial}{\partial x} + \mathbf{e}_{\perp}\frac{\partial}{\partial y} \mathrel{\hat{=}} \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ 0 \end{Bmatrix} \tag{2.176}$$

where

$$\frac{\partial}{\partial x} = (\mathbf{a}^{\beta} \cdot \mathbf{e})\frac{\partial}{\partial\theta^{\beta}} \quad ; \quad \frac{\partial}{\partial y} = (\mathbf{a}^{\beta} \cdot \mathbf{e}_{\perp})\frac{\partial}{\partial\theta^{\beta}} \tag{2.177}$$

It follows that (in P)

$$\boldsymbol{\nabla}_n w = \mathbf{a}^\beta \frac{\partial w}{\partial \theta^\beta} = \mathbf{e}\frac{\partial w}{\partial x} + \mathbf{e}_\perp \frac{\partial w}{\partial y} \hat{=} \begin{Bmatrix} \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y} \\ 0 \end{Bmatrix} \tag{2.178}$$

The strain tensor $\widetilde{\boldsymbol{\gamma}}$ reads in point P

$$\widetilde{\boldsymbol{\gamma}} = \widetilde{\varepsilon}_x \mathbf{e}\otimes\mathbf{e} + \frac{1}{2}\widetilde{\gamma}_{xy}(\mathbf{e}\otimes\mathbf{e}_\perp + \mathbf{e}_\perp\otimes\mathbf{e}) + \widetilde{\varepsilon}_y \mathbf{e}_\perp\otimes\mathbf{e}_\perp \hat{=} \begin{bmatrix} \widetilde{\varepsilon}_x & \dfrac{1}{2}\widetilde{\gamma}_{xy} & 0 \\ \dfrac{1}{2}\widetilde{\gamma}_{xy} & \widetilde{\varepsilon}_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.179}$$

Obviously, a 2-dimensional surface in space has no thickness, which means that the components $\widetilde{\varepsilon}_z, \widetilde{\gamma}_{xz}$ and $\widetilde{\gamma}_{yz}$ do not exist. In point P, the curvature tensor $\mathbf{B}$ reads with $\Gamma^3_{\alpha\beta} = \Gamma^3_{\beta\alpha}$

$$\begin{aligned} \mathbf{B} = -\boldsymbol{\nabla}_n \otimes \mathbf{n} &= n_3\Gamma^3_{\alpha\beta}\mathbf{a}^\alpha\otimes\mathbf{a}^\beta = n_3\Gamma^3_{\alpha\beta}[(\mathbf{a}^\alpha\cdot\mathbf{e})\mathbf{e} + (\mathbf{a}^\alpha\cdot\mathbf{e}_\perp)\mathbf{e}_\perp]\otimes[(\mathbf{a}^\beta\cdot\mathbf{e})\mathbf{e} + (\mathbf{a}^\beta\cdot\mathbf{e}_\perp)\mathbf{e}_\perp] \\ &= n_3\Gamma^3_{\alpha\beta}[(\mathbf{a}^\alpha\cdot\mathbf{e})(\mathbf{a}^\beta\cdot\mathbf{e})\mathbf{e}\otimes\mathbf{e} + (\mathbf{a}^\alpha\cdot\mathbf{e})(\mathbf{a}^\beta\cdot\mathbf{e}_\perp)(\mathbf{e}\otimes\mathbf{e}_\perp + \mathbf{e}_\perp\otimes\mathbf{e}) + (\mathbf{a}^\alpha\cdot\mathbf{e}_\perp)(\mathbf{a}^\beta\cdot\mathbf{e}_\perp)\mathbf{e}_\perp\otimes\mathbf{e}_\perp \\ &= \kappa_x\mathbf{e}\otimes\mathbf{e} + \kappa_{xy}(\mathbf{e}\otimes\mathbf{e}_\perp + \mathbf{e}_\perp\otimes\mathbf{e}) + \kappa_y\mathbf{e}_\perp\otimes\mathbf{e}_\perp \\ &\hat{=} \begin{bmatrix} \kappa_x & \kappa_{xy} & 0 \\ \kappa_{xy} & \kappa_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{2.180}$$

where κ_x and κ_y are the normal curvatures of the surface relatively to the x and y coordinates. Remember that $\Gamma^3_{\alpha\beta} = \mathbf{g}^3\cdot\mathbf{a}_{\alpha,\beta}$ and $n_3 = \pm 1/\sqrt{g^{33}}$. The mean curvature of the surface is $H_B = \operatorname{tr}\mathbf{B}/2 = \mathbf{B}\cdot\mathbf{N}/2 = (\kappa_x+\kappa_y)/2 = (\kappa_1+\kappa_2)/2$ and the Gaussian curvature $K_B = \det(\mathbf{B}+\mathbf{n}\otimes\mathbf{n}) = -\mathbf{B}\cdot\mathbf{EBE}/2 = \kappa_x\kappa_y - \kappa_{xy}^2 = \kappa_1\kappa_2$, where κ_1 and κ_2 are the principal curvatures of the surface. In point P, the interior part of the tensor $\boldsymbol{\nabla}_n\otimes\mathbf{v}$ reads

$$\begin{aligned} (\boldsymbol{\nabla}_n\otimes\mathbf{v})\mathbf{N} &= (v_{\beta,\alpha} - \Gamma^\gamma_{\alpha\beta}v_\gamma + n_3^2 g^{3\gamma}\Gamma^3_{\alpha\beta}v_\gamma)\mathbf{a}^\alpha\otimes\mathbf{a}^\beta \\ &= (v_{\beta,\alpha} - \Gamma^\gamma_{\alpha\beta}v_\gamma + n_3^2 g^{3\gamma}\Gamma^3_{\alpha\beta}v_\gamma)[(\mathbf{a}^\alpha\cdot\mathbf{e})\mathbf{e} + (\mathbf{a}^\alpha\cdot\mathbf{e}_\perp)\mathbf{e}_\perp]\otimes[(\mathbf{a}^\beta\cdot\mathbf{e})\mathbf{e} + (\mathbf{a}^\beta\cdot\mathbf{e}_\perp)\mathbf{e}_\perp] \\ &= \frac{\partial u}{\partial x}\mathbf{e}\otimes\mathbf{e} + \frac{\partial v}{\partial x}\mathbf{e}\otimes\mathbf{e}_\perp + \frac{\partial u}{\partial y}\mathbf{e}_\perp\otimes\mathbf{e} + \frac{\partial v}{\partial y}\mathbf{e}_\perp\otimes\mathbf{e}_\perp \\ &\hat{=} \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} & 0 \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{2.181}$$

where the cartesian components $\partial u/\partial x$, $\partial v/\partial x$, $\partial u/\partial y$ and $\partial v/\partial y$ contain the covariant derivatives $v_{\beta;\alpha} = v_{\beta,\alpha} - \Gamma^{\gamma}_{\alpha\beta} v_\gamma + n_3^2 g^{3\gamma} \Gamma^3_{\alpha\beta} v_\gamma$ of the components of $\mathbf{v} = v_\alpha \mathbf{a}^\alpha$ on the curved surface. With equations (1.110a), (2.173) and (2.177), we have

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}[(\mathbf{a}^\alpha \cdot \mathbf{e}) v_\alpha] = (\mathbf{a}^\beta \cdot \mathbf{e}) \frac{\partial}{\partial \theta^\beta}[(\mathbf{a}^\alpha \cdot \mathbf{e}) v_\alpha] \\
&= (\mathbf{a}^\beta \cdot \mathbf{e}) \left[(\mathbf{a}^\alpha \cdot \mathbf{e}) \frac{\partial v_\alpha}{\partial \theta^\beta} + \left(\mathbf{e} \cdot \frac{\partial \mathbf{a}^\alpha}{\partial \theta^\beta} \right) v_\alpha + \left(\mathbf{a}^\alpha \cdot \frac{\partial \mathbf{e}}{\partial \theta^\beta} \right) v_\alpha \right] \\
&= (\mathbf{a}^\beta \cdot \mathbf{e}) \left[(\mathbf{a}^\alpha \cdot \mathbf{e}) v_{\alpha,\beta} + (\mathbf{a}^\gamma \cdot \mathbf{e}) \mathbf{a}_\gamma \cdot \frac{\partial \mathbf{a}^\alpha}{\partial \theta^\beta} v_\alpha \right] \\
&= (\mathbf{a}^\beta \cdot \mathbf{e})(\mathbf{a}^\alpha \cdot \mathbf{e}) v_{\alpha,\beta} + (\mathbf{a}^\beta \cdot \mathbf{e})(\mathbf{a}^\gamma \cdot \mathbf{e})(-\Gamma^\alpha_{\beta\gamma} + n_3^2 g^{3\alpha} \Gamma^3_{\beta\gamma}) v_\alpha \\
&= (\mathbf{a}^\alpha \cdot \mathbf{e})(\mathbf{a}^\beta \cdot \mathbf{e})[v_{\beta,\alpha} - \Gamma^\gamma_{\alpha\beta} v_\gamma + n_3^2 g^{3\gamma} \Gamma^3_{\alpha\beta} v_\gamma]
\end{aligned}
\tag{2.182}
$$

$$
\begin{aligned}
\frac{\partial v}{\partial x} &= \frac{\partial}{\partial x}[(\mathbf{a}^\alpha \cdot \mathbf{e}_\perp) v_\alpha] = (\mathbf{a}^\beta \cdot \mathbf{e}) \frac{\partial}{\partial \theta^\beta}[(\mathbf{a}^\alpha \cdot \mathbf{e}_\perp) v_\alpha] \\
&= (\mathbf{a}^\beta \cdot \mathbf{e}) \left[(\mathbf{a}^\alpha \cdot \mathbf{e}_\perp) \frac{\partial v_\alpha}{\partial \theta^\beta} + \left(\mathbf{e}_\perp \cdot \frac{\partial \mathbf{a}^\alpha}{\partial \theta^\beta} \right) v_\alpha + \left(\mathbf{a}^\alpha \cdot \frac{\partial \mathbf{e}_\perp}{\partial \theta^\beta} \right) v_\alpha \right] \\
&= (\mathbf{a}^\beta \cdot \mathbf{e}) \left[(\mathbf{a}^\alpha \cdot \mathbf{e}_\perp) v_{\alpha,\beta} + (\mathbf{a}^\gamma \cdot \mathbf{e}_\perp) \mathbf{a}_\gamma \cdot \frac{\partial \mathbf{a}^\alpha}{\partial \theta^\beta} v_\alpha \right] \\
&= (\mathbf{a}^\beta \cdot \mathbf{e})(\mathbf{a}^\alpha \cdot \mathbf{e}_\perp) v_{\alpha,\beta} + (\mathbf{a}^\beta \cdot \mathbf{e})(\mathbf{a}^\gamma \cdot \mathbf{e}_\perp)(-\Gamma^\alpha_{\beta\gamma} + n_3^2 g^{3\alpha} \Gamma^3_{\beta\gamma}) v_\alpha \\
&= (\mathbf{a}^\alpha \cdot \mathbf{e})(\mathbf{a}^\beta \cdot \mathbf{e}_\perp)(v_{\beta,\alpha} - \Gamma^\gamma_{\alpha\beta} v_\gamma + n_3^2 g^{3\gamma} \Gamma^3_{\alpha\beta} v_\gamma)
\end{aligned}
\tag{2.183}
$$

$$
\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}[(\mathbf{a}^\alpha \cdot \mathbf{e}) v_\alpha] = (\mathbf{a}^\beta \cdot \mathbf{e}_\perp) \frac{\partial}{\partial \theta^\beta}[(\mathbf{a}^\alpha \cdot \mathbf{e}) v_\alpha] \\
&= (\mathbf{a}^\beta \cdot \mathbf{e}_\perp) \left[(\mathbf{a}^\alpha \cdot \mathbf{e}) \frac{\partial v_\alpha}{\partial \theta^\beta} + \left(\mathbf{e} \cdot \frac{\partial \mathbf{a}^\alpha}{\partial \theta^\beta} \right) v_\alpha + \left(\mathbf{a}^\alpha \cdot \frac{\partial \mathbf{e}}{\partial \theta^\beta} \right) v_\alpha \right] \\
&= (\mathbf{a}^\beta \cdot \mathbf{e}_\perp) \left[(\mathbf{a}^\alpha \cdot \mathbf{e}) v_{\alpha,\beta} + (\mathbf{a}^\gamma \cdot \mathbf{e}) \mathbf{a}_\gamma \cdot \frac{\partial \mathbf{a}^\alpha}{\partial \theta^\beta} v_\alpha \right] \\
&= (\mathbf{a}^\beta \cdot \mathbf{e}_\perp)(\mathbf{a}^\alpha \cdot \mathbf{e}) v_{\alpha,\beta} + (\mathbf{a}^\beta \cdot \mathbf{e}_\perp)(\mathbf{a}^\gamma \cdot \mathbf{e})(-\Gamma^\alpha_{\beta\gamma} + n_3^2 g^{3\alpha} \Gamma^3_{\beta\gamma}) v_\alpha \\
&= (\mathbf{a}^\alpha \cdot \mathbf{e}_\perp)(\mathbf{a}^\beta \cdot \mathbf{e})(v_{\beta,\alpha} - \Gamma^\gamma_{\alpha\beta} v_\gamma + n_3^2 g^{3\gamma} \Gamma^3_{\alpha\beta} v_\gamma)
\end{aligned}
\tag{2.184}
$$

$$
\begin{aligned}
\frac{\partial v}{\partial y} &= \frac{\partial}{\partial y}[(\mathbf{a}^\alpha \cdot \mathbf{e}_\perp) v_\alpha] = (\mathbf{a}^\beta \cdot \mathbf{e}_\perp) \frac{\partial}{\partial \theta^\beta}[(\mathbf{a}^\alpha \cdot \mathbf{e}_\perp) v_\alpha] \\
&= (\mathbf{a}^\beta \cdot \mathbf{e}_\perp) \left[(\mathbf{a}^\alpha \cdot \mathbf{e}_\perp) \frac{\partial v_\alpha}{\partial \theta^\beta} + \left(\mathbf{e}_\perp \cdot \frac{\partial \mathbf{a}^\alpha}{\partial \theta^\beta} \right) v_\alpha + \left(\mathbf{a}^\alpha \cdot \frac{\partial \mathbf{e}_\perp}{\partial \theta^\beta} \right) v_\alpha \right] \\
&= (\mathbf{a}^\beta \cdot \mathbf{e}_\perp) \left[(\mathbf{a}^\alpha \cdot \mathbf{e}_\perp) v_{\alpha,\beta} + (\mathbf{a}^\gamma \cdot \mathbf{e}_\perp) \mathbf{a}_\gamma \cdot \frac{\partial \mathbf{a}^\alpha}{\partial \theta^\beta} v_\alpha \right] \\
&= (\mathbf{a}^\beta \cdot \mathbf{e}_\perp)(\mathbf{a}^\alpha \cdot \mathbf{e}_\perp) v_{\alpha,\beta} + (\mathbf{a}^\beta \cdot \mathbf{e}_\perp)(\mathbf{a}^\gamma \cdot \mathbf{e}_\perp)(-\Gamma^\alpha_{\beta\gamma} + n_3^2 g^{3\alpha} \Gamma^3_{\beta\gamma}) v_\alpha \\
&= (\mathbf{a}^\alpha \cdot \mathbf{e}_\perp)(\mathbf{a}^\beta \cdot \mathbf{e}_\perp)(v_{\beta,\alpha} - \Gamma^\gamma_{\alpha\beta} v_\gamma + n_3^2 g^{3\gamma} \Gamma^3_{\alpha\beta} v_\gamma)
\end{aligned}
\tag{2.185}
$$

where we have used the following relations (in point P)

$$\mathbf{e} = \mathbf{N}\mathbf{e} = (\mathbf{a}_\gamma \otimes \mathbf{a}^\gamma)\mathbf{e} = \mathbf{a}_\gamma(\mathbf{a}^\gamma \cdot \mathbf{e}) \quad ; \quad \mathbf{e}_\perp = \mathbf{N}\mathbf{e}_\perp = (\mathbf{a}_\gamma \otimes \mathbf{a}^\gamma)\mathbf{e}_\perp = \mathbf{a}_\gamma(\mathbf{a}^\gamma \cdot \mathbf{e}_\perp)$$
$$\frac{\partial v_\alpha}{\partial \theta^\beta} = v_{\alpha,\beta} \quad ; \quad \frac{\partial \mathbf{e}}{\partial \theta^\beta} = 0 \quad ; \quad \frac{\partial \mathbf{e}_\perp}{\partial \theta^\beta} = 0 \quad ; \quad \frac{\partial \mathbf{a}^\alpha}{\partial \theta^\beta} \cdot \mathbf{a}_\gamma = \mathbf{a}^\alpha_{,\beta} \cdot \mathbf{a}_\gamma = -\Gamma^\alpha_{\beta\gamma} + n_3^2 g^{3\alpha}\Gamma^3_{\beta\gamma} \tag{2.186}$$

The semi-exterior part of $\boldsymbol{\nabla}_n \otimes \mathbf{v}$ reads (with $\mathbf{n} = \mathbf{e}_3$ in point P)

$$\begin{aligned}\mathbf{B}\mathbf{v} \otimes \mathbf{n} &= n_3 a^{\gamma\beta}\Gamma^3_{\alpha\beta} v_\gamma \mathbf{a}^\alpha \otimes \mathbf{n} = (\kappa_x u + \kappa_{xy} v)\mathbf{e} \otimes \mathbf{n} + (\kappa_{xy} u + \kappa_y v)\mathbf{e}_\perp \otimes \mathbf{n} \\ &\hat{=} \left(\begin{bmatrix} \kappa_x & \kappa_{xy} & 0 \\ \kappa_{xy} & \kappa_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ 0 \end{Bmatrix} \right) \otimes (\,0 \quad 0 \quad 1\,) = \begin{bmatrix} 0 & 0 & \kappa_x u + \kappa_{xy} v \\ 0 & 0 & \kappa_{xy} u + \kappa_y v \\ 0 & 0 & 0 \end{bmatrix}\end{aligned} \tag{2.187}$$

It follows that

$$\boldsymbol{\nabla}_n \otimes \mathbf{v} = (\boldsymbol{\nabla}_n \otimes \mathbf{v})\mathbf{N} + \mathbf{B}\mathbf{v} \otimes \mathbf{n} \;\hat{=}\; \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} & \kappa_x u + \kappa_{xy} v \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} & \kappa_{xy} u + \kappa_y v \\ 0 & 0 & 0 \end{bmatrix} \tag{2.188}$$

In the special case where $\mathbf{g}_3 = \mathbf{g}^3 = \mathbf{n}$ with $n_3 = 1$, the contravariant base vectors $\mathbf{g}^\beta$ are normal to $\mathbf{n}$, which means that $\mathbf{N}\mathbf{g}^\beta = \mathbf{g}^\beta = \mathbf{a}^\beta$ and $g^{3\gamma} = \mathbf{g}^3 \cdot \mathbf{g}^\gamma = \mathbf{n} \cdot \mathbf{a}^\gamma = 0$. Therefore, we have $v_{\beta;\alpha} = v_{\beta,\alpha} - \Gamma^\gamma_{\alpha\beta} v_\gamma$ and $-\Gamma^\gamma_{\alpha 3} = \mathbf{g}^\gamma_{,\alpha} \cdot \mathbf{g}_3 = \mathbf{a}^\gamma_{,\alpha} \cdot \mathbf{n} = a^{\gamma\beta}\Gamma^3_{\alpha\beta}$, which means that $\boldsymbol{\nabla}_n \otimes \mathbf{v} = (v_{\beta,\alpha} - \Gamma^\gamma_{\alpha\beta} v_\gamma)\mathbf{a}^\alpha \otimes \mathbf{a}^\beta - \Gamma^\gamma_{\alpha 3} v_\gamma \mathbf{a}^\alpha \otimes \mathbf{n}$.

Remark: On a curved surface, one must first compute the quantities with the intrinsic tensor notation (or with the notation with indices), paying special attention on derivatives, before to write any components in some matrix! For example, it is wrong to write

$$\boldsymbol{\nabla}_n \otimes \mathbf{v} \neq \left(\mathbf{e}\frac{\partial}{\partial x} + \mathbf{e}_\perp \frac{\partial}{\partial y} \right) \otimes (u\mathbf{e} + v\mathbf{e}_\perp) \;\hat{=}\; \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} & 0 \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.189}$$

Thus we have on the curved surface in point P

$$(\boldsymbol{\nabla}_n \otimes \mathbf{v})\mathbf{N} \;\hat{=}\; \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} & \kappa_x u + \kappa_{xy} v \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} & \kappa_{xy} u + \kappa_y v \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} & 0 \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.190}$$

$$\mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{v})^T \mathrel{\hat{=}} \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & 0 \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ; \quad w\mathbf{B} \mathrel{\hat{=}} \begin{bmatrix} \kappa_x & \kappa_{xy} & 0 \\ \kappa_{xy} & \kappa_y & 0 \\ 0 & 0 & 0 \end{bmatrix} w \tag{2.191}$$

$$w^2\mathbf{B}^2 \mathrel{\hat{=}} \begin{bmatrix} \kappa_x & \kappa_{xy} & 0 \\ \kappa_{xy} & \kappa_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \kappa_x & \kappa_{xy} & 0 \\ \kappa_{xy} & \kappa_y & 0 \\ 0 & 0 & 0 \end{bmatrix} w^2 = \begin{bmatrix} \kappa_x^2 + \kappa_{xy}^2 & \kappa_{xy}(\kappa_x + \kappa_y) & 0 \\ \kappa_{xy}(\kappa_x + \kappa_y) & \kappa_y^2 + \kappa_{xy}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} w^2 \tag{2.192}$$

$$(\boldsymbol{\nabla}_n \otimes \mathbf{v})\mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{v})^T \mathrel{\hat{=}} \begin{bmatrix} \left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial x}\right)^2 & \dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\dfrac{\partial v}{\partial y} & 0 \\ \dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\dfrac{\partial v}{\partial y} & \left(\dfrac{\partial u}{\partial y}\right)^2 + \left(\dfrac{\partial v}{\partial y}\right)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.193}$$

$$(\boldsymbol{\nabla}_n \otimes \mathbf{v})\mathbf{B} \mathrel{\hat{=}} \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} & 0 \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \kappa_x & \kappa_{xy} & 0 \\ \kappa_{xy} & \kappa_y & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \kappa_x\dfrac{\partial u}{\partial x} + \kappa_{xy}\dfrac{\partial v}{\partial x} & \kappa_{xy}\dfrac{\partial u}{\partial x} + \kappa_y\dfrac{\partial v}{\partial x} & 0 \\ \kappa_x\dfrac{\partial u}{\partial y} + \kappa_{xy}\dfrac{\partial v}{\partial y} & \kappa_{xy}\dfrac{\partial u}{\partial y} + \kappa_y\dfrac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.194}$$

$$\mathbf{B}(\boldsymbol{\nabla}_n \otimes \mathbf{v})^T \mathrel{\hat{=}} \begin{bmatrix} \kappa_x\dfrac{\partial u}{\partial x} + \kappa_{xy}\dfrac{\partial v}{\partial x} & \kappa_x\dfrac{\partial u}{\partial y} + \kappa_{xy}\dfrac{\partial v}{\partial y} & 0 \\ \kappa_{xy}\dfrac{\partial u}{\partial x} + \kappa_y\dfrac{\partial v}{\partial x} & \kappa_{xy}\dfrac{\partial u}{\partial y} + \kappa_y\dfrac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.195}$$

$$\mathbf{B}\mathbf{v} + \boldsymbol{\nabla}_n w \mathrel{\hat{=}} \begin{bmatrix} \kappa_x & \kappa_{xy} & 0 \\ \kappa_{xy} & \kappa_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ 0 \end{Bmatrix} + \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ 0 \end{Bmatrix} w = \begin{Bmatrix} \kappa_x u + \kappa_{xy} v + \dfrac{\partial w}{\partial x} \\ \kappa_{xy} u + \kappa_y v + \dfrac{\partial w}{\partial y} \\ 0 \end{Bmatrix} \tag{2.196}$$

$$(\mathbf{Bv}+\boldsymbol{\nabla}_n w)\otimes(\mathbf{Bv}+\boldsymbol{\nabla}_n w)\ \hat{=}$$

$$\hat{=}\left\{\begin{matrix}\kappa_x u+\kappa_{xy}v+\dfrac{\partial w}{\partial x}\\ \kappa_{xy}u+\kappa_y v+\dfrac{\partial w}{\partial y}\\ 0\end{matrix}\right\}\otimes\left(\kappa_x u+\kappa_{xy}v+\frac{\partial w}{\partial x}\quad \kappa_{xy}u+\kappa_y v+\frac{\partial w}{\partial y}\quad 0\right) \tag{2.197}$$

$$\hat{=}\begin{bmatrix}\left(\kappa_x u+\kappa_{xy}v+\dfrac{\partial w}{\partial x}\right)^2 & \left(\kappa_x u+\kappa_{xy}v+\dfrac{\partial w}{\partial x}\right)\left(\kappa_{xy}u+\kappa_y v+\dfrac{\partial w}{\partial y}\right) & 0\\ \left(\kappa_x u+\kappa_{xy}v+\dfrac{\partial w}{\partial x}\right)\left(\kappa_{xy}u+\kappa_y v+\dfrac{\partial w}{\partial y}\right) & \left(\kappa_{xy}u+\kappa_y v+\dfrac{\partial w}{\partial y}\right)^2 & 0\\ 0 & 0 & 0\end{bmatrix}$$

The components $\widetilde{\varepsilon}_x, \widetilde{\varepsilon}_y$ and $\widetilde{\gamma}_{xy}/2$ of the surface strain tensor $\widetilde{\boldsymbol{\gamma}}$ can now exactly be written as follows

$$\begin{aligned}\widetilde{\varepsilon}_x=\mathbf{e}\cdot\widetilde{\boldsymbol{\gamma}}\mathbf{e}=\frac{\partial u}{\partial x}-\kappa_x w+\frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]-w\left(\kappa_x\frac{\partial u}{\partial x}+\kappa_{xy}\frac{\partial v}{\partial x}\right)\\ +\frac{1}{2}w^2(\kappa_x^2+\kappa_{xy}^2)+\frac{1}{2}\left(\kappa_x u+\kappa_{xy}v+\frac{\partial w}{\partial x}\right)^2\end{aligned} \tag{2.198}$$

$$\begin{aligned}\widetilde{\varepsilon}_y=\mathbf{e}_\perp\cdot\widetilde{\boldsymbol{\gamma}}\mathbf{e}_\perp=\frac{\partial v}{\partial y}-\kappa_y w+\frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]-w\left(\kappa_y\frac{\partial v}{\partial y}+\kappa_{xy}\frac{\partial u}{\partial y}\right)\\ +\frac{1}{2}w^2(\kappa_y^2+\kappa_{xy}^2)+\frac{1}{2}\left(\kappa_y v+\kappa_{xy}u+\frac{\partial w}{\partial y}\right)^2\end{aligned} \tag{2.199}$$

$$\begin{aligned}\frac{1}{2}\widetilde{\gamma}_{xy}=\mathbf{e}\cdot\widetilde{\boldsymbol{\gamma}}\mathbf{e}_\perp=\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-\kappa_{xy}w+\frac{1}{2}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right)\\ -\frac{1}{2}w\left[\kappa_x\frac{\partial u}{\partial y}+\kappa_{xy}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\kappa_y\frac{\partial v}{\partial x}\right]+\frac{1}{2}w^2\kappa_{xy}(\kappa_x+\kappa_y)\\ +\frac{1}{2}\left(\kappa_x u+\kappa_{xy}v+\frac{\partial w}{\partial x}\right)\left(\kappa_y v+\kappa_{xy}u+\frac{\partial w}{\partial y}\right)\end{aligned} \tag{2.200}$$

To get the linear and angular dilatation ε_x, ε_y and $\gamma_{xy}/2$, we must take into account the following nonlinear expressions

$$\begin{aligned}\varepsilon_x&=\sqrt{1+2\widetilde{\varepsilon}_x}-1\\ \varepsilon_y&=\sqrt{1+2\widetilde{\varepsilon}_y}-1\end{aligned}\quad;\qquad \frac{1}{2}\gamma_{xy}=\frac{1}{2}\arcsin\left(\frac{\widetilde{\gamma}_{xy}}{\sqrt{(1+2\widetilde{\varepsilon}_x)(1+2\widetilde{\varepsilon}_y)}}\right) \tag{2.201}$$

which read after development up to the second order terms

$$
\begin{aligned}
\varepsilon_x &= \sqrt{1+2\widetilde{\varepsilon}_x} - 1 \simeq \widetilde{\varepsilon}_x - \frac{1}{2}\widetilde{\varepsilon}_x^2 + \dots \\
\varepsilon_y &= \sqrt{1+2\widetilde{\varepsilon}_y} - 1 \simeq \widetilde{\varepsilon}_y - \frac{1}{2}\widetilde{\varepsilon}_y^2 + \dots \\
\frac{1}{2}\gamma_{xy} &= \frac{1}{2}\arcsin\left(\frac{\widetilde{\gamma}_{xy}}{\sqrt{(1+2\widetilde{\varepsilon}_x)(1+2\widetilde{\varepsilon}_y)}}\right) \simeq \frac{1}{2}\widetilde{\gamma}_{xy} - \frac{1}{2}\widetilde{\gamma}_{xy}(\widetilde{\varepsilon}_x + \widetilde{\varepsilon}_y) + \dots
\end{aligned}
\tag{2.202}
$$

Considering surfaces where the displacements are much smaller than the principal radii of curvature ($|\mathbf{u}| \ll |1/\kappa_{1,2}|$), we get explicitly [2.9]

$$
\varepsilon_x \simeq \frac{\partial u}{\partial x} - \kappa_x w + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 - w\kappa_{xy}\frac{\partial v}{\partial x} + \frac{1}{2}w^2\kappa_{xy}^2 + \frac{1}{2}\left(\kappa_x u + \kappa_{xy} v + \frac{\partial w}{\partial x}\right)^2 \tag{2.203}
$$

$$
\varepsilon_y \simeq \frac{\partial v}{\partial y} - \kappa_y w + \frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2 - w\kappa_{xy}\frac{\partial u}{\partial y} + \frac{1}{2}w^2\kappa_{xy}^2 + \frac{1}{2}\left(\kappa_y v + \kappa_{xy} u + \frac{\partial w}{\partial y}\right)^2 \tag{2.204}
$$

$$
\begin{aligned}
\frac{1}{2}\gamma_{xy} \simeq{}& \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \kappa_{xy} w - \frac{1}{2}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) \\
&+ \frac{1}{2}w\left[\kappa_x\frac{\partial v}{\partial x} + \kappa_{xy}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \kappa_y\frac{\partial u}{\partial y}\right] - \frac{1}{2}w^2\kappa_{xy}(\kappa_x + \kappa_y) \\
&+ \frac{1}{2}\left(\kappa_x u + \kappa_{xy} v + \frac{\partial w}{\partial x}\right)\left(\kappa_y v + \kappa_{xy} u + \frac{\partial w}{\partial y}\right)
\end{aligned}
\tag{2.205}
$$

For plane surfaces with no curvature, we have $\mathbf{B} = 0$ and

$$
\begin{aligned}
\varepsilon_x &\simeq \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \\
\varepsilon_y &\simeq \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \\
\frac{1}{2}\gamma_{xy} &\simeq \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \frac{1}{2}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} - \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)
\end{aligned}
\tag{2.206}
$$

2.4.26 Rotation components of a 2-dimensional curved surface in the 3-dimensional space written in a Cartesian system up to the first-order terms

The rotation of a surface element is fully described by the exact rotation vector $\boldsymbol{\omega}_S = \Omega\mathbf{n} + \mathbf{E}\boldsymbol{\omega}$ or by the orthogonal tensor $\mathbf{Q}_S$, which reads up to the second-order terms

$$
\begin{aligned}
\mathbf{Q}_S \cong{}& \mathbf{I} - \Omega\mathbf{E} - \boldsymbol{\omega}\otimes\mathbf{n} + \mathbf{n}\otimes\boldsymbol{\omega} + \frac{1}{2}\Omega\mathbf{E}\,\boldsymbol{\omega}\otimes\mathbf{n} + \frac{1}{2}\mathbf{n}\otimes\Omega\mathbf{E}\,\boldsymbol{\omega} \\
&- \frac{1}{2}[\Omega^2\mathbf{N} + \boldsymbol{\omega}\otimes\boldsymbol{\omega} + (\boldsymbol{\omega}\cdot\boldsymbol{\omega})\mathbf{n}\otimes\mathbf{n}]
\end{aligned}
\tag{2.207}
$$

where

$$
\begin{aligned}
\boldsymbol{\omega} &\cong (\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{n} - \left[\frac{3}{4}(\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{N} + \frac{1}{4}\mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T\right](\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{n} \\
\Omega\mathbf{E} &\cong \frac{1}{2}[(\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{N} - \mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T] - \frac{1}{4}\big((\boldsymbol{\nabla}_n \otimes \mathbf{u})^2\mathbf{N} - \mathbf{N}[(\boldsymbol{\nabla}_n \otimes \mathbf{u})^2]^T\big) \\
\Omega &\cong -\frac{1}{4}[(\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{N} - \mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{u})^T] \cdot \mathbf{E} + \frac{1}{8}\big((\boldsymbol{\nabla}_n \otimes \mathbf{u})^2\mathbf{N} - \mathbf{N}[(\boldsymbol{\nabla}_n \otimes \mathbf{u})^2]^T\big) \cdot \mathbf{E}
\end{aligned}
\tag{2.208}
$$

By only considering the first-order terms, we have

$$
\begin{aligned}
\boldsymbol{\omega} &\cong (\boldsymbol{\nabla}_n \otimes \mathbf{u})\mathbf{n} = \mathbf{B}\mathbf{v} + \boldsymbol{\nabla}_n w \\
\Omega\mathbf{n} &\cong -\frac{1}{4}\left([(\boldsymbol{\nabla}_n \otimes \mathbf{v})\mathbf{N} - \mathbf{N}(\boldsymbol{\nabla}_n \otimes \mathbf{v})^T] \cdot \mathbf{E}\right)\mathbf{n}
\end{aligned}
\tag{2.209}
$$

which gives in components

$$
\boldsymbol{\omega} \mathrel{\hat{=}} \begin{bmatrix} \kappa_x & \kappa_{xy} & 0 \\ \kappa_{xy} & \kappa_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ 0 \end{Bmatrix} + \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ 0 \end{Bmatrix} w = \begin{Bmatrix} \kappa_x u + \kappa_{xy} v + \dfrac{\partial w}{\partial x} \\ \kappa_{xy} u + \kappa_y v + \dfrac{\partial w}{\partial y} \\ 0 \end{Bmatrix} = \begin{Bmatrix} \omega_x \\ \omega_y \\ 0 \end{Bmatrix}
\tag{2.210}
$$

$$
\mathbf{E}\boldsymbol{\omega} \mathrel{\hat{=}} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \kappa_x u + \kappa_{xy} v + \dfrac{\partial w}{\partial x} \\ \kappa_{xy} u + \kappa_y v + \dfrac{\partial w}{\partial y} \\ 0 \end{Bmatrix} = \begin{Bmatrix} \kappa_{xy} u + \kappa_y v + \dfrac{\partial w}{\partial y} \\ -\kappa_x u - \kappa_{xy} v - \dfrac{\partial w}{\partial x} \\ 0 \end{Bmatrix} = \begin{Bmatrix} \omega_y \\ -\omega_x \\ 0 \end{Bmatrix}
\tag{2.211}
$$

$$
\Omega\mathbf{E} \mathrel{\hat{=}} \frac{1}{2} \begin{bmatrix} 0 & \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} & 0 \\ \dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial x} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \Omega & 0 \\ -\Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\tag{2.212}
$$

$$
\Omega\mathbf{n} \mathrel{\hat{=}} -\frac{1}{4} \left(\begin{bmatrix} 0 & \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} & 0 \\ \dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial x} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}\right) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \Omega \end{Bmatrix}
\tag{2.213}
$$

$$\boldsymbol{\omega}_S = \mathit{\Omega}\mathbf{n} + \mathbf{E}\boldsymbol{\omega} \mathrel{\hat{=}} \left\{ \begin{array}{c} \kappa_{xy}u + \kappa_y v + \dfrac{\partial w}{\partial y} \\ -\kappa_x u - \kappa_{xy} v - \dfrac{\partial w}{\partial x} \\ \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}\right) \end{array} \right\} = \left\{ \begin{array}{c} \omega_y \\ -\omega_x \\ \mathit{\Omega} \end{array} \right\} = \left\{ \begin{array}{c} \mathit{\Omega}_x \\ \mathit{\Omega}_y \\ \mathit{\Omega}_z \end{array} \right\} \tag{2.214}$$

$$\boldsymbol{\omega} \otimes \mathbf{n} \mathrel{\hat{=}} \left\{ \begin{array}{c} \omega_x \\ \omega_y \\ 0 \end{array} \right\} \otimes (\,0 \quad 0 \quad 1\,) = \begin{bmatrix} 0 & 0 & \omega_x \\ 0 & 0 & \omega_y \\ 0 & 0 & 0 \end{bmatrix} \tag{2.215}$$

$$\mathbf{Q}_S \cong \mathbf{I} - \mathit{\Omega}\mathbf{E} - \boldsymbol{\omega} \otimes \mathbf{n} + \mathbf{n} \otimes \boldsymbol{\omega} \mathrel{\hat{=}} \begin{bmatrix} 1 & -\mathit{\Omega} & -\omega_x \\ \mathit{\Omega} & 1 & -\omega_y \\ \omega_x & \omega_y & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\mathit{\Omega}_z & \mathit{\Omega}_y \\ \mathit{\Omega}_z & 1 & -\mathit{\Omega}_x \\ -\mathit{\Omega}_y & \mathit{\Omega}_x & 1 \end{bmatrix} \tag{2.216}$$

The components $\mathit{\Omega}_x$, $\mathit{\Omega}_y$ and $\mathit{\Omega}_z$ of the rotation vector $\boldsymbol{\omega}_S$ describe both the out-of-plane and the in-plane rotation of a surface element. We have up to the first-order terms

$$\begin{aligned} \mathit{\Omega}_x &= \omega_y = \kappa_{xy}u + \kappa_y v + \frac{\partial w}{\partial y} \\ \mathit{\Omega}_y &= -\omega_x = -\kappa_x u - \kappa_{xy} v - \frac{\partial w}{\partial x} \\ \mathit{\Omega}_z &= \mathit{\Omega} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \end{aligned} \tag{2.217}$$

3. Recovery of interference fringes in holographic interferometry

3.1 Introduction

This section first assumes that the reader already knows the basic concepts of holographic interferometry [3.1–3.55], and second shows an example of application of the intrinsic tensor calculus presented in the previous sections. Its purpose is to explain how to apply holographic interferometry to large deformation measurement and how to deal with the related problem of vanishing fringe patterns.

In common industrial environment, large deformation measurements of opaque bodies by means of holographic interferometry are often related to the problem of decreasing fringe spacing and contrast, causing the loss of the interference fringe pattern, which contains the whole information on the corresponding deformation. Therefore, the only way to determine the surface strain, rotation and displacement components of a structure element under load relatively to the unloaded state is first to recover the interference fringes – at least locally – and then to use the correct adequate relations to process the recovered fringe pattern properly.

The main purpose of this section is to explicitly and quantitatively present the general equation system for a systematic fringe recovery procedure in the general case of a large unknown object deformation. The relations for the quantitative evaluation of the recovered fringes, i.e. the optical path difference and the exact fringe vector of the modified interference pattern, are explicitly presented. All needed relations are first introduced in form of general vector and tensor equations. Then, equations for fringe recovery are written in cartesian components and used within a quantitative practical experiment to demonstrate the reliability of the theory. These relations are general and may also be used in other application fields (with their related problems) of holographic interferometry, when the loss of fringe spacing and contrast should be compensated.

3.2 Optical path difference by a geometrical modification

To enable fringe recovery in a general geometrical case, three important points must be considered:

1) The holographic setup should allow a geometrical and/or an optical modification to adequately compensate the unknown mechanical deformation and the optical image aberrations.
2) The holographic method should be one allowing independent acting on the holographic images, e.g. one of the type belonging either the real-time technique or the double exposure technique on two holograms (because not all holographic methods are suitable for this purpose [3.45]).
3) The holographic images of the deformed and undeformed body must sufficiently overlap, the fringe spacing must be large enough and the fringe visibility must have sufficient quality to be analysed properly.

Considering the formation of fringes in a general case, we choose here, without restricting the generality, both the real-time technique and the double exposure technique on two holograms with a geometrical modification at the reconstruction, that means the possibility of moving the reference source and the hologram corresponding to the undeformed configuration. No move of the object source, no change of the wavelength λ and no modification on the deformed configuration will be considered here. This enables to present an analogous situation for both real-time and double exposure techniques (Fig. 3.1). For an observer in $\widetilde{\mathrm{R}}$ (point $\widetilde{\mathrm{R}}$ is the collineation center of the observing system), the intensity J in some point $\widetilde{\mathrm{K}}$ in the 3-dimensional space is given as follows [3.29,3.43,3.44]

$$J = A_0 + A_1 \cos\left(\frac{2\pi}{\lambda} D_d\right) = A_0 - A_1 \cos\left(\frac{2\pi}{\lambda} D_r\right) = A_0 + A_1 \cos\Phi \tag{3.1}$$

where D_d and D_r respectively are the optical path differences in the case of the double exposure and of the real-time technique and where Φ is the phase (or phase difference) used as alternative to the optical path difference. The observing system, e.g. a CCD camera, is focused in point $\widetilde{\mathrm{K}}$ where the fringes are observed, which means that point $\widetilde{\mathrm{K}}$ is in the object plane of our optical system. The minus sign in equation (3.1) explains that white interference fringes in double exposure correspond to dark fringes in real-time.

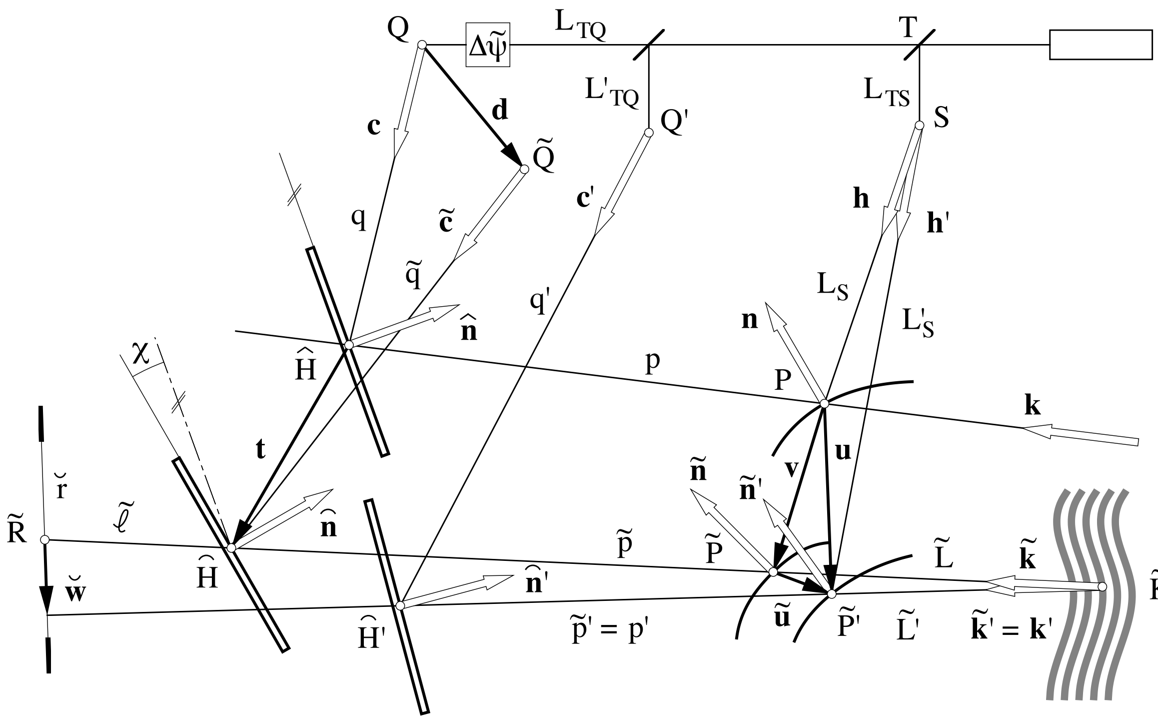

Fig. 3.1: Holographic setup at recording and reconstruction with modification

The phases in some points $\widetilde{\mathrm{P}}$ and $\widetilde{\mathrm{P}}'$ on the image surfaces are given by the two following equations of interference identity

$$\begin{aligned}\widetilde{\varphi} &= \varphi + \pi + \frac{2\pi}{\lambda}[(p - q) - (\widetilde{p} - \widetilde{q})] + (\widetilde{\psi} - \psi)\\ \widetilde{\varphi}' &= \varphi' + \pi + \frac{2\pi}{\lambda}[(p' - q') - (\widetilde{p}' - \widetilde{q}')] + (\widetilde{\psi}' - \psi')\end{aligned} \tag{3.2}$$

where φ and φ' respectively are the phases in P and P$'$ on the real surfaces at the recording, where $\widetilde{\varphi}$ and $\widetilde{\varphi}'$ respectively are the phases in $\widetilde{\mathrm{P}}$ and $\widetilde{\mathrm{P}}'$ on the image surfaces, where ψ and ψ' respectively are the phases in the reference sources Q and Q$'$, where $p, p', \widetilde{p}$ and $\widetilde{p}'$ are the lengths between the object surface and the hologram(s), and where $q, q', \widetilde{q}$ and $\widetilde{q}'$ are the length between the reference sources Q and Q′ and the hologram(s). As a convention, the sign $'$ (prime) describes a quantity in a deformed configuration and the sign $\widetilde{}$ (tilde) a quantity in a modified configuration. In the case of real-time, we have $\widetilde{\varphi}' = \varphi'$ because the deformed object surface is identical with its image. The optical path difference D_d for the double exposure and D_r for the real-time in some point $\widetilde{\mathrm{K}}$ in the space $\mathbb{R}^3$ respectively are defined as follows

$$\begin{aligned} D_d &= \left(\frac{\lambda}{2\pi}\widetilde{\varphi} - \widetilde{L}\right) - \left(\frac{\lambda}{2\pi}\widetilde{\varphi}' - \widetilde{L}'\right) \\ D_r &= \left(\frac{\lambda}{2\pi}(\widetilde{\varphi} - \pi) - \widetilde{L}\right) - \left(\frac{\lambda}{2\pi}\widetilde{\varphi}' - \widetilde{L}'\right) \end{aligned} \tag{3.3}$$

and the phase relations reads

$$\begin{aligned} \varphi &= \frac{2\pi}{\lambda}(L_{TS} + L_S) + \varphi_T \\ \varphi' &= \frac{2\pi}{\lambda}(L_{TS} + L'_S) + \varphi'_T \end{aligned} \quad ; \quad \begin{aligned} \psi &= \frac{2\pi}{\lambda}L_{TQ} + \varphi_T \\ \psi' &= \frac{2\pi}{\lambda}L'_{TQ} + \varphi'_T \end{aligned} \quad ; \quad \begin{aligned} \widetilde{\psi} &= \frac{2\pi}{\lambda}L_{TQ} + \widetilde{\varphi}_T + \Delta\widetilde{\psi} \\ \widetilde{\psi}' &= \frac{2\pi}{\lambda}L'_{TQ} + \widetilde{\varphi}_T \end{aligned} \tag{3.4}$$

where φ_T, φ'_T and $\widetilde{\varphi}_T$ respectively are the phases at the first beam-splitter at recording (undeformed and deformed configuration) and at reconstruction. The phases $\widetilde{\psi}$ and $\widetilde{\psi}'$ respectively are the phases in the “new” point sources $\widetilde{\mathrm{Q}}$ and $\widetilde{\mathrm{Q}}' = \mathrm{Q}'$ at reconstruction. The phase increment $\Delta\widetilde{\psi}$ is used at reconstruction in the phase-shifting method. In case of the real-time technique, we have $\varphi'_T = \widetilde{\varphi}_T$.

Because no modification is performed on the deformed configuration, the point P′ is identical with its image $\widetilde{\mathrm{P}}'$ and we have $\widetilde{p}' = p'$ and $\widetilde{q}' = q'$. Introducing equations (3.4) in equations (3.3), we find the same expression $D = D_d = D_r$ for the optical path difference of the double exposure and the real-time technique, both expressed as a sum of length differences.

$$D = \lambda\nu = \frac{\lambda}{2\pi}\Phi = (\widetilde{L}' - \widetilde{L}) - (L'_S - L_S) + (\widetilde{q} - q) - (\widetilde{p} - p) + \frac{\lambda}{2\pi}\Delta\widetilde{\psi} \tag{3.5}$$

where ν is the so-called fringe order. Assuming a large deformation and a large modification, we can develop the length $L'_S, \widetilde{q}$ and $\widetilde{p}$ in equation (3.5) up to the second-order nonlinear terms as follows

$$\begin{aligned} L'_S &= L_S + \mathbf{u}\cdot\mathbf{h} + \frac{1}{2L_S}\mathbf{u}\cdot\mathbf{H}\mathbf{u} + \ldots && ; \quad |\mathbf{u}| \ll L_S \\ \widetilde{q} &= q + \widetilde{\mathbf{c}}\cdot(\mathbf{t}-\mathbf{d}) - \frac{1}{2\widetilde{q}}(\mathbf{t}-\mathbf{d})\cdot\widetilde{\mathbf{C}}(\mathbf{t}-\mathbf{d}) + \ldots && ; \quad |\mathbf{t}|, |\mathbf{d}| \ll \widetilde{q} \\ \frac{1}{\widetilde{q}} &= \frac{1}{q} - \frac{1}{q^2}\widetilde{\mathbf{c}}\cdot(\mathbf{t}-\mathbf{d}) + \frac{1}{q^3}\left[\frac{1}{2}(\mathbf{t}-\mathbf{d})\cdot\widetilde{\mathbf{C}}(\mathbf{t}-\mathbf{d}) + [\widetilde{\mathbf{c}}\cdot(\mathbf{t}-\mathbf{d})]^2\right] + \ldots \\ \widetilde{p} &= p + \widetilde{\mathbf{k}}\cdot(\mathbf{t}-\mathbf{v}) - \frac{1}{2\widetilde{p}}(\mathbf{t}-\mathbf{v})\cdot\widetilde{\mathbf{K}}(\mathbf{t}-\mathbf{v}) + \ldots && ; \quad |\mathbf{t}|, |\mathbf{v}| \ll \widetilde{p} \\ \frac{1}{\widetilde{p}} &= \frac{1}{p} - \frac{1}{p^2}\widetilde{\mathbf{k}}\cdot(\mathbf{t}-\mathbf{v}) + \frac{1}{p^3}\left[\frac{1}{2}(\mathbf{t}-\mathbf{v})\cdot\widetilde{\mathbf{K}}(\mathbf{t}-\mathbf{v}) + [\widetilde{\mathbf{k}}\cdot(\mathbf{t}-\mathbf{v})]^2\right] + \ldots \end{aligned} \tag{3.6}$$

where

$$\mathbf{H} = \mathbf{I} - \mathbf{h}\otimes\mathbf{h} \quad ; \quad \widetilde{\mathbf{C}} = \mathbf{I} - \widetilde{\mathbf{c}}\otimes\widetilde{\mathbf{c}} \quad ; \quad \widetilde{\mathbf{K}} = \mathbf{I} - \widetilde{\mathbf{k}}\otimes\widetilde{\mathbf{k}} \tag{3.7}$$

respectively are normal projectors relative to the directions $\mathbf{h}, \widetilde{\mathbf{c}}$ and $\widetilde{\mathbf{k}}$. The observing direction is given by the unit vector $\widetilde{\mathbf{k}}$, the illuminating direction by the unit vector $\mathbf{h}$ ($\mathbf{h}'$ for the deformed object surface), and the direction of the corresponding reference ray at reconstruction by the unit vector $\widetilde{\mathbf{c}}$. The vector $\mathbf{u}$ represents the unknown displacement of the surface point P to the new position $\mathrm{P}' = \widetilde{\mathrm{P}}'$, the modification vector $\mathbf{t}$ represents the displacement of the hologram point $\widehat{\mathrm{H}}$ to the new position $\widetilde{\mathrm{H}}$ and the modification vector $\mathbf{d}$ represents the displacement of the reference source Q in the new position $\widetilde{\mathrm{Q}}$. The aberration vector $\mathbf{v}$ is the displacement of the object surface point P to the image point $\widetilde{\mathrm{P}}$ caused by both the modification of the reference source and the hologram.

As it has been observed, a contribution to the interference phenomenon in point $\widetilde{\mathrm{K}}$ is given by the whole bundle of rays coming from $\widetilde{\mathrm{K}}$ and entering the aperture $\breve{A}$ of our observing system, which is assumed to be small compared to the distances involved in our holographic setup. In order that both image areas $\widetilde{A}$ and $\widetilde{A}'$ around the points $\widetilde{\mathrm{P}}$ and $\widetilde{\mathrm{P}}'$ on the object image surfaces contribute to the interference, the rays coming from $\widetilde{\mathrm{K}}$ and going through the corresponding points $\widetilde{\mathrm{P}}$ and $\widetilde{\mathrm{P}}'$ must enter the aperture of the observing system [3.45]. It follows that both areas $\widetilde{A}$ and $\widetilde{A}'$ on the object image surfaces, which are related to the aperture $\breve{A}$ of the optical system, must overlap respectively to the observing direction $\widetilde{\mathbf{k}}$ while remaining very small (same order as the aperture). With the aperture radius $\breve{r}$ and the small vector $\breve{\mathbf{w}}$ in the aperture plane, which connect both corresponding rays coming from $\widetilde{\mathrm{K}}$, we have to consider the overlapping condition

$$|\breve{\mathbf{w}}| \ll \breve{r} \qquad \Rightarrow \qquad \left|\frac{\widetilde{\mathbf{K}}\widetilde{\mathbf{u}}}{\widetilde{L}}\right| \simeq \left|\frac{\widetilde{\mathbf{K}}\breve{\mathbf{w}}}{\widetilde{\ell}+\widetilde{p}+\widetilde{L}}\right| \ll \frac{\breve{r}}{\widetilde{\ell}+\widetilde{p}+\widetilde{L}} \qquad ; \qquad |\widetilde{\mathbf{u}}| \ll \ell_0 \tag{3.8}$$

where $\widetilde{\mathbf{K}}\widetilde{\mathbf{u}}$ is the lateral offset of the image points $\widetilde{\mathrm{P}}$ and $\widetilde{\mathrm{P}}$ contained in the so-called superposition vector $\widetilde{\mathbf{f}}_S = \widetilde{\mathbf{K}}\widetilde{\mathbf{u}}$ and where ℓ_0 is some characteristic length of the setup. Equation (3.8) gives a condition on the order of magnitude of the vector $\widetilde{\mathbf{K}}\widetilde{\mathbf{u}}$ respectively to the length $\widetilde{L}$ and the vector $\widetilde{\mathbf{K}}\breve{\mathbf{w}}$. Assuming now in our further considerations that both rays $\widetilde{\mathrm{K}}\widetilde{\mathrm{P}}$ and $\widetilde{\mathrm{K}}\widetilde{\mathrm{P}}'$ enter the aperture of the observing system, we can develop the length $\widetilde{L}'$ relatively to the length $\widetilde{L}$ up to the second-order nonlinear terms as follows

$$\widetilde{L}' = \widetilde{L} + \widetilde{\mathbf{u}}\cdot\widetilde{\mathbf{k}} + \frac{1}{2\widetilde{L}}\widetilde{\mathbf{u}}\cdot\widetilde{\mathbf{K}}\widetilde{\mathbf{u}} + \ldots = \widetilde{L} + \widetilde{\mathbf{f}}_L\cdot\widetilde{\mathbf{k}} + \frac{1}{2\widetilde{L}}\widetilde{\mathbf{f}}_S\cdot\widetilde{\mathbf{f}}_S + \ldots \tag{3.9}$$

where $\widetilde{\mathbf{u}}\cdot\widetilde{\mathbf{k}}$ is the longitudinal offset contained in the so-called longitudinal superposition vector $\widetilde{\mathbf{f}}_L = (\widetilde{\mathbf{u}}\cdot\widetilde{\mathbf{k}})\widetilde{\mathbf{k}}$. Introducing equations (3.6) and the above development (3.9) in equation (3.5), we get with $\widetilde{\mathbf{u}} = \mathbf{u} - \mathbf{v}$

$$\begin{aligned} D = \mathbf{u}\cdot\widetilde{\mathbf{g}} - \mathbf{t}\cdot(\widetilde{\mathbf{k}} - \widetilde{\mathbf{c}}) - \mathbf{d}\cdot\widetilde{\mathbf{c}} - \frac{1}{2L_S}\mathbf{u}\cdot\mathbf{H}\mathbf{u} - \frac{1}{2\widetilde{q}}(\mathbf{t}-\mathbf{d})\cdot\widetilde{\mathbf{C}}(\mathbf{t}-\mathbf{d}) \\ + \frac{1}{2p}(\mathbf{t}-\mathbf{v})\cdot\widetilde{\mathbf{K}}(\mathbf{t}-\mathbf{v}) + \frac{1}{2\widetilde{L}}\widetilde{\mathbf{f}}_S\cdot\widetilde{\mathbf{f}}_S + \frac{\lambda}{2\pi}\Delta\widetilde{\psi} + \ldots \end{aligned} \tag{3.10}$$

where $\widetilde{\mathbf{g}} = \widetilde{\mathbf{k}} - \mathbf{h}$ is the modified sensitivity vector and where the vector $\widetilde{\mathbf{K}}\mathbf{v}$ represents the unknown lateral aberration of the image point $\widetilde{\mathrm{P}}$ relatively to P caused by our geometrical modification.

3.3 Aberration and astigmatism of holographic images

In equation (3.10), the lateral aberration $\widetilde{\mathbf{K}}\mathbf{v}$ appears in the second-order nonlinear terms and can be determined according to the Bragg condition for thin holograms. Assuming the existence of an image point $\widetilde{\mathrm{P}}$, which can be easily verified in an experiment, the interference identity equation (3.2) must have a stationary behavior up to the second derivative in a small area (related to the aperture of the optical system) around the corresponding hologram point $\overset{\frown}{\mathrm{H}}$ (Fig. 3.2). With the constant phases $\varphi, \widetilde{\varphi}, \psi$ and $\widetilde{\psi}$, we write the stationary behavior of the phase difference function $\Theta_P = 2\pi[(\widetilde{p} - \widetilde{q}) - (p - q)]/\lambda$ for both points P and $\widetilde{\mathrm{P}}$ assumed to be fixed in space. By setting the first total differential equal to zero, we get with $\widetilde{p} = \widetilde{p}(\overset{\frown}{\mathbf{r}}), \widetilde{q} = \widetilde{q}(\overset{\frown}{\mathbf{r}}), p = p(\widehat{\mathbf{r}})$ and $q = q(\widehat{\mathbf{r}})$

$$0 = \mathrm{d}\Theta_P = \frac{2\pi}{\lambda}[\mathrm{d}\overset{\frown}{\mathbf{r}} \cdot \boldsymbol{\nabla}_{\overset{\frown}{n}}(\widetilde{p} - \widetilde{q}) - \mathrm{d}\widehat{\mathbf{r}} \cdot \boldsymbol{\nabla}_{\hat{n}}(p - q)] = \frac{2\pi}{\lambda}[\mathrm{d}\overset{\frown}{\mathbf{r}} \cdot \overset{\frown}{\mathbf{N}}(\widetilde{\mathbf{k}} - \widetilde{\mathbf{c}}) - \mathrm{d}\widehat{\mathbf{r}} \cdot \widehat{\mathbf{N}}(\mathbf{k} - \mathbf{c})] \quad (3.11)$$

where $\mathrm{d}\widehat{\mathbf{r}} = \widehat{\mathbf{N}}\mathrm{d}\widehat{\mathbf{r}}$ and $\mathrm{d}\overset{\frown}{\mathbf{r}} = \overset{\frown}{\mathbf{N}}\mathrm{d}\overset{\frown}{\mathbf{r}}$ respectively are the vector increments (or first differentials) on the hologram before and after modification [3.29]. The 2-dimensional derivative operators $\boldsymbol{\nabla}_{\hat{n}}$ and $\boldsymbol{\nabla}_{\overset{\frown}{n}}$ enable us to compute the gradients of the lengths on the hologram planes, which lead to units vectors associated with their normal projectors. As we can see in the second part of equation (3.11), we have $\boldsymbol{\nabla}_{\overset{\frown}{n}}(\widetilde{p} - \widetilde{q}) = \overset{\frown}{\mathbf{N}}(\widetilde{\mathbf{k}} - \widetilde{\mathbf{c}})$ and $\boldsymbol{\nabla}_{\hat{n}}(p - q) = \widehat{\mathbf{N}}(\mathbf{k} - \mathbf{c})$. The two normal projectors $\widehat{\mathbf{N}} = \mathbf{I} - \widehat{\mathbf{n}} \otimes \widehat{\mathbf{n}}$ and $\overset{\frown}{\mathbf{N}} = \mathbf{I} - \overset{\frown}{\mathbf{n}} \otimes \overset{\frown}{\mathbf{n}}$ respectively project onto the hologram plane before and after modification. The two unit vectors $\widehat{\mathbf{n}}$ and $\overset{\frown}{\mathbf{n}}$ are normal to their corresponding hologram plane.

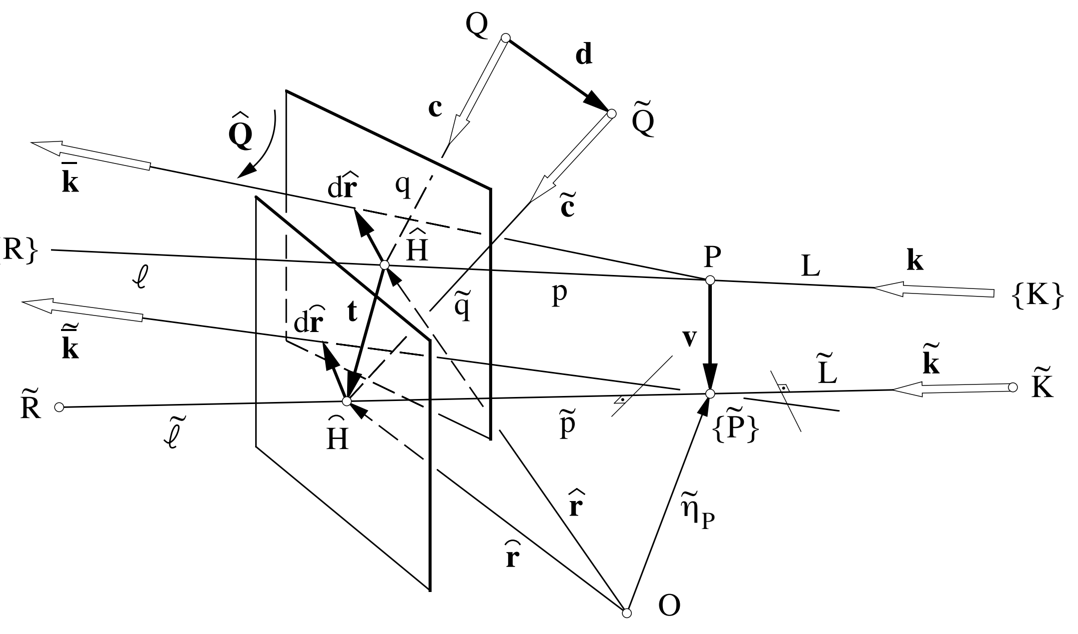

Fig. 3.2: Aberration and astigmatism of an image point due to a modification at reconstruction

Because of the rigid body motion of the hologram, both vector increments $\mathrm{d}\widehat{\mathbf{r}}$ and $\mathrm{d}\overset{\frown}{\mathbf{r}}$ are related by the following relations

$$\mathrm{d}\widehat{\mathbf{r}} = \widehat{\mathbf{Q}}^T \mathrm{d}\overset{\frown}{\mathbf{r}} = \mathrm{d}\overset{\frown}{\mathbf{r}}\widehat{\mathbf{Q}} \qquad ; \qquad \mathrm{d}\overset{\frown}{\mathbf{r}} = \widehat{\mathbf{Q}}\mathrm{d}\widehat{\mathbf{r}} = \mathrm{d}\widehat{\mathbf{r}}\widehat{\mathbf{Q}}^T \quad (3.12)$$

where $\widehat{\mathbf{Q}}$ is the orthogonal tensor describing the rotation of the hologram, that means $\widehat{\mathbf{Q}}^T\widehat{\mathbf{Q}} = \mathbf{I}$ or $\widehat{\mathbf{Q}}^{-1} = \widehat{\mathbf{Q}}^T$. It follows that we also have $\widehat{\mathbf{n}} = \widehat{\mathbf{Q}}^T\widehat{\mathbf{n}}$, $\widehat{\mathbf{n}} = \widehat{\mathbf{Q}}\widehat{\mathbf{n}}$, $\widehat{\mathbf{N}} = \widehat{\mathbf{Q}}^T\widehat{\mathbf{N}}\widehat{\mathbf{Q}}$ and $\widehat{\mathbf{N}} = \widehat{\mathbf{Q}}\widehat{\mathbf{N}}\widehat{\mathbf{Q}}^T$. For moderate and large rotations, the tensor $\widehat{\mathbf{Q}}$, which is still close to the indentity tensor $\mathbf{I}$, can be developed up to the second-order terms as follows

$$\widehat{\mathbf{Q}} = \mathbf{I} + (\widehat{\mathbf{Q}} - \mathbf{I}) = \mathbf{I} - \hat{\chi}\widehat{\mathbf{E}}_H + \frac{1}{2}\hat{\chi}^2\widehat{\mathbf{E}}_H^2 + \ldots \tag{3.13}$$

where $\widehat{\mathbf{E}}_H = -\widehat{\mathbf{E}}_H^T$ is the 2-dimensional antimetric tensor describing a $-\pi/2$-rotation in a plane normal to the rotation axis $\widehat{\Delta}_H$ of unit direction vector $\widehat{\mathbf{n}}_H$ and where $\hat{\chi}$ is the exact angle of rotation around the rotation axis in radians. The tensor $\widehat{\mathbf{E}}_H^2 = (\widehat{\mathbf{E}}_H^2)^T$ is symmetric and can also be written in the form of a negative normal projection $\widehat{\mathbf{E}}_H^2 = -\widehat{\mathbf{N}}_H = -\mathbf{I} + \widehat{\mathbf{n}}_H \otimes \widehat{\mathbf{n}}_H$ onto the plane normal to $\widehat{\mathbf{n}}_H$. Remember that $\widehat{\mathbf{E}}_H = \boldsymbol{\mathcal{E}}\widehat{\mathbf{n}}_H$ can be determined by contracting the vector $\widehat{\mathbf{n}}_H$ with the so-called third-rank permutation tensor $\boldsymbol{\mathcal{E}}$. It follows that the rotation is completely described by the exact rotation vector $\widehat{\boldsymbol{\omega}}_H = \hat{\chi}\widehat{\mathbf{n}}_H$.

Introducing equation (3.12) in equation (3.11), which must be valid for all vector increments $\mathrm{d}\widehat{\mathbf{r}}$ and $\mathrm{d}\widehat{\mathbf{r}}$, we get the well known ray-tracing equations

$$\widehat{\mathbf{N}}[\widehat{\mathbf{Q}}^T(\widetilde{\mathbf{k}} - \widetilde{\mathbf{c}}) - (\mathbf{k} - \mathbf{c})] = 0 \qquad ; \qquad \widehat{\mathbf{N}}[(\widetilde{\mathbf{k}} - \widetilde{\mathbf{c}}) - \widehat{\mathbf{Q}}(\mathbf{k} - \mathbf{c})] = 0 \tag{3.14}$$

which have the same meaning but are expressed either relatively to the unmodified or to the modified hologram. For a given modification, because all the units vectors $\widetilde{\mathbf{c}}$, $\mathbf{c}$, $\widehat{\mathbf{n}}$, $\widehat{\mathbf{n}}$ and the rotation tensor $\widehat{\mathbf{Q}}$ can be measured, equations (3.14) give, together with the auxiliary conditions $|\mathbf{k}| = |\widetilde{\mathbf{k}}| = 1$, a relation between the observing direction $\mathbf{k}$ at recording and the corresponding observing direction $\widetilde{\mathbf{k}}$ at reconstruction.

Because our modification can be large, we develop the directions $\mathbf{k}$ and $\mathbf{c}$ respectively to the configuration at reconstruction up to the second-order terms as follows

$$\begin{aligned} \mathbf{k} &= \widetilde{\mathbf{k}} - \frac{1}{\widetilde{p}}\widetilde{\mathbf{K}}(\mathbf{t} - \mathbf{v}) - \frac{1}{2\widetilde{p}^2}(\mathbf{t} - \mathbf{v})\widetilde{\boldsymbol{\mathcal{K}}}(\mathbf{t} - \mathbf{v}) + \ldots \\ \mathbf{c} &= \widetilde{\mathbf{c}} - \frac{1}{\widetilde{q}}\widetilde{\mathbf{C}}(\mathbf{t} - \mathbf{d}) - \frac{1}{2\widetilde{q}^2}(\mathbf{t} - \mathbf{d})\widetilde{\boldsymbol{\mathcal{C}}}(\mathbf{t} - \mathbf{d}) + \ldots \end{aligned} \tag{3.15}$$

with the two superprojectors (triadics) $\widetilde{\boldsymbol{\mathcal{K}}} = \widetilde{\mathbf{K}} \otimes \widetilde{\mathbf{k}} + \widetilde{\mathbf{K}} \otimes \widetilde{\mathbf{k}})^T + \widetilde{\mathbf{k}} \otimes \widetilde{\mathbf{K}}$ and $\widetilde{\boldsymbol{\mathcal{C}}} = \widetilde{\mathbf{C}} \otimes \widetilde{\mathbf{c}} + \widetilde{\mathbf{C}} \otimes \widetilde{\mathbf{c}})^T + \widetilde{\mathbf{c}} \otimes \widetilde{\mathbf{C}}$. By eliminating the lengths $\widetilde{p}$ and $\widetilde{q}$ with equations (3.6), equations (3.15) read

$$\begin{aligned} \mathbf{k} &= \widetilde{\mathbf{k}} - \frac{1}{p}\widetilde{\mathbf{K}}(\mathbf{t} - \mathbf{v}) - \frac{1}{2p^2}[(\mathbf{t} - \mathbf{v}) \cdot \widetilde{\mathbf{K}}(\mathbf{t} - \mathbf{v})]\widetilde{\mathbf{k}} + \ldots \\ \mathbf{c} &= \widetilde{\mathbf{c}} - \frac{1}{q}\widetilde{\mathbf{C}}(\mathbf{t} - \mathbf{d}) - \frac{1}{2q^2}[(\mathbf{t} - \mathbf{d}) \cdot \widetilde{\mathbf{C}}(\mathbf{t} - \mathbf{d})]\widetilde{\mathbf{c}} + \ldots \end{aligned} \tag{3.16}$$

Considering the configuration at reconstruction, we define the non-symmetric oblique projector $\widehat{\mathbf{M}} = \mathbf{I} - \widehat{\mathbf{n}} \otimes \widetilde{\mathbf{k}}/\widehat{\mathbf{n}} \cdot \widetilde{\mathbf{k}}$, which acts as a projection along the direction $\widehat{\mathbf{n}}$ onto a plane perpendicular to the observing direction $\widetilde{\mathbf{k}}$ (if applied from the left onto an arbitrary vector on its right). Its transpose $\widehat{\mathbf{M}}^T = \mathbf{I} - \widetilde{\mathbf{k}} \otimes \widehat{\mathbf{n}}/\widetilde{\mathbf{k}} \cdot \widehat{\mathbf{n}}$ is also an oblique projector and acts as a projection along the observing direction $\widetilde{\mathbf{k}}$ onto a plane perpendicular to the unit normal $\widehat{\mathbf{n}}$ of the hologram.

Introducing equations (3.16) in the right equation (3.14), we can write the lateral aberration $\widetilde{\mathbf{K}}\mathbf{v}$ up to the second-order terms as follows

$$\frac{1}{p}\widetilde{\mathbf{K}}(\mathbf{t}-\mathbf{v}) = \widehat{\mathbf{M}}\widetilde{\mathbf{v}}_1 + \widehat{\mathbf{M}}\widetilde{\mathbf{v}}_2 \qquad \Rightarrow \qquad \widetilde{\mathbf{K}}\mathbf{v} = \widetilde{\mathbf{K}}\mathbf{t} - p\widehat{\mathbf{M}}\widetilde{\mathbf{v}}_1 - p\widehat{\mathbf{M}}\widetilde{\mathbf{v}}_2 \tag{3.17}$$

with

$$\begin{aligned}\widetilde{\mathbf{v}}_1 &= \frac{1}{\widetilde{q}}\widetilde{\mathbf{C}}(\mathbf{t}-\mathbf{d}) + (\widehat{\mathbf{Q}}-\mathbf{I})(\widetilde{\mathbf{k}}-\widetilde{\mathbf{c}}) \\ \widetilde{\mathbf{v}}_2 &= (\widehat{\mathbf{Q}}-\mathbf{I})\left(\frac{1}{\widetilde{q}}(\mathbf{I}-\widehat{\mathbf{M}})\widetilde{\mathbf{C}}(\mathbf{t}-\mathbf{d}) - \widehat{\mathbf{M}}(\widehat{\mathbf{Q}}-\mathbf{I})(\widetilde{\mathbf{k}}-\widetilde{\mathbf{c}})\right) \\ &\quad + \frac{1}{2}\widehat{\mathbf{Q}}\left(\frac{1}{\widetilde{q}^2}(\mathbf{t}-\mathbf{d})\widetilde{\boldsymbol{\mathcal{C}}}(\mathbf{t}-\mathbf{d}) - (\widetilde{\mathbf{v}}_1\cdot\widehat{\mathbf{M}}^T\widehat{\mathbf{M}}\widetilde{\mathbf{v}}_1)\widetilde{\mathbf{k}}\right) + \ldots\end{aligned} \tag{3.18}$$

where $|\widetilde{\mathbf{v}}_2| \ll |\widetilde{\mathbf{v}}_1|$, $\widehat{\mathbf{M}}\widehat{\mathbf{N}} = \widehat{\mathbf{M}}$ and $\widehat{\mathbf{M}}\widetilde{\mathbf{K}} = \widetilde{\mathbf{K}}$. The first and second-order terms are explicitly defined as function of the modification $\mathbf{t}$, $\mathbf{d}$ and $\widehat{\mathbf{Q}}$.

Introducing equation (3.17) in equation (3.10), the optical path difference reads

$$\begin{aligned}D &= \mathbf{u}\cdot\widetilde{\mathbf{g}} - \mathbf{t}\cdot(\widetilde{\mathbf{k}}-\widetilde{\mathbf{c}}) - \mathbf{d}\cdot\widetilde{\mathbf{c}} - \frac{1}{2L_S}\mathbf{u}\cdot\mathbf{H}\mathbf{u} - \frac{1}{2\widetilde{q}}(\mathbf{t}-\mathbf{d})\cdot\widetilde{\mathbf{C}}(\mathbf{t}-\mathbf{d}) \\ &\quad + \frac{p}{2}\widetilde{\mathbf{v}}_1\cdot\widehat{\mathbf{M}}^T\widehat{\mathbf{M}}\widetilde{\mathbf{v}}_1 + \frac{1}{2\widetilde{L}}\widetilde{\mathbf{f}}_S\cdot\widetilde{\mathbf{f}}_S + \frac{\lambda}{2\pi}\Delta\widetilde{\psi} + \ldots\end{aligned} \tag{3.19}$$

and can be used to calculate the displacement $\mathbf{u}$ from the interference fringe pattern with more accuracy by still using conventional methods.

By setting the second total differential of the function Θ_P equal to zero, we get

$$\begin{aligned}0 = \frac{\lambda}{2\pi}\mathrm{d}^2\Theta_P &= \mathrm{d}\widehat{\mathbf{r}}\cdot[\boldsymbol{\nabla}_{\widehat{\mathrm{n}}}\otimes\boldsymbol{\nabla}_{\widehat{\mathrm{n}}}(\widetilde{p}-\widetilde{q})]\mathrm{d}\widehat{\mathbf{r}} + \mathrm{d}^2\widehat{\mathbf{r}}\cdot\boldsymbol{\nabla}_{\widehat{\mathrm{n}}}(\widetilde{p}-\widetilde{q}) \\ &\quad - \mathrm{d}\widehat{\mathbf{r}}\cdot[\boldsymbol{\nabla}_{\hat{n}}\otimes\boldsymbol{\nabla}_{\hat{n}}(p-q)]\mathrm{d}\widehat{\mathbf{r}} - \mathrm{d}^2\widehat{\mathbf{r}}\cdot\boldsymbol{\nabla}_{\hat{n}}(p-q) \\ &= \mathrm{d}\widehat{\mathbf{r}}\cdot\widehat{\mathbf{N}}\left(\frac{1}{\widetilde{p}}\widetilde{\mathbf{K}} - \frac{1}{\widetilde{q}}\widetilde{\mathbf{C}}\right)\widehat{\mathbf{N}}\mathrm{d}\widehat{\mathbf{r}} - \mathrm{d}\widehat{\mathbf{r}}\cdot\widehat{\mathbf{N}}\left(\frac{1}{p}\mathbf{K} - \frac{1}{q}\mathbf{C}\right)\widehat{\mathbf{N}}\mathrm{d}\widehat{\mathbf{r}} \\ &\quad + \mathrm{d}^2\widehat{\mathbf{r}}\cdot\widehat{\mathbf{N}}(\widetilde{\mathbf{k}}-\widetilde{\mathbf{c}}) - \mathrm{d}^2\widehat{\mathbf{r}}\cdot\widehat{\mathbf{N}}(\mathbf{k}-\mathbf{c})\end{aligned} \tag{3.20}$$

Because second derivatives of lengths or better to say gradients of unit vectors lead to normal projectors, we get the two normal projectors $\mathbf{K} = \mathbf{I}-\mathbf{k}\otimes\mathbf{k}$ and $\mathbf{C} = \mathbf{I}-\mathbf{c}\otimes\mathbf{c}$. The second differentials $\mathrm{d}^2\widehat{\mathbf{r}} = \widehat{\mathbf{N}}\mathrm{d}^2\widehat{\mathbf{r}}$ and $\mathrm{d}^2\widehat{\mathbf{r}} = \widehat{\mathbf{N}}\mathrm{d}^2\widehat{\mathbf{r}}$ of the position vectors $\widehat{\mathbf{r}}$ and $\widehat{\mathbf{r}}$ on the hologram planes before and after modification depend on the set of curvilinear coordinates on the object surface and are generally not equal to zero. Because of the rigid body motion of the hologram, the rotation tensor $\widehat{\mathbf{Q}}$ is constant on the hologram plane and the deformation gradient of the hologram surface reads $\widehat{\mathbf{F}}_H = \widehat{\mathbf{Q}}\widehat{\mathbf{N}} = \widehat{\mathbf{N}} + (\boldsymbol{\nabla}_{\hat{n}}\otimes\mathbf{t})^T$. Thus, the second differentials are related by the equation $\mathrm{d}^2\widehat{\mathbf{r}} = \widehat{\mathbf{Q}}\mathrm{d}^2\widehat{\mathbf{r}}$, which implies with equation (3.14) that the last two terms in equation (3.20) give zero. Introducing the variable unit vector $\widetilde{\mathbf{m}} = \widetilde{\mathbf{K}}\widetilde{\mathbf{m}}$ perpendicular to the observing direction $\widetilde{\mathbf{k}}$, we may write with the vector increment $\mathrm{d}\widetilde{\mathbf{k}} = \widetilde{\mathbf{m}}\mathrm{d}\widetilde{\phi}$ the affine connection $\mathrm{d}\widehat{\mathbf{r}} = \widetilde{p}\widehat{\mathbf{M}}^T\mathrm{d}\widetilde{\mathbf{k}} = \widetilde{p}\mathrm{d}\widetilde{\phi}\widetilde{\mathbf{m}}\widehat{\mathbf{M}}$. With equation (3.12) and $\widetilde{\mathbf{m}} \cdot \widehat{\mathbf{M}}\widetilde{\mathbf{K}}\widehat{\mathbf{M}}^T\widetilde{\mathbf{m}} = \widetilde{\mathbf{m}} \cdot \widetilde{\mathbf{K}}\widetilde{\mathbf{m}} = 1$, equation (3.20) becomes

$$-\frac{1}{\widetilde{p}} = \widetilde{\mathbf{m}} \cdot \widetilde{\mathbf{T}}\widetilde{\mathbf{m}} \quad ; \quad \widetilde{\mathbf{T}} = \widehat{\mathbf{M}}\left[-\frac{1}{\widetilde{q}}\widetilde{\mathbf{C}} - \widehat{\mathbf{Q}}\left(\frac{1}{p}\mathbf{K} - \frac{1}{q}\mathbf{C}\right)\widehat{\mathbf{Q}}^T\right]\widehat{\mathbf{M}}^T \tag{3.21}$$

where $\widetilde{\mathbf{T}}$ is the 2-dimensional symmetric curvature tensor of the wavefront at point $\widehat{\mathrm{H}}$ at reconstruction, which travels along *diverging* rays in the direction $\widetilde{\mathbf{k}}$ from the point $\widetilde{\mathrm{P}}$. As we can see, equation (3.21) describes an astigmatic interval $\{\widetilde{\mathrm{P}}\}$ while rotating $\widetilde{\mathbf{m}}$ in the plane normal to $\widetilde{\mathbf{k}}$. Both minimum and maximum values $-1/\widetilde{p}_1$ and $-1/\widetilde{p}_2$ corresponding to the directions $\widetilde{\mathbf{m}}_1$ and $\widetilde{\mathbf{m}}_2$ ($\widetilde{\mathbf{m}}_2 \perp \widetilde{\mathbf{m}}_1$) are the eigenvalues of $\widetilde{\mathbf{T}}$ defining the endpoints $\widetilde{\mathrm{P}}_1$ and $\widetilde{\mathrm{P}}_2$ of the astigmatic interval $\{\widetilde{\mathrm{P}}\}$. The two invariants of the curvature tensor $\widetilde{\mathbf{T}}$ of the wavefront, i.e. the mean curvature $\widetilde{H}$ and the Gaussian curvature $\widetilde{K}$, can be calculated as follows

$$\mathrm{tr}\,\widetilde{\mathbf{T}} = \widetilde{\mathbf{T}} \cdot \widetilde{\mathbf{K}} = -\frac{1}{\widetilde{p}_1} - \frac{1}{\widetilde{p}_2} = 2\widetilde{H} \quad ; \quad \det\,(\widetilde{\mathbf{T}} + \widetilde{\mathbf{k}} \otimes \widetilde{\mathbf{k}}) = -\frac{1}{2}\widetilde{\mathbf{T}} \cdot \widetilde{\mathbf{E}}_k\widetilde{\mathbf{T}}\widetilde{\mathbf{E}}_k = \frac{1}{\widetilde{p}_1\widetilde{p}_2} = \widetilde{K} \tag{3.22}$$

where $2\widetilde{H}$ represents the trace and $\widetilde{K}$ the determinant of $\widetilde{\mathbf{T}}$. The tensor $\widetilde{\mathbf{E}}_k = -\widetilde{\mathbf{E}}_k^T$ is a 2-dimensional antimetric tensor describing a $-\pi/2$-rotation in the plane normal to the observing direction $\widetilde{\mathbf{k}}$. By introducing the equations (3.6) and (3.17) in equation (3.21), we get

$$\widetilde{\mathbf{k}} \cdot \mathbf{v} = p + \frac{1}{\widetilde{\mathbf{m}} \cdot \widetilde{\mathbf{T}}\widetilde{\mathbf{m}}} + \widetilde{\mathbf{k}} \cdot \mathbf{t} + \frac{\widetilde{\mathbf{v}}_1 \cdot \widehat{\mathbf{M}}^T\widehat{\mathbf{M}}\widetilde{\mathbf{v}}_1}{2\widetilde{\mathbf{m}} \cdot \widetilde{\mathbf{T}}\widetilde{\mathbf{m}}} + \ldots \tag{3.23}$$

Equation (3.23) gives the longitudinal part $\widetilde{\mathbf{k}} \cdot \mathbf{v}$ up to the second-order terms of the point aberration $\mathbf{v}$ relatively to the observing direction $\widetilde{\mathbf{k}}$ by taking into account the astigmatism of point $\{\widetilde{\mathrm{P}}\}$. Because of this astigmatism, the length $\widetilde{p}$ varies within the astigmatic interval as function of the direction of $\widetilde{\mathbf{m}}$. As such, any arbitrary point on the ray of direction $\widetilde{\mathbf{k}}$ in the astigmatic interval can be chosen as a reference point for $\{\widetilde{\mathrm{P}}\}$ in our considerations. According to the mean curvature of the wavefront at $\widehat{\mathrm{H}}$, the position of the particular point $\widetilde{\mathrm{P}}_0$ in the astigmatic interval can be determined for example by writing its distance $\widetilde{p}_0$ to the hologram as follows

$$\frac{1}{2}\left(\frac{1}{\widetilde{p}_0}\widetilde{\mathbf{K}} + \widetilde{\mathbf{T}}\right) \cdot \widetilde{\mathbf{K}} = 0 \quad ; \quad \widetilde{\mathbf{K}} \cdot \widetilde{\mathbf{K}} = 2 \qquad \Rightarrow \qquad -\frac{1}{\widetilde{p}_0} = \frac{1}{2}\widetilde{\mathbf{T}} \cdot \widetilde{\mathbf{K}} = -\frac{1}{2}\left(\frac{1}{\widetilde{p}_1} + \frac{1}{\widetilde{p}_2}\right) \tag{3.24}$$

With the corresponding point aberration $\mathbf{v}_0$, equation (3.23) can be related to $\widetilde{\mathrm{P}}_0$ by writing

$$\widetilde{\mathbf{k}} \cdot \mathbf{v}_0 = p + \frac{2}{\widetilde{\mathbf{T}} \cdot \widetilde{\mathbf{K}}} + \widetilde{\mathbf{k}} \cdot \mathbf{t} + \frac{\widetilde{\mathbf{v}}_1 \cdot \widehat{\mathbf{M}}^T\widehat{\mathbf{M}}\widetilde{\mathbf{v}}_1}{\widetilde{\mathbf{T}} \cdot \widetilde{\mathbf{K}}} + \ldots \tag{3.25}$$

3.4 Superposition, visibility and fringe vectors

In the unpleasant case where no fringe pattern is present (because of our large deformation), we need first to recover the interference fringes before being able to apply equation (3.19). Fringe recovery can be achieved with a suitable geometrical modification of the hologram and the reference source at reconstruction [3.46–3.51]. In order to find this modification, we must now deal with the concepts of superposition, visibility and fringe vectors.

3.4.1 Superposition vectors

Considering the second-order terms, the superposition vector $\widetilde{\mathbf{f}}_S$ (Fig. 3.3) describing the lateral overlapping of the image points $\widetilde{\mathrm{P}}$ and $\widetilde{\mathrm{P}}'$ relatively to the observing direction $\widetilde{\mathbf{k}}$ reads with equation (3.17)

$$\begin{aligned}\widetilde{\mathbf{f}}_S &= \widetilde{\mathbf{K}}\widetilde{\mathbf{u}} = \widetilde{\mathbf{K}}\mathbf{u} - \widetilde{\mathbf{K}}\mathbf{v} = \widetilde{\mathbf{K}}(\mathbf{u}-\mathbf{t}) + p\widehat{\mathbf{M}}\widetilde{\mathbf{v}}_1 + p\widehat{\mathbf{M}}\widetilde{\mathbf{v}}_2 + \dots \\ &= \widetilde{\mathbf{K}}(\mathbf{u}-\mathbf{t}) + p\widehat{\mathbf{M}}\left[\frac{1}{\widetilde{q}}\widetilde{\mathbf{C}}(\mathbf{t}-\mathbf{d}) + (\widehat{\mathbf{Q}}-\mathbf{I})(\widetilde{\mathbf{k}}-\widetilde{\mathbf{c}})\right] \\ &\quad + p\widehat{\mathbf{M}}\left[(\widehat{\mathbf{Q}}-\mathbf{I})\left(\frac{1}{\widetilde{q}}(\mathbf{I}-\widehat{\mathbf{M}})\widetilde{\mathbf{C}}(\mathbf{t}-\mathbf{d}) - \widehat{\mathbf{M}}(\widehat{\mathbf{Q}}-\mathbf{I})(\widetilde{\mathbf{k}}-\widetilde{\mathbf{c}})\right)\right. \\ &\qquad \left. + \frac{1}{2}\widehat{\mathbf{Q}}\left(\frac{1}{\widetilde{q}^2}(\mathbf{t}-\mathbf{d})\widetilde{\boldsymbol{\mathcal{C}}}(\mathbf{t}-\mathbf{d}) - (\widetilde{\mathbf{v}}_1\cdot\widehat{\mathbf{M}}^T\widehat{\mathbf{M}}\widetilde{\mathbf{v}}_1)\widetilde{\mathbf{k}}\right)\right] + \dots\end{aligned} \tag{3.26}$$

Considering the second-order terms, the longitudinal superposition vector $\widetilde{\mathbf{f}}_L$ (Fig. 3.3) describing the longitudinal offset of the images relatively to the direction $\widetilde{\mathbf{k}}$ reads with equations (3.23)

$$\widetilde{\mathbf{f}}_L = (\widetilde{\mathbf{k}}\cdot\widetilde{\mathbf{u}})\widetilde{\mathbf{k}} = [\widetilde{\mathbf{k}}\cdot(\mathbf{u}-\mathbf{v})]\widetilde{\mathbf{k}} = \left[-p - \frac{1}{\widetilde{\mathbf{m}}\cdot\widetilde{\mathbf{T}}\widetilde{\mathbf{m}}} + \widetilde{\mathbf{k}}\cdot(\mathbf{u}-\mathbf{t}) - \frac{\widetilde{\mathbf{v}}_1\cdot\widehat{\mathbf{M}}^T\widehat{\mathbf{M}}\widetilde{\mathbf{v}}_1}{2\widetilde{\mathbf{m}}\cdot\widetilde{\mathbf{T}}\widetilde{\mathbf{m}}}\right]\widetilde{\mathbf{k}} + \dots \tag{3.27}$$

The modification must be chosen according to equation (3.8), which implies that the superposition vector remains very small, i.e. $\widetilde{\mathbf{f}}_S \simeq 0$. In this case, the small areas corresponding to the aperture of the optical system around the image points on the object surface lateraly overlap, which is one of the necessary (but alone not sufficient) conditions for the recovery of interference fringes. The longitudinal superposition vector $\widetilde{\mathbf{f}}_L$ can be used to "adjust" the longitudinal aberration and the astigmatism of point $\{\widetilde{\mathrm{P}}\}$. Introducing equation (3.26) in equation (3.19), the optical path difference reads

$$\begin{aligned}D &= \mathbf{u}\cdot\widetilde{\mathbf{g}} - \mathbf{t}\cdot(\widetilde{\mathbf{k}}-\widetilde{\mathbf{c}}) - \mathbf{d}\cdot\widetilde{\mathbf{c}} - \frac{1}{2L_S}\mathbf{u}\cdot\mathbf{H}\mathbf{u} - \frac{1}{2\widetilde{q}}(\mathbf{t}-\mathbf{d})\cdot\widetilde{\mathbf{C}}(\mathbf{t}-\mathbf{d}) \\ &\quad + \frac{1}{2p}(\mathbf{u}-\mathbf{t})\cdot\widetilde{\mathbf{K}}(\mathbf{u}-\mathbf{t}) + \widetilde{\mathbf{f}}_S\cdot\left[\frac{1}{2}\left(\frac{1}{p}+\frac{1}{\widetilde{\widetilde{L}}}\right)\widetilde{\mathbf{f}}_S - \frac{1}{p}\widetilde{\mathbf{K}}(\mathbf{u}-\mathbf{t})\right] + \frac{\lambda}{2\pi}\Delta\widetilde{\psi} + \dots\end{aligned} \tag{3.28}$$

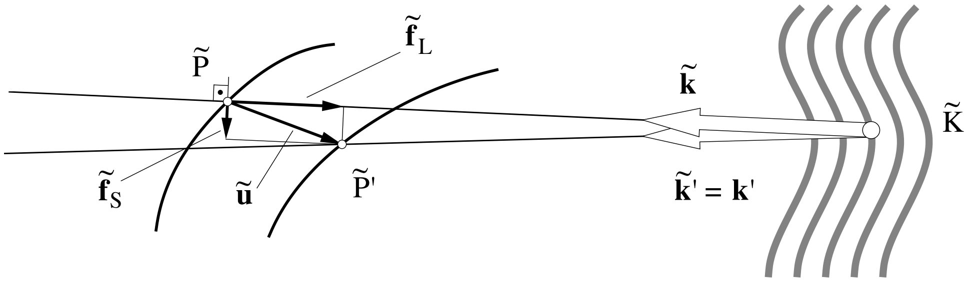

Fig. 3.3: Superposition vectors $\widetilde{\mathbf{f}}_S$ and $\widetilde{\mathbf{f}}_L$

3.4.2 Visibility and fringe vectors

For the recovery of interference fringes, two suplementary necessary conditions (as single condition not sufficient) may be found by writing the quasi-stationary behavior of the optical path difference D respectively to the collineation centers $\widetilde{\mathrm{K}}$ and $\widetilde{\mathrm{R}}$. This leads to the visibility vector $\widetilde{\mathbf{f}}_K$ and to the fringe vector $\widetilde{\mathbf{f}}_R$, which must be kept very small, i.e. $\widetilde{\mathbf{f}}_K \simeq 0$ and $\widetilde{\mathbf{f}}_R \simeq 0$. In order to see interference fringes, the intensity contribution in point $\widetilde{\mathrm{K}}$ of all rays included in the bundle going through the aperture of the optical system must be similar. This property is described by the first derivative of the optical path difference D relatively to the observing direction $\widetilde{\mathbf{k}}$ and the fixed point $\widetilde{\mathrm{K}}$ (collineation center) at reconstruction (Fig. 3.4a). This case will be treated as case (A).

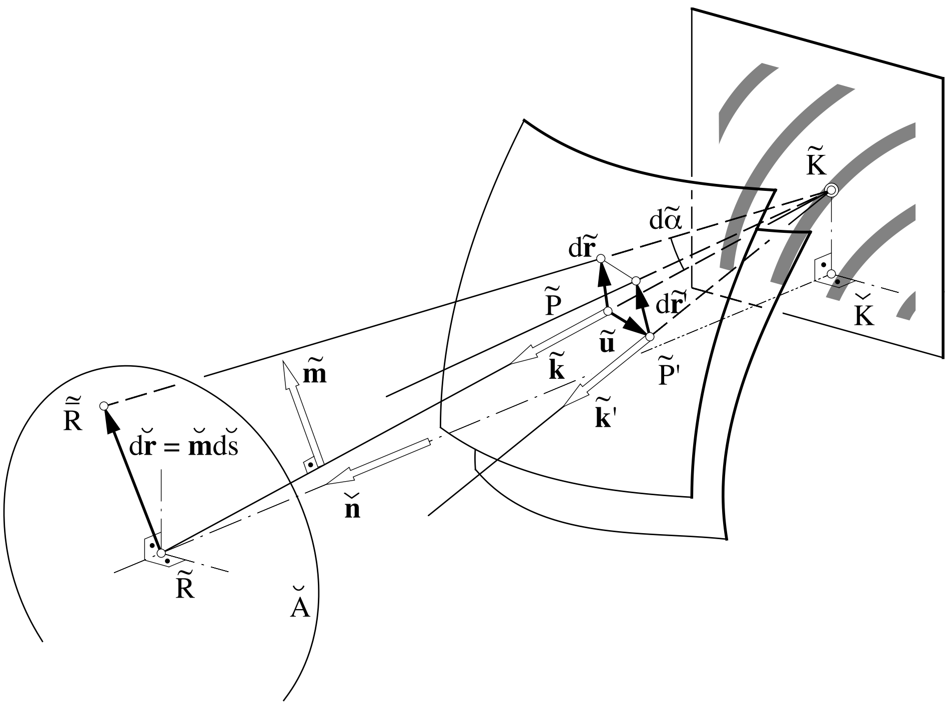

Fig. 3.4a: Case (A): Configuration at reconstruction with $\widetilde{\mathrm{K}}$ as collineation center

On another hand, considering the variable point $\widetilde{\widetilde{\mathrm{K}}}$ in the neighborhood of $\widetilde{\mathrm{K}}$ on the object plane of the optical system, the intensity contribution of the rays in $\widetilde{\widetilde{\mathrm{K}}}$ must be similar to that in $\widetilde{\mathrm{K}}$ in order to have a sufficient fringe spacing. This property is described by the first derivative of the optical path difference D relatively to the observing direction $\widetilde{\mathbf{k}}$ and the fixed point $\widetilde{\mathrm{R}}$ (collineation center) at reconstruction (Fig. 3.4b). This case will be treated as case (B).

For the recovery of interference fringes, we only need to consider the first-order linear terms in the derivatives of equations (3.10) or (3.19), whereas for the determination of the deformation, we have to consider the derivatives of equation (3.5) at least up to the second-order nonlinear terms.

By only considering the first-order linear terms in equation (3.19), the first total differential of the optical

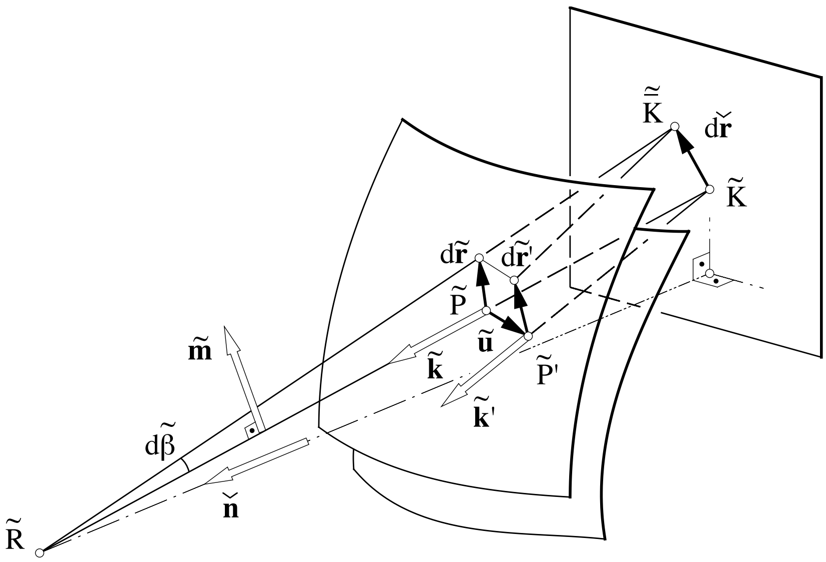

Fig. 3.4b: Case (B): Configuration at reconstruction with $\widetilde{\mathrm{R}}$ as collineation center

path difference reads in both cases (A) and (B)

$$\mathrm{d}D = \mathrm{d}\mathbf{u}\cdot(\widetilde{\mathbf{k}}-\mathbf{h}) + \mathbf{u}\cdot(\mathrm{d}\widetilde{\mathbf{k}}-\mathrm{d}\mathbf{h}) - \mathrm{d}\mathbf{t}\cdot(\widetilde{\mathbf{k}}-\widetilde{\mathbf{c}}) - \mathbf{t}\cdot(\mathrm{d}\widetilde{\mathbf{k}}-\mathrm{d}\widetilde{\mathbf{c}}) - \mathbf{d}\cdot\mathrm{d}\widetilde{\mathbf{c}} + \dots \tag{3.29}$$

with

$$\begin{aligned}
&\mathrm{d}\mathbf{u} = \mathrm{d}\mathbf{r}(\boldsymbol{\nabla}_n \otimes \mathbf{u}) &&; \quad \mathrm{d}\widetilde{\mathbf{k}} = \mathrm{d}\widehat{\mathbf{r}}(\boldsymbol{\nabla}_{\widehat{\mathrm{n}}} \otimes \widetilde{\mathbf{k}}) \\
&\mathrm{d}\widetilde{\mathbf{c}} = \mathrm{d}\widehat{\mathbf{r}}(\boldsymbol{\nabla}_{\widehat{\mathrm{n}}} \otimes \widetilde{\mathbf{c}}) = \frac{1}{\widetilde{q}}\mathrm{d}\widehat{\mathbf{r}}\widehat{\mathbf{N}}\widetilde{\mathbf{C}} &&; \quad \mathrm{d}\mathbf{h} = \mathrm{d}\mathbf{r}(\boldsymbol{\nabla}_n \otimes \mathbf{h}) = \frac{1}{L_S}\mathrm{d}\mathbf{r}\mathbf{N}\mathbf{H} \\
&\mathbf{t} = \widehat{\mathbf{r}} - \widehat{\mathbf{r}} &&; \quad \mathrm{d}\mathbf{t} = \mathrm{d}\widehat{\mathbf{r}} - \mathrm{d}\widehat{\mathbf{r}} = \mathrm{d}\widehat{\mathbf{r}}(\mathbf{I} - \widehat{\mathbf{Q}})
\end{aligned} \tag{3.30}$$

where the vector increments $\mathrm{d}\widetilde{\mathbf{k}}$, $\mathrm{d}\widetilde{\mathbf{c}}$ and $\mathrm{d}\mathbf{h}$ are the first total differentials respectively corresponding to the collineation centers $\widetilde{\mathrm{K}}$ in case (A) or $\widetilde{\mathrm{R}}$ in case (B), $\widetilde{\mathrm{C}}$ and S. Note that the vector increment $\mathrm{d}\mathbf{r} = \mathbf{N}\mathrm{d}\mathbf{r}$ is perpendicular to the unit normal $\mathbf{n}$ of the object surface. The first total differential $\mathrm{d}\mathbf{u}$ of the displacement vector $\mathbf{u}$ of a point P on the object surface contains the dilatation and the rotation of a surface element. For moderate deformations, the tensor $\boldsymbol{\nabla}_n \otimes \mathbf{u}$ reads

$$\boldsymbol{\nabla}_n \otimes \mathbf{u} = \widetilde{\boldsymbol{\gamma}} + \Omega\mathbf{E} + \boldsymbol{\omega}\otimes\mathbf{n} - \frac{1}{2}(\widetilde{\boldsymbol{\gamma}}^2 + \Omega^2\mathbf{N} + \boldsymbol{\omega}\otimes\boldsymbol{\omega}) + \Omega\widetilde{\boldsymbol{\gamma}}\mathbf{E} + \widetilde{\boldsymbol{\gamma}}\boldsymbol{\omega}\otimes\mathbf{n} + \frac{1}{2}\Omega\mathbf{E}\boldsymbol{\omega}\otimes\mathbf{n} \tag{3.31}$$

where $\widetilde{\boldsymbol{\gamma}}$ is the 2-dimensional symmetric surface strain tensor and $\boldsymbol{\omega}_S \equiv \chi\mathbf{n}_S \equiv \Omega\mathbf{n} + \mathbf{E}\boldsymbol{\omega}$ the exact rotation vector of a surface element (with $\boldsymbol{\omega} = \mathbf{N}\boldsymbol{\omega}$). The direction vector $\mathbf{n}_S$ is parallel to the rotation

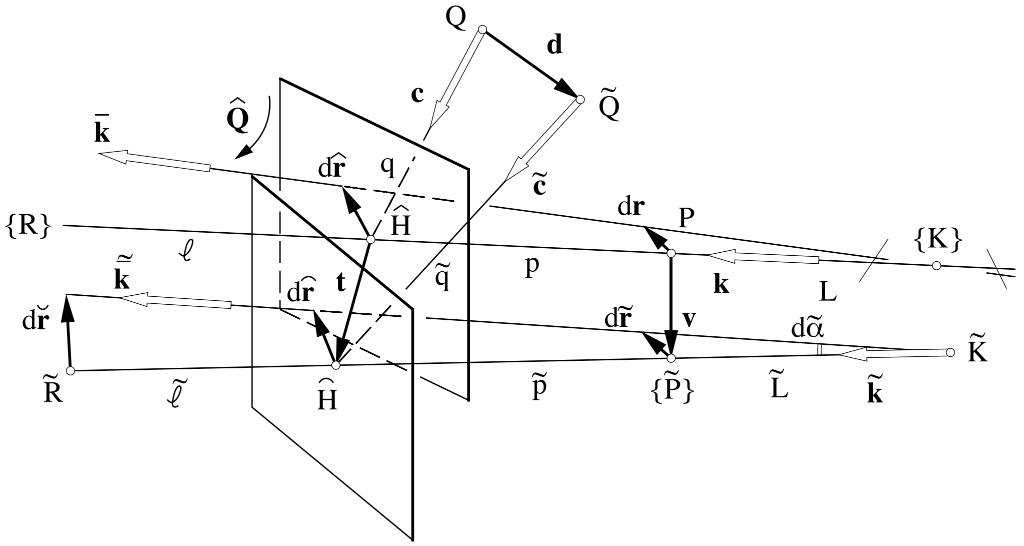

Fig. 3.5a: Case (A): Apparent image deformation of an point neighborhood with fixed point $\tilde{\mathrm{K}}$

axis Δ_S and χ is the angle of rotation in radians. The 2-dimensional interior antimetric tensor $\Omega\mathbf{E}$ first describes the "in-plane" rotation and the 2-dimensional semi-exterior tensor $\boldsymbol{\omega}\otimes\mathbf{n}$ first describes the "out-of-plane" rotation of a surface element.

In case (A), we have (Fig. 3.5a)

$$d\tilde{\mathbf{k}} = \frac{1}{\tilde{p}+\tilde{L}}\, d\hat{\mathbf{r}}\hat{\mathbf{N}}\tilde{\mathbf{K}} = \tilde{\mathbf{m}}\, d\tilde{\alpha} \qquad ; \qquad d\hat{\mathbf{r}} = (\tilde{p}+\tilde{L})\, d\tilde{\alpha}\,\tilde{\mathbf{m}}\hat{\mathbf{M}} = \frac{\tilde{p}+\tilde{L}}{\tilde{L}}\hat{\mathbf{M}}^T d\tilde{\mathbf{r}} \tag{3.32}$$

and in case (B), we have (Fig. 3.5b)

$$d\tilde{\mathbf{k}} = -\frac{1}{\tilde{\ell}}\, d\hat{\mathbf{r}}\hat{\mathbf{N}}\tilde{\mathbf{K}} = -\tilde{\mathbf{m}}\, d\tilde{\beta} \qquad ; \qquad d\hat{\mathbf{r}} = \tilde{\ell}\, d\tilde{\beta}\,\tilde{\mathbf{m}}\hat{\mathbf{M}} = \frac{\tilde{\ell}}{\tilde{\ell}+\tilde{p}}\hat{\mathbf{M}}^T d\tilde{\mathbf{r}} \tag{3.33}$$

We now have to explicit the vector increment $d\mathbf{r}$ on the object surface as function of the vector increment $d\hat{\mathbf{r}}$ on the hologram in both cases (A) and (B). This must be done by considering the first-order transverse ray aberration $d\boldsymbol{\eta}$ of the skewed rays in the corresponding virtual points $\{\mathrm{K}\}$ and $\{\mathrm{R}\}$ at recording. We write therefore in a plane perpendicular to $\mathbf{k}$ the first-order transverse ray aberration relatively to both normal projections $\mathbf{K}d\mathbf{r}$ and $\mathbf{K}d\hat{\mathbf{r}}$. In case (A), we have

$$d\boldsymbol{\eta}_K + L\, d\bar{\mathbf{k}} = \mathbf{K}d\mathbf{r} \qquad ; \qquad d\boldsymbol{\eta}_K + (L+p)\, d\bar{\mathbf{k}} = \mathbf{K}d\hat{\mathbf{r}} \tag{3.34}$$

and in case (B)

$$d\boldsymbol{\eta}_R - (\ell+p)\, d\bar{\mathbf{k}} = \mathbf{K}d\mathbf{r} \qquad ; \qquad d\boldsymbol{\eta}_R - \ell\, d\bar{\mathbf{k}} = \mathbf{K}d\hat{\mathbf{r}} \tag{3.35}$$

where $d\bar{\mathbf{k}} = \mathbf{K}d\bar{\mathbf{k}}$ represents the first-order vector increment between the skewed directions $\mathbf{k}$ and $\bar{\mathbf{k}}$ at recording (Figs. 3.5a and 3.5b). Because the corresponding rays are skew, no collineation center exists for the directions $\mathbf{k}$ and $\bar{\mathbf{k}}$. We therefore write $d\bar{\mathbf{k}}$ (instead of $d\mathbf{k}$) to remind us that corresponding

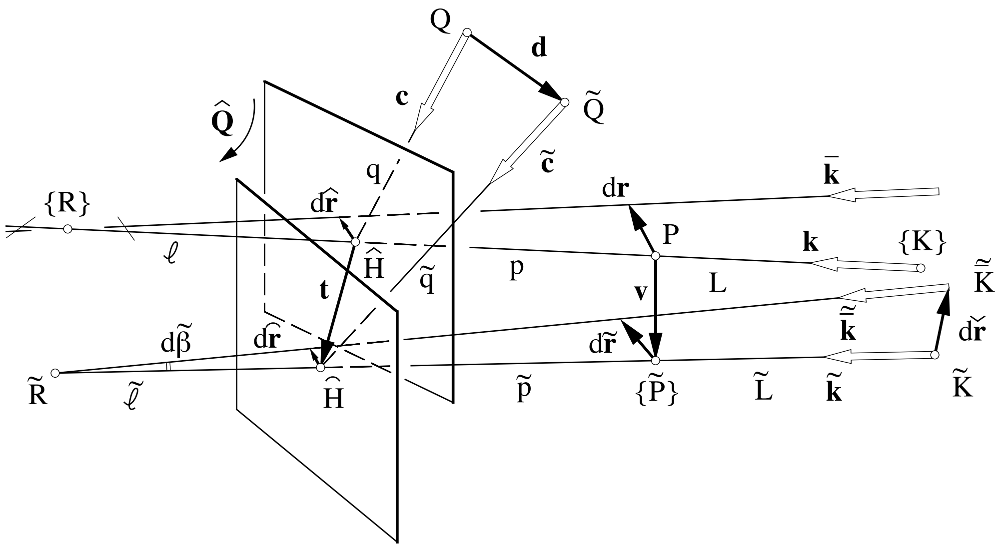

Fig. 3.5b: Case (B): Apparent image deformation of an point neighborhood with fixed point $\widetilde{\mathrm{R}}$

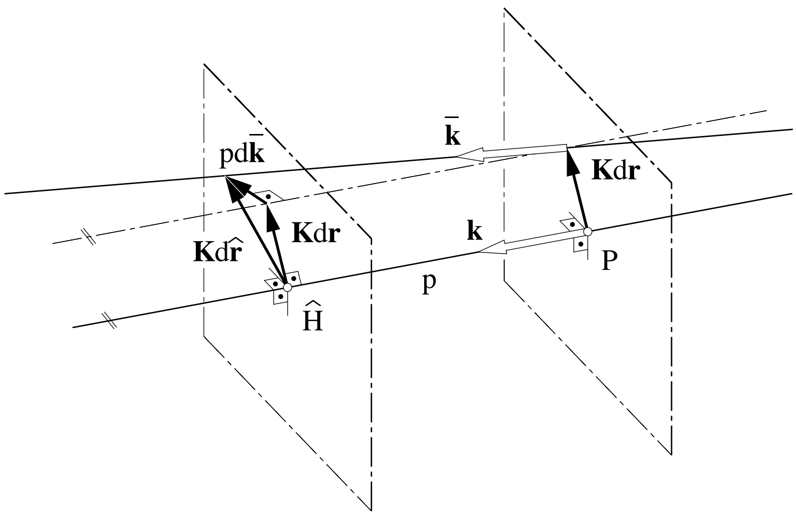

Fig. 3.6: Cases (A) and (B): Affine connection between $\mathbf{K}\mathrm{d}\mathbf{r}$ and $\mathbf{K}\mathrm{d}\hat{\mathbf{r}}$ at recording

virtual points $\{\mathrm{K}\}$ and $\{\mathrm{R}\}$ are astigmatic. Substracting the right equation (3.34) and (3.35) from the corresponding left equation, we can eliminate the “astigmatic” lengths L and ℓ. In both cases cases (A) and (B), we get (Fig. 3.6)

$$\mathbf{K}\mathrm{d}\mathbf{r} = \mathbf{K}\mathrm{d}\hat{\mathbf{r}} - p\mathrm{d}\bar{\mathbf{k}} \tag{3.36}$$

To determine the still unknown vector increment $\mathrm{d}\bar{\mathbf{k}}$, we can write, with $\widehat{\widetilde{\mathbf{N}}} = \widehat{\mathbf{N}}$ and $\widehat{\widetilde{\mathbf{Q}}} = \widehat{\mathbf{Q}}$, the corresponding ray-tracing equation (3.14) for the neighboring rays of directions $\bar{\mathbf{k}}$ and $\tilde{\bar{\mathbf{k}}}$

$$\widehat{\mathbf{N}}[\widehat{\mathbf{Q}}^T(\tilde{\bar{\mathbf{k}}} - \tilde{\bar{\mathbf{c}}}) - (\bar{\mathbf{k}} - \bar{\mathbf{c}})] = 0 \tag{3.37}$$

Substracting equation (3.14) from equation (3.37), we get by only considering the first-order terms

$$\widehat{\mathbf{N}}[\widehat{\mathbf{Q}}^T(\mathrm{d}\widetilde{\mathbf{k}} - \mathrm{d}\widetilde{\mathbf{c}}) - (\mathrm{d}\bar{\mathbf{k}} - \mathrm{d}\mathbf{c})] = 0 \qquad \Rightarrow \qquad \mathrm{d}\bar{\mathbf{k}} = \widehat{\mathbf{M}}[\widehat{\mathbf{Q}}^T(\mathrm{d}\widetilde{\mathbf{k}} - \mathrm{d}\widetilde{\mathbf{c}}) + \mathrm{d}\mathbf{c}] \tag{3.38}$$

where $\widehat{\mathbf{M}} = \mathbf{I} - \widehat{\mathbf{n}} \otimes \mathbf{k}/\widehat{\mathbf{n}} \cdot \mathbf{k}$ is an oblique projector and where we have used the calculation rules $\widehat{\mathbf{M}}\widehat{\mathbf{N}} = \widehat{\mathbf{M}}$ and $\widehat{\mathbf{M}}\mathbf{K} = \mathbf{K}$. The vector increment $\mathrm{d}\mathbf{c}$ is the first total differential corresponding to the collineation center Q, that means

$$\mathrm{d}\mathbf{c} = \mathrm{d}\widehat{\mathbf{r}}(\boldsymbol{\nabla}_{\hat{n}} \otimes \mathbf{c}) = \frac{1}{q}\mathrm{d}\widehat{\mathbf{r}}\widehat{\mathbf{N}}\mathbf{C} \tag{3.39}$$

Introducing equations (3.12), (3.30), (3.32), (3.33) and (3.39) in equation (3.38), we get with $\mathrm{d}\widehat{\mathbf{r}} = \widehat{\mathbf{N}}\mathrm{d}\widehat{\mathbf{r}} = \widehat{\mathbf{M}}^T\mathrm{d}\widehat{\mathbf{r}}$ for case (A)

$$\mathrm{d}\bar{\mathbf{k}} = -\mathbf{T}_K\mathrm{d}\widehat{\mathbf{r}} \qquad \text{with} \qquad \mathbf{T}_K = \widehat{\mathbf{M}}\left[\widehat{\mathbf{Q}}^T\left(-\frac{1}{\widetilde{p}+\widetilde{L}}\widetilde{\mathbf{K}} + \frac{1}{\widetilde{q}}\widetilde{\mathbf{C}}\right)\widehat{\mathbf{Q}} - \frac{1}{q}\mathbf{C}\right]\widehat{\mathbf{M}}^T \tag{3.40}$$

and for case (B)

$$\mathrm{d}\bar{\mathbf{k}} = -\mathbf{T}_R\mathrm{d}\widehat{\mathbf{r}} \qquad \text{with} \qquad \mathbf{T}_R = \widehat{\mathbf{M}}\left[\widehat{\mathbf{Q}}^T\left(\frac{1}{\widetilde{\ell}}\widetilde{\mathbf{K}} + \frac{1}{\widetilde{q}}\widetilde{\mathbf{C}}\right)\widehat{\mathbf{Q}} - \frac{1}{q}\mathbf{C}\right]\widehat{\mathbf{M}}^T \tag{3.41}$$

The 2-dimensional symmetric tensor $\mathbf{T}_K$ represents the virtual tensor of curvature of the wavefront at point $\widehat{\mathrm{H}}$ at recording, which travels along *diverging* rays in the direction $\mathbf{k}$ from the astigmatic interval $\{\mathrm{K}\}$. The 2-dimensional symmetric tensor $\mathbf{T}_R$ represents the virtual tensor of curvature of the wavefront at point $\widehat{\mathrm{H}}$ at recording, which travels along *converging* rays in the direction $\mathbf{k}$ to the astigmatic interval $\{\mathrm{R}\}$. With the variable unit vector $\mathbf{m}$ perpendicular to the direction $\mathbf{k}$, we have similar to equation (3.21)

$$-\frac{1}{p+L} = \mathbf{m} \cdot \mathbf{T}_K\mathbf{m} \qquad ; \qquad \frac{1}{\ell} = \mathbf{m} \cdot \mathbf{T}_R\mathbf{m} \tag{3.42}$$

which respectively define the distances from the hologram to the virtual astigmatic intervals $\{\mathrm{K}\}$ and $\{\mathrm{R}\}$ at recording. By introducing equations (3.40) and (3.41) in equation (3.36), we get with equations (3.12), (3.32) and (3.33) in case (A)

$$\mathrm{d}\mathbf{r} = \mathbf{M}^T(\mathbf{K} + p\mathbf{T}_K)\mathrm{d}\widehat{\mathbf{r}} \qquad ; \qquad \mathbf{K}\mathrm{d}\mathbf{r} = \frac{\widetilde{p}+\widetilde{L}}{\widetilde{L}}(\mathbf{K} + p\mathbf{T}_K)\widehat{\mathbf{Q}}^T\widehat{\mathbf{M}}^T\widetilde{\mathbf{K}}\mathrm{d}\widetilde{\mathbf{r}} \tag{3.43a}$$

and in case (B)

$$\mathrm{d}\mathbf{r} = \mathbf{M}^T(\mathbf{K} + p\mathbf{T}_R)\mathrm{d}\widehat{\mathbf{r}} \qquad ; \qquad \mathbf{K}\mathrm{d}\mathbf{r} = \frac{\widetilde{\ell}}{\widetilde{\ell}+\widetilde{p}}(\mathbf{K} + p\mathbf{T}_R)\widehat{\mathbf{Q}}^T\widehat{\mathbf{M}}^T\widetilde{\mathbf{K}}\mathrm{d}\widetilde{\mathbf{r}} \tag{3.43b}$$

where $\mathbf{M} = \mathbf{I} - \mathbf{n} \otimes \mathbf{k}/\mathbf{n} \cdot \mathbf{k}$ is an oblique projector and where we have used the calculation rules $\mathbf{K}\mathbf{M}^T = \mathbf{K}$ and $\mathrm{d}\mathbf{r} = \mathbf{M}^T\mathbf{K}\mathrm{d}\mathbf{r}$. In equations (3.43a) and (3.43b), the first part describes the mapping $\mathrm{d}\widehat{\mathbf{r}} \rightarrow \mathrm{d}\mathbf{r}$ of the hologram surface onto the object surface at recording, and the second part describes the inverse mapping $\widetilde{\mathbf{K}}\mathrm{d}\widetilde{\mathbf{r}} \rightarrow \mathbf{K}\mathrm{d}\mathbf{r}$ of the apparent projected image surface deformation at reconstruction onto the projected object surface at recording.

On the other hand, both tensors $\mathbf{T}_K$ and $\mathbf{T}_R$ are related to the tensor $\widetilde{\mathbf{T}}$ by the following dual equations

$$\begin{aligned} \widehat{\mathbf{M}}\widehat{\mathbf{Q}}\left(\frac{1}{p}\mathbf{K}+\mathbf{T}_K\right)\widehat{\mathbf{M}}^T &= \widehat{\mathbf{M}}\left(\frac{-1}{\widetilde{p}+\widetilde{L}}\widetilde{\mathbf{K}}-\widetilde{\mathbf{T}}\right)\widehat{\mathbf{Q}}\widehat{\mathbf{M}}^T \\ \widehat{\mathbf{M}}\widehat{\mathbf{Q}}\left(\frac{1}{p}\mathbf{K}+\mathbf{T}_R\right)\widehat{\mathbf{M}}^T &= \widehat{\mathbf{M}}\left(\frac{1}{\widetilde{\ell}}\widetilde{\mathbf{K}}-\widetilde{\mathbf{T}}\right)\widehat{\mathbf{Q}}\widehat{\mathbf{M}}^T \end{aligned} \tag{3.44}$$

where we have used the calculation rules $\widehat{\mathbf{M}}\widehat{\mathbf{Q}}\widehat{\mathbf{M}} = \widehat{\mathbf{M}}\widehat{\mathbf{Q}}$ and $\widehat{\mathbf{M}}^T\widehat{\mathbf{Q}}\widehat{\mathbf{M}}^T = \widehat{\mathbf{Q}}\widehat{\mathbf{M}}^T$. With equations (3.12), (3.29), (3.30), (3.32), (3.33) (3.43a) and (3.43b), the derivatives of the optical path difference D relatively to the fixed collineation centers $\widetilde{\mathrm{K}}$ and $\widetilde{\mathrm{R}}$ read respectively

$$\frac{\mathrm{d}D_K}{\mathrm{d}\widetilde{\alpha}} = \widetilde{\mathbf{m}}\cdot\widetilde{\mathbf{f}}_K \qquad ; \qquad \frac{\mathrm{d}D_R}{\mathrm{d}\widetilde{\beta}} = \widetilde{\mathbf{m}}\cdot\widetilde{\mathbf{f}}_R \tag{3.45}$$

with

$$\widetilde{\mathbf{f}}_K \simeq (\widetilde{p}+\widetilde{L})\widehat{\mathbf{M}}\left[\widehat{\mathbf{Q}}(\mathbf{K}+p\mathbf{T}_K)\mathbf{M}\widetilde{\mathbf{w}}+\frac{1}{\widetilde{q}}\widetilde{\mathbf{C}}(\mathbf{t}-\mathbf{d})+(\widehat{\mathbf{Q}}-\mathbf{I})(\widetilde{\mathbf{k}}-\widetilde{\mathbf{c}})\right]+\widetilde{\mathbf{K}}(\mathbf{u}-\mathbf{t}) \tag{3.46}$$

$$\widetilde{\mathbf{f}}_R \simeq \widetilde{\ell}\widehat{\mathbf{M}}\left[\widehat{\mathbf{Q}}(\mathbf{K}+p\mathbf{T}_R)\mathbf{M}\widetilde{\mathbf{w}}+\frac{1}{\widetilde{q}}\widetilde{\mathbf{C}}(\mathbf{t}-\mathbf{d})+(\widehat{\mathbf{Q}}-\mathbf{I})(\widetilde{\mathbf{k}}-\widetilde{\mathbf{c}})\right]-\widetilde{\mathbf{K}}(\mathbf{u}-\mathbf{t}) \tag{3.47}$$

and

$$\widetilde{\mathbf{w}} = \mathbf{N}\widetilde{\mathbf{w}} = (\boldsymbol{\nabla}_n\otimes\mathbf{u})\widetilde{\mathbf{g}}-\frac{1}{L_S}\mathbf{NHu} \tag{3.48}$$

where $\widetilde{\mathbf{f}}_K$ and $\widetilde{\mathbf{f}}_R$ are respectively the visibility vector and the fringe vector (here only written up to the first-order terms), and where $\widetilde{\mathbf{w}}$ is a 2-dimensional vector containing the deformation of the object surface. Both vectors $\widetilde{\mathbf{f}}_K$ and $\widetilde{\mathbf{f}}_R$ always are perpendicular to the observing direction $\widetilde{\mathbf{k}}$.

3.5 Recovery of interference fringes

It has already been said that for moderate or large deformation no interference fringes appear. Practically, this fact may be observed for example in real-time by continually increasing the deformation of the object. During this process, the interference fringes lose contrast and become less visible; they also at the same time become closer and closer to each other. In addition, the holographic images may move away and get more separated from each other. Finally, no convenient observation may be done because the fringes have totally disapeared. The purpose of this section is to explain how to quantitatively compensate this problem.

3.5.1 Fringe contrast and visibility

The visibility vector $\widetilde{\mathbf{f}}_K$ gives an information on the fringe contrast and visibility. For optical systems with a small aperture $\breve{A}$ compared to the distances involved in the holographic setup, the visibility V of the interference fringes in point $\widetilde{\mathrm{K}}$ can be written as follows

$$V = \frac{J_{max} - J_{min}}{J_{max} + J_{min}} = \left| \frac{1}{\breve{A}} \iint\limits_{\breve{A}} \exp\Big(-\frac{2\pi i}{\lambda} \underbrace{\Delta D_K}_{= \mathrm{d}D_K + \frac{1}{2}\mathrm{d}^2 D_K + \dots} \Big) \mathrm{d}\breve{A} \right| \qquad (3.49)$$

where $\Delta D_K \cong \mathrm{d}D_K + \mathrm{d}^2 D_K/2$ represents the increment of the optical path difference D in point $\widetilde{\mathrm{K}}$ related to the corresponding neighboring rays of directions $\widetilde{\widetilde{\mathbf{k}}}$ and $\widetilde{\widetilde{\mathbf{k}}}'$ going through the aperture (Fig. 3.7). Considering only the first-order terms in equation (3.49), the visibility can be increased by reducing the value of $\mathrm{d}D_K$ to zero. According to equations (3.45) and (3.47), this can be done by setting at least the first-order terms in the development of the visibility vector $\widetilde{\mathbf{f}}_K$ equal to zero.

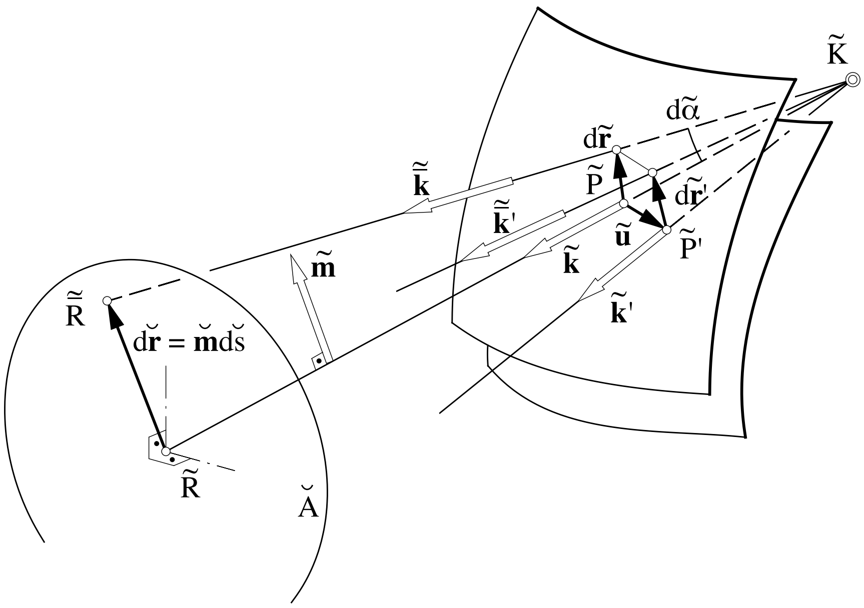

Fig. 3.7: Contribution of the corresponding neighboring rays to the visibility

3.5.2 Fringe spacing and direction

Along a fringe, that means in the direction $\widetilde{\mathbf{m}}_{\|}$ parallel to the fringe, the optical path difference D is constant and we write according to equation (3.45)

$$\frac{\mathrm{d}D_R}{\mathrm{d}\widetilde{\beta}} = \widetilde{\mathbf{m}}_{\|} \cdot \widetilde{\mathbf{f}}_R = 0 \qquad \Rightarrow \qquad \widetilde{\mathbf{m}}_{\|} \perp \widetilde{\mathbf{f}}_R \qquad (3.50)$$

which means that the fringe vector $\widetilde{\mathbf{f}}_R$ is always perpendicular to the fringes. For two neighboring interference fringes, we can write with the fringe spacing d_f in the direction $\widetilde{\mathbf{m}}_{\perp}$ perpendicular to the fringe

$$\frac{\Delta D_R}{\Delta\widetilde{\beta}} \simeq \frac{\lambda(\widetilde{\ell} + \widetilde{p} + \widetilde{L})}{d_f} = \widetilde{\mathbf{m}}_{\perp} \cdot \widetilde{\mathbf{f}}_R = |\widetilde{\mathbf{f}}_R| \qquad \Rightarrow \qquad \frac{d_f}{\widetilde{\ell} + \widetilde{p} + \widetilde{L}} = \frac{\lambda}{|\widetilde{\mathbf{f}}_R|} \qquad (3.51)$$

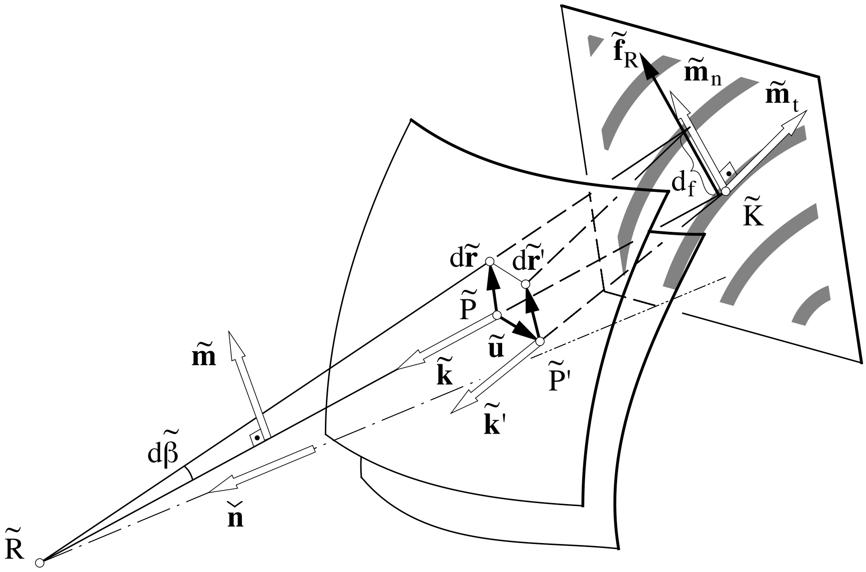

Fig. 3.7a: Geometrical meaning of the fringe vector

which means that by setting the fringe vector $\widetilde{\mathbf{f}}_R$ of equation (3.47) near to zero, the fringe spacing d_f can be increased.

3.5.3 Conditions for a fringe recovery

The developments of the equations (3.44) up to the first-order terms read

$$\begin{aligned} \widehat{\mathbf{M}}\widehat{\mathbf{Q}}(\mathbf{K}+p\mathbf{T}_K) &= p\widehat{\mathbf{M}}\left(\frac{-1}{\widetilde{p}+\widetilde{L}}\widetilde{\mathbf{K}}-\widetilde{\mathbf{T}}\right)\widehat{\mathbf{Q}}\widehat{\mathbf{M}}^T = \frac{\widetilde{p}+\widetilde{L}-p}{\widetilde{p}+\widetilde{L}}\widetilde{\mathbf{K}}\widehat{\mathbf{Q}}\widehat{\mathbf{M}}^T+\dots \\ \widehat{\mathbf{M}}\widehat{\mathbf{Q}}(\mathbf{K}+p\mathbf{T}_R) &= p\widehat{\mathbf{M}}\left(\frac{1}{\widetilde{\ell}}\widetilde{\mathbf{K}}-\widetilde{\mathbf{T}}\right)\widehat{\mathbf{Q}}\widehat{\mathbf{M}}^T = \frac{\widetilde{\ell}+p}{\widetilde{\ell}}\widetilde{\mathbf{K}}\widehat{\mathbf{Q}}\widehat{\mathbf{M}}^T+\dots \end{aligned} \tag{3.52}$$

and can be introduced in equations (3.46) and (3.47). We get by only considering the first-order terms

$$\widetilde{\mathbf{f}}_K \simeq (\widetilde{p}+\widetilde{L}-p)\widetilde{\mathbf{K}}\widehat{\mathbf{Q}}\widehat{\mathbf{M}}^T\mathbf{M}\widetilde{\mathbf{w}} + (\widetilde{p}+\widetilde{L})\widehat{\mathbf{M}}\left[\frac{1}{\widetilde{q}}\widetilde{\mathbf{C}}(\mathbf{t}-\mathbf{d})+(\widehat{\mathbf{Q}}-\mathbf{I})(\widetilde{\mathbf{k}}-\widetilde{\mathbf{c}})\right]+\widetilde{\mathbf{K}}(\mathbf{u}-\mathbf{t}) \tag{3.53}$$

$$\widetilde{\mathbf{f}}_R \simeq (\widetilde{\ell}+p)\widetilde{\mathbf{K}}\widehat{\mathbf{Q}}\widehat{\mathbf{M}}^T\mathbf{M}\widetilde{\mathbf{w}} + \widetilde{\ell}\widehat{\mathbf{M}}\left[\frac{1}{\widetilde{q}}\widetilde{\mathbf{C}}(\mathbf{t}-\mathbf{d})+(\widehat{\mathbf{Q}}-\mathbf{I})(\widetilde{\mathbf{k}}-\widetilde{\mathbf{c}})\right]-\widetilde{\mathbf{K}}(\mathbf{u}-\mathbf{t}) \tag{3.54}$$

Fringe recovery can be achieved by setting simultaneously the three vectors $\widetilde{\mathbf{f}}_S$, $\widetilde{\mathbf{f}}_K$ and $\widetilde{\mathbf{f}}_R$ near to zero, that means setting equations (3.53) and (3.54) and the first-order terms in equation (3.26) equal to zero. Because of the linear interdependance of the three vectors $\widetilde{\mathbf{f}}_S$, $\widetilde{\mathbf{f}}_K$ and $\widetilde{\mathbf{f}}_R$, both lengths $\widetilde{p}+\widetilde{L}$ and $\widetilde{\ell}$ can be eliminated and we get the so-called *equations for fringe recovery*

$$\begin{aligned} \widetilde{\mathbf{f}}_S &\simeq 0 \\ \widetilde{\mathbf{f}}_K &\simeq 0 \\ \widetilde{\mathbf{f}}_R &\simeq 0 \end{aligned} \quad\Leftrightarrow\quad \boxed{\begin{aligned} &\frac{1}{p}\widetilde{\mathbf{K}}(\mathbf{u}-\mathbf{t})-\widetilde{\mathbf{K}}\widehat{\mathbf{Q}}\widehat{\mathbf{M}}^T\mathbf{M}\widetilde{\mathbf{w}}=0 \\ &\frac{1}{p}\widetilde{\mathbf{K}}(\mathbf{u}-\mathbf{t})+\frac{1}{\widetilde{q}}\widehat{\mathbf{M}}\widetilde{\mathbf{C}}(\mathbf{t}-\mathbf{d})+\widehat{\mathbf{M}}(\widehat{\mathbf{Q}}-\mathbf{I})(\widetilde{\mathbf{k}}-\widetilde{\mathbf{c}})=0 \end{aligned}} \tag{3.55}$$

which represent a system of two bidimensional vector equations (four scalar equations) [3.44]. According to equations (3.25) and (3.27), we may also write an auxiliary condition of longitudinal superposition (not required for fringe recovery) as follows

$$\widetilde{\mathbf{f}}_L \simeq 0 \qquad \Leftrightarrow \qquad \widetilde{\mathbf{k}} \cdot (\mathbf{u} - \mathbf{t}) - p - \frac{2}{\widetilde{\mathbf{T}} \cdot \widetilde{\mathbf{K}}} = 0 \tag{3.56}$$

3.6 Fringe Analysis

In order to properly analyse the interference fringes, we have to consider the exact fringe vector $\widetilde{\mathbf{f}}_R$. It follows that not only the first-order terms, but also the higher-order terms must be considered if a development is performed. This can be achieved by directly writing the first differential of the optical path difference [3.51,3.54] from equation (3.5) as follows

$$\mathrm{d}D = (\mathrm{d}\widetilde{L}' - \mathrm{d}\widetilde{L}) - (\mathrm{d}L'_S - \mathrm{d}L_S) + (\mathrm{d}\widetilde{q} - \mathrm{d}q) - (\mathrm{d}\widetilde{p} - \mathrm{d}p) \tag{3.57}$$

With $\widetilde{\mathrm{R}}$ as fixed collineation center, we have with $\widetilde{L}' = \widetilde{L}'(\widetilde{\mathbf{r}}', \check{\mathbf{r}})$, $\widetilde{L} = \widetilde{L}(\widetilde{\mathbf{r}}, \check{\mathbf{r}})$, $L'_S = L'_S(\widetilde{\mathbf{r}}')$, $L_S = L_S(\mathbf{r}), \widetilde{q} = \widetilde{q}(\widehat{\mathbf{r}}), q = q(\widehat{\mathbf{r}}), \widetilde{p} = \widetilde{p}(\widehat{\mathbf{r}}, \widetilde{\mathbf{r}})$ and $p = p(\widehat{\mathbf{r}}, \mathbf{r})$

$$\begin{aligned}
\mathrm{d}\widetilde{L}' &= \mathrm{d}\widetilde{\mathbf{r}}' \cdot \boldsymbol{\partial}_{\tilde{n}'} \widetilde{L}' + \mathrm{d}\check{\mathbf{r}} \cdot \boldsymbol{\partial}_{\check{n}} \widetilde{L}' = \mathrm{d}\widetilde{\mathbf{r}}' \cdot \widetilde{\mathbf{N}}'\widetilde{\mathbf{k}}' - \mathrm{d}\check{\mathbf{r}} \cdot \check{\mathbf{N}}\widetilde{\mathbf{k}}' \\
\mathrm{d}\widetilde{L} &= \mathrm{d}\widetilde{\mathbf{r}} \cdot \boldsymbol{\partial}_{\tilde{n}} \widetilde{L} + \mathrm{d}\check{\mathbf{r}} \cdot \boldsymbol{\partial}_{\check{n}} \widetilde{L} = \mathrm{d}\widetilde{\mathbf{r}} \cdot \widetilde{\mathbf{N}}\widetilde{\mathbf{k}} - \mathrm{d}\check{\mathbf{r}} \cdot \check{\mathbf{N}}\widetilde{\mathbf{k}} \\
\mathrm{d}L'_S &= \mathrm{d}\widetilde{\mathbf{r}}' \cdot \boldsymbol{\nabla}_{\tilde{n}'} \widetilde{L}'_S = \mathrm{d}\widetilde{\mathbf{r}}' \cdot \widetilde{\mathbf{N}}'\mathbf{h}' \\
\mathrm{d}L_S &= \mathrm{d}\mathbf{r} \cdot \boldsymbol{\nabla}_n L_S = \mathrm{d}\mathbf{r} \cdot \mathbf{N}\mathbf{h} \\
\mathrm{d}\widetilde{q} &= \mathrm{d}\widehat{\mathbf{r}} \cdot \boldsymbol{\nabla}_{\widehat{\mathrm{n}}} \widetilde{q} = \mathrm{d}\widehat{\mathbf{r}} \cdot \widehat{\mathbf{N}}\widetilde{\mathbf{c}} \\
\mathrm{d}q &= \mathrm{d}\widehat{\mathbf{r}} \cdot \boldsymbol{\nabla}_{\hat{n}} q = \mathrm{d}\widehat{\mathbf{r}} \cdot \widehat{\mathbf{N}}\mathbf{c} \\
\mathrm{d}\widetilde{p} &= \mathrm{d}\widehat{\mathbf{r}} \cdot \boldsymbol{\partial}_{\widehat{\mathrm{n}}} \widetilde{p} + \mathrm{d}\widetilde{\mathbf{r}} \cdot \boldsymbol{\partial}_{\tilde{n}} \widetilde{p} = \mathrm{d}\widehat{\mathbf{r}} \cdot \widehat{\mathbf{N}}\widetilde{\mathbf{k}} - \mathrm{d}\widetilde{\mathbf{r}} \cdot \widetilde{\mathbf{N}}\widetilde{\mathbf{k}} \\
\mathrm{d}p &= \mathrm{d}\widehat{\mathbf{r}} \cdot \boldsymbol{\partial}_{\hat{n}} p + \mathrm{d}\mathbf{r} \cdot \boldsymbol{\partial}_n p = \mathrm{d}\widehat{\mathbf{r}} \cdot \widehat{\mathbf{N}}\mathbf{k} - \mathrm{d}\mathbf{r} \cdot \mathbf{N}\mathbf{k}
\end{aligned} \tag{3.58}$$

where $\check{\mathbf{N}} = \mathbf{I} - \check{\mathbf{n}} \otimes \check{\mathbf{n}}$ is a normal projector, which projects onto a plane parallel to the object plane of the optical system. The vector increment $\mathrm{d}\check{\mathbf{r}}$ represents the first total differential on the object plane of our observing system (Fig. 3.7b), which is normal to the optical axis of unit direction $\check{\mathbf{n}}$. The corresponding affine connections read with their relative oblique projectors

$$\begin{aligned}
\mathrm{d}\check{\mathbf{r}} &= (\widetilde{\ell} + \widetilde{p} + \widetilde{L})\mathrm{d}\widetilde{\beta}\widetilde{\mathbf{m}}\check{\mathbf{M}} \qquad ; \qquad \check{\mathbf{M}} = \mathbf{I} - \frac{\check{\mathbf{n}} \otimes \widetilde{\mathbf{k}}}{\check{\mathbf{n}} \cdot \widetilde{\mathbf{k}}} \\
\mathrm{d}\widetilde{\mathbf{r}} &= (\widetilde{\ell} + \widetilde{p})\mathrm{d}\widetilde{\beta}\widetilde{\mathbf{m}}\widetilde{\mathbf{M}} \qquad ; \qquad \widetilde{\mathbf{M}} = \mathbf{I} - \frac{\widetilde{\mathbf{n}} \otimes \widetilde{\mathbf{k}}}{\widetilde{\mathbf{n}} \cdot \widetilde{\mathbf{k}}}
\end{aligned} \tag{3.59}$$

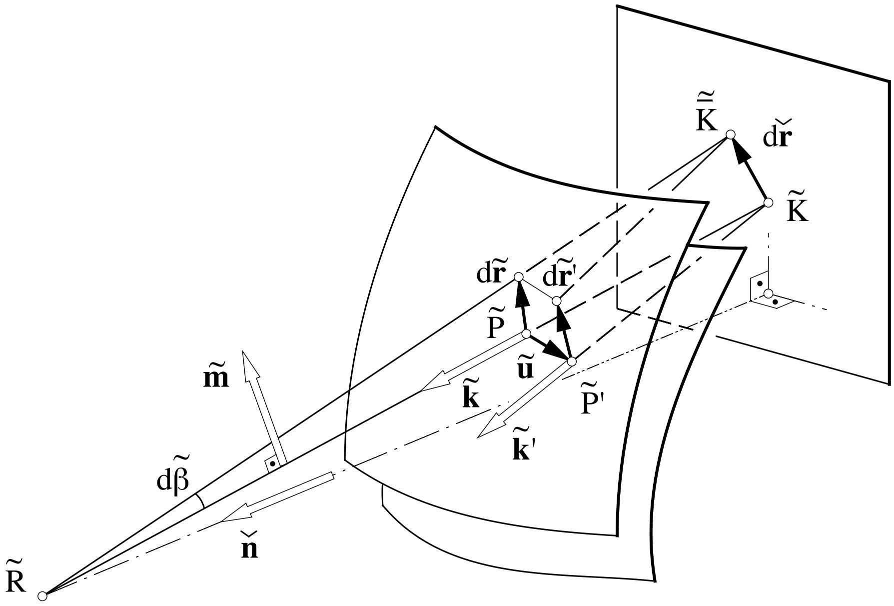

Fig. 3.7b: Derivative relatively to the fixed point $\widetilde{\mathrm{R}}$ on the image surfaces

Let us now recall the total differential $\mathrm{d}\mathbf{r}'$ on the deformed object surface, which is related to the total differential $\mathrm{d}\mathbf{r}$ on the undeformed object surface by the deformation gradient $\mathbf{F}_S$ as follows

$$\mathbf{r}' = \mathbf{r} + \mathbf{u} \qquad \Rightarrow \qquad \mathrm{d}\mathbf{r}' = \mathrm{d}\mathbf{r} + \mathrm{d}\mathbf{u} = \mathbf{N}\mathrm{d}\mathbf{r} + \mathrm{d}\mathbf{r}(\boldsymbol{\nabla}_n \otimes \mathbf{u}) = \mathbf{F}_S \mathrm{d}\mathbf{r} \tag{3.60}$$

where $\mathbf{F}_S = \mathbf{N}'\mathbf{F}\mathbf{N} = \mathbf{N} + (\boldsymbol{\nabla}_n \otimes \mathbf{u})^T$ is a mixed semi-projection of the deformation gradient $\mathbf{F}$ of a 3-dimensional body. Because no optical or geometrical modification is performed on the deformed configuration, we have $\widetilde{\mathbf{k}}' = \mathbf{k}'$, $\mathrm{d}\widetilde{\mathbf{r}}' = \mathrm{d}\mathbf{r}'$ and $\widetilde{\mathbf{N}}' = \mathbf{N}' = \mathbf{I} - \mathbf{n}' \otimes \mathbf{n}'$. Introducing equations (3.12), (3.14), (3.58) and (3.60) in equation (3.57) gives

$$\mathrm{d}D_R = \mathrm{d}\mathbf{r} \cdot \mathbf{N}[\mathbf{F}_S^T(\mathbf{k}' - \mathbf{h}') - (\mathbf{k} - \mathbf{h})] - \mathrm{d}\check{\mathbf{r}} \cdot \check{\mathbf{N}}(\widetilde{\mathbf{k}}' - \widetilde{\mathbf{k}}) \tag{3.61}$$

Considering the affine connections (3.43b), (3.59) and the dual equation (3.44), we get

$$\frac{\mathrm{d}D_R}{\mathrm{d}\widetilde{\beta}} = \widetilde{\mathbf{m}} \cdot \widetilde{\mathbf{f}}_R \tag{3.62}$$

with the exact fringe vector containing the deformation

$$\widetilde{\mathbf{f}}_R = p(\widetilde{\mathbf{K}} - \widetilde{\ell}\,\widetilde{\mathbf{T}})\widehat{\mathbf{Q}}\widehat{\mathbf{M}}^T\mathbf{M}[\mathbf{F}_S^T(\mathbf{k}' - \mathbf{h}') - (\mathbf{k} - \mathbf{h})] - (\widetilde{\ell} + \widetilde{p} + \widetilde{L})\check{\mathbf{M}}(\widetilde{\mathbf{k}}' - \widetilde{\mathbf{k}}) \tag{3.63}$$

Once the fringe pattern has been recovered with equations (3.55), the expression (3.63) must be used (at least up to the second-order terms) for the fringe analysis. Note that, by only considering the first-order terms, the approximate expressions (3.47) and (3.54) for the fringe vector may be obtained by introducing equations (3.15), (3.16), (3.17), (3.18), (3.44) and (3.52) in equation (3.63) together with the calculation rule $\check{\mathbf{M}}\widetilde{\mathbf{K}} = \widetilde{\mathbf{K}}$ and the development

$$\widetilde{\mathbf{k}}' = \mathbf{k}' = \widetilde{\mathbf{k}} + \frac{1}{\widetilde{L}}\widetilde{\mathbf{K}}\widetilde{\mathbf{u}} + \ldots \tag{3.64}$$

3.7 Experimental verification

3.7.1 Theoretical verification

Equations for fringe recovery (3.55) can be used to recover the interference fringe pattern when the object surface deformation is large and unknown! This can be achieved by first pointing the observing system, e.g. our CCD-camera, on the deformed object surface to get approximately the observing direction $\widetilde{\mathbf{k}} \simeq \widetilde{\mathbf{k}}'$ for a given point $\widetilde{\mathrm{P}}'$. Because the deformation is unknown, it is not possible to directly find quantitatively the required modification. However, this problem can still be solved by doing a systematic search, while reducing the independent modification parameters to a strict minimum. Assuming that the geometrical quantities in our holographic setup are known (at least approximately), we can measure the lateral displacement $\widetilde{\mathbf{K}}\mathbf{u}$ of the object surface by image processing and introduce its value in equations (3.55). Because $\widetilde{\mathbf{K}}\widehat{\mathbf{Q}}\widehat{\mathbf{M}}^T\mathbf{M}\widetilde{\mathbf{w}}$ is bidimensional, we can write a parametric representation of this unknown "projected deformation vector" as follows

$$\widetilde{\mathbf{K}}\widehat{\mathbf{Q}}\widehat{\mathbf{M}}^T\mathbf{M}\widetilde{\mathbf{w}} = -\sigma_1\widetilde{\mathbf{m}}_1 - \sigma_2\widetilde{\mathbf{m}}_2 \qquad ; \qquad \widetilde{\mathbf{m}}_1 \perp \widetilde{\mathbf{m}}_2 \tag{3.65}$$

where σ_1 and σ_2 are two linear independent factors and where $\widetilde{\mathbf{m}}_1$ and $\widetilde{\mathbf{m}}_2$ are two unit vectors perpendicular to the observing direction $\widetilde{\mathbf{k}}$. Introducing equations (3.13) and (3.65) in equation (3.55), we get by only considering the first-order terms

$$\begin{aligned} &\frac{1}{p}\widetilde{\mathbf{K}}\mathbf{t} = \frac{1}{p}\widetilde{\mathbf{K}}\mathbf{u} + \sigma_1\widetilde{\mathbf{m}}_1 + \sigma_2\widetilde{\mathbf{m}}_2 \\ &\frac{1}{\widetilde{q}}\widehat{\mathbf{M}}\widetilde{\mathbf{C}}(\mathbf{t}-\mathbf{d}) - \hat{\chi}\widehat{\mathbf{M}}\widehat{\mathbf{E}}_H(\widetilde{\mathbf{k}}-\widetilde{\mathbf{c}}) = \sigma_1\widetilde{\mathbf{m}}_1 + \sigma_2\widetilde{\mathbf{m}}_2 \end{aligned} \tag{3.66}$$

Because of the linear interdependance of the components contained in the modification terms $\mathbf{t}$, $\mathbf{d}$ and $\hat{\chi}\widehat{\mathbf{E}}_H$, the equation system (3.66) has several solutions, which give us some flexibility for the fringe recovery procedure. By only varying the modification terms while keeping in a first approximation the other geometrical quantities in equations (3.66) constant, we get the required modification as function of the two "modification" parameters σ_1 and σ_2. The search must then be done around the zero value in the parametric plane (σ_1, σ_2) until fringes appear.

$$\sigma_1, \sigma_2 \qquad \rightarrow \qquad \begin{cases} \mathbf{t} = \mathbf{t}(\sigma_1, \sigma_2) \\ \mathbf{d} = \mathbf{d}(\sigma_1, \sigma_2) \\ \hat{\chi}\widehat{\mathbf{E}}_H = \hat{\chi}\widehat{\mathbf{E}}_H(\sigma_1, \sigma_2) \end{cases} \tag{3.67}$$

Equations (3.66) are independent from the choice of the coordinate system. In order to write them in components, we first choose a right handed cartesian vector base system $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ such that $\mathbf{e}_1 /\!/ \widetilde{\mathbf{m}}_1$, $\mathbf{e}_2 /\!/ \widetilde{\mathbf{m}}_2$ and $\mathbf{e}_3 /\!/ \widetilde{\mathbf{k}}$ for a selected observing direction $\widetilde{\mathbf{k}}$ (often the optical axis of the observing system). In the system $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, we have for the geometry of the optical setup

$$\widetilde{\mathbf{m}}_1 \hat{=} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \quad ; \quad \widetilde{\mathbf{m}}_2 \hat{=} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \quad ; \quad \widetilde{\mathbf{k}} \hat{=} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \quad ; \quad \widetilde{\mathbf{K}} \hat{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\widetilde{\mathbf{c}} \mathrel{\hat{=}} \begin{Bmatrix} \tilde{c}_1 \\ \tilde{c}_2 \\ \tilde{c}_3 \end{Bmatrix} \quad ; \quad \widetilde{\mathbf{C}} \mathrel{\hat{=}} \begin{bmatrix} 1-\tilde{c}_1^2 & -\tilde{c}_1\tilde{c}_2 & -\tilde{c}_1\tilde{c}_3 \\ -\tilde{c}_2\tilde{c}_1 & 1-\tilde{c}_2^2 & -\tilde{c}_2\tilde{c}_3 \\ -\tilde{c}_3\tilde{c}_1 & -\tilde{c}_3 c_2 & 1-\tilde{c}_3^2 \end{bmatrix} \tag{3.68}$$

$$\widehat{\mathbf{n}} \mathrel{\hat{=}} \begin{Bmatrix} \widehat{\mathrm{n}}_1 \\ \widehat{\mathrm{n}}_2 \\ \widehat{\mathrm{n}}_3 \end{Bmatrix} \quad ; \quad \widehat{\mathbf{M}} \mathrel{\hat{=}} \begin{bmatrix} 1 & 0 & -\widehat{\mathrm{n}}_1/\widehat{\mathrm{n}}_3 \\ 0 & 1 & -\widehat{\mathrm{n}}_2/\widehat{\mathrm{n}}_3 \\ 0 & 0 & 0 \end{bmatrix}$$

and for the displacement and the modification

$$\mathbf{u} \mathrel{\hat{=}} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad ; \quad \mathbf{t} \mathrel{\hat{=}} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} \quad ; \quad \mathbf{d} \mathrel{\hat{=}} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} \quad ; \quad \widehat{\boldsymbol{\omega}}_H = \hat{\chi}\widehat{\mathbf{n}}_H \mathrel{\hat{=}} \hat{\chi} \begin{Bmatrix} \hat{n}_{H1} \\ \hat{n}_{H2} \\ \hat{n}_{H3} \end{Bmatrix} \tag{3.69}$$

$$\hat{\chi}\widehat{\mathbf{E}}_H = \hat{\chi}\boldsymbol{\mathfrak{E}}\widehat{\mathbf{n}}_H \mathrel{\hat{=}} \hat{\chi} \begin{bmatrix} 0 & \hat{n}_{H3} & -\hat{n}_{H2} \\ -\hat{n}_{H3} & 0 & \hat{n}_{H1} \\ \hat{n}_{H2} & -\hat{n}_{H1} & 0 \end{bmatrix} \quad ; \quad \begin{aligned} \boldsymbol{\mathfrak{E}} = {} & \widetilde{\mathbf{m}}_1 \otimes \widetilde{\mathbf{m}}_2 \otimes \widetilde{\mathbf{k}} + \widetilde{\mathbf{m}}_2 \otimes \widetilde{\mathbf{k}} \otimes \widetilde{\mathbf{m}}_1 \\ & + \widetilde{\mathbf{k}} \otimes \widetilde{\mathbf{m}}_1 \otimes \widetilde{\mathbf{m}}_2 - \widetilde{\mathbf{m}}_2 \otimes \widetilde{\mathbf{m}}_1 \otimes \widetilde{\mathbf{k}} \\ & - \widetilde{\mathbf{m}}_1 \otimes \widetilde{\mathbf{k}} \otimes \widetilde{\mathbf{m}}_2 - \widetilde{\mathbf{k}} \otimes \widetilde{\mathbf{m}}_2 \otimes \widetilde{\mathbf{m}}_1 \end{aligned}$$

where the sign $\hat{=}$ draws attention to the fact that the base vectors are omitted in the component notation. Introducing the relations (3.68) and (3.69) in equations (3.66) gives

$$\begin{aligned} \frac{1}{p}t_1 &= \frac{1}{p}u_1 + \sigma_1 \\ \frac{1}{p}t_2 &= \frac{1}{p}u_2 + \sigma_2 \\ A_{31}(t_1-d_1) + A_{32}(t_2-d_2) &+ A_{33}(t_3-d_3) + A_{34}\hat{\chi} = \sigma_1 \\ A_{41}(t_1-d_1) + A_{42}(t_2-d_2) &+ A_{43}(t_3-d_3) + A_{44}\hat{\chi} = \sigma_2 \end{aligned} \tag{3.70}$$

where

$$A_{31} = \frac{1}{\widetilde{q}}\left(1-\tilde{c}_1^2 + \frac{\widehat{\mathrm{n}}_1}{\widehat{\mathrm{n}}_3}\tilde{c}_1\tilde{c}_3\right) \quad ; \quad A_{32} = \frac{1}{\widetilde{q}}\left(-\tilde{c}_1\tilde{c}_2 + \frac{\widehat{\mathrm{n}}_1}{\widehat{\mathrm{n}}_3}\tilde{c}_2\tilde{c}_3\right)$$

$$A_{33} = \frac{1}{\widetilde{q}}\left(-\tilde{c}_1\tilde{c}_3 - \frac{\widehat{\mathrm{n}}_1}{\widehat{\mathrm{n}}_3}(1-\tilde{c}_3^2)\right) \quad ; \quad A_{34} = -\frac{\widehat{\mathrm{n}}_1}{\widehat{\mathrm{n}}_3}\hat{n}_{H2}\tilde{c}_1 + \left(\hat{n}_{H3} + \frac{\widehat{\mathrm{n}}_1}{\widehat{\mathrm{n}}_3}\hat{n}_{H1}\right)\tilde{c}_2 + \hat{n}_{H2}(1-\tilde{c}_3)$$

$$A_{41} = \frac{1}{\widetilde{q}}\left(-\tilde{c}_1\tilde{c}_2 + \frac{\widehat{\mathrm{n}}_2}{\widehat{\mathrm{n}}_3}\tilde{c}_1\tilde{c}_3\right) \quad ; \quad A_{42} = \frac{1}{\widetilde{q}}\left(1-\tilde{c}_2^2 + \frac{\widehat{\mathrm{n}}_2}{\widehat{\mathrm{n}}_3}\tilde{c}_2\tilde{c}_3\right)$$

$$A_{43} = \frac{1}{\widetilde{q}}\left(-\tilde{c}_2\tilde{c}_3 - \frac{\widehat{\mathrm{n}}_2}{\widehat{\mathrm{n}}_3}(1-\tilde{c}_3^2)\right) \quad ; \quad A_{44} = -\left(\hat{n}_{H3} + \frac{\widehat{\mathrm{n}}_2}{\widehat{\mathrm{n}}_3}\hat{n}_{H2}\right)\tilde{c}_1 + \frac{\widehat{\mathrm{n}}_2}{\widehat{\mathrm{n}}_3}\hat{n}_{H1}\tilde{c}_2 - \hat{n}_{H1}(1-\tilde{c}_3)$$

By choosing the geometry in the optical setup and the modification such that $\mathbf{d} = 0$ and $A_{ij} \neq 0$, which means that only the position of the hologram in space can be modified while the reference source Q remains fixed, we get with $A_{11} = A_{22} = 1/p$ the following matrix equation system

$$\begin{bmatrix} A_{11} & 0 & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ \hat{\chi} \end{Bmatrix} = \frac{1}{p} \begin{Bmatrix} u_1 \\ u_2 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_1 \\ \sigma_2 \end{Bmatrix} \quad \Leftrightarrow \quad [A]\{t\} = \frac{1}{p}\{u\} + \{\sigma\} \tag{3.71}$$

For $[A]$ not singular, equation (3.71) gives the modification "vector" $\{t\}$ as function of the measured lateral "vector" $\{u\}$ and the parametric "vector" $\{\sigma\}$

$$\{t\} = \frac{1}{p}[A]^{-1}\{u\} + [A]^{-1}\{\sigma\} \tag{3.72}$$

In some special geometrical cases, the matrix $[A]$ may be singular. However, equation (3.71) can be solved in most cases by setting $t_3 = at_1 + bt_2$, where a and b are constant proportionality factors, or by slightly changing the geometry.

In order to numerically verify the validity of equation (3.72), we still have to determine the values of σ_1 and σ_2 in the deformation vector $\widetilde{\mathbf{K}}\widehat{\mathbf{Q}}\widehat{\mathbf{M}}^T\mathbf{M}\widetilde{\mathbf{w}}$ of equation (3.65). This step is not necessary for fringe recovery, but will be performed here to prove that equations (3.66) are correct. With the developments (3.13) and (3.16), the oblique projectors $\widehat{\mathbf{M}}$ and $\mathbf{M}$ can be developed up to the first-order terms as follows

$$\begin{aligned}\widehat{\mathbf{M}} &= \widehat{\mathbf{M}} + \frac{1}{\widehat{\mathbf{n}}\cdot\widetilde{\mathbf{k}}}[(\widehat{\mathbf{n}}\otimes\widehat{\mathbf{M}}\widetilde{\mathbf{v}}_1)\widehat{\mathbf{M}} + \widehat{\mathbf{M}}(\hat{\chi}\widehat{\mathbf{n}}\widehat{\mathbf{E}}_H\otimes\widetilde{\mathbf{k}})] + \ldots \\ \mathbf{M} &= \mathbf{I} - \frac{\mathbf{n}\otimes\widetilde{\mathbf{k}}}{\mathbf{n}\cdot\widetilde{\mathbf{k}}} + \frac{1}{(\mathbf{n}\cdot\widetilde{\mathbf{k}})^2}[(\mathbf{n}\cdot\widetilde{\mathbf{k}})\mathbf{n}\otimes\widehat{\mathbf{M}}\widetilde{\mathbf{v}}_1 - (\mathbf{n}\cdot\widehat{\mathbf{M}}\widetilde{\mathbf{v}}_1)\mathbf{n}\otimes\widetilde{\mathbf{k}}] + \ldots\end{aligned} \tag{3.73}$$

A first approximation of equation (3.65) can now be written with equations (3.13), (3.48) and (3.73)

$$\widetilde{\mathbf{K}}\widehat{\mathbf{Q}}\widehat{\mathbf{M}}^T\mathbf{M}\widetilde{\mathbf{w}} = \left(\mathbf{I} - \frac{\mathbf{n}\otimes\widetilde{\mathbf{k}}}{\mathbf{n}\cdot\widetilde{\mathbf{k}}}\right)\left[(\boldsymbol{\nabla}_n\otimes\mathbf{u})(\widetilde{\mathbf{k}}-\mathbf{h}) - \frac{1}{L_S}\mathbf{NHu}\right] + \ldots \tag{3.74}$$

In order to write equation (3.74) in components, we introduce in some selected point P on the object surface another right handed cartesian vector base system $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ such that $\mathbf{e}_x \perp \mathbf{e}_y, \mathbf{e}_x \perp \mathbf{n}, \mathbf{e}_y \perp \mathbf{n}$ and $\mathbf{e}_z \mathbin{/\!/} \mathbf{n}$. The bridge between the component notation of any arbitrary vector $\{v\}$ or 3×3 matrix $[M]$ in either the coordinate system $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ or $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is given by the 3×3 orthogonal matrix $[Z]$ as follows

$$\begin{aligned}\{v\}_{\langle\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\rangle} &= [Z]\{v\}_{\langle\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z\rangle} \qquad & [M]_{\langle\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\rangle} &= [Z][M]_{\langle\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z\rangle}[Z]^T \\ & \qquad ; & & \\ \{v\}_{\langle\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z\rangle} &= [Z]^T\{v\}_{\langle\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\rangle} \qquad & [M]_{\langle\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z\rangle} &= [Z]^T[M]_{\langle\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\rangle}[Z]\end{aligned}$$

where

$$[Z] = \begin{bmatrix} e_{x1} & e_{y1} & e_{z1} \\ e_{x2} & e_{y2} & e_{z2} \\ e_{x3} & e_{y3} & e_{z3} \end{bmatrix} \tag{3.75}$$

with the columns of $[Z]$ representing the components of the base vectors $\mathbf{e}_x, \mathbf{e}_y$ and $\mathbf{e}_z$ written relatively to the system $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Because $[Z]$ is orthogonal, it describes a rotation and we have $[Z]^{-1} = [Z]^T$. In the system $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$, we have for the geometry of the optical setup

$$\mathbf{e}_x \mathrel{\hat{=}} \begin{Bmatrix}1\\0\\0\end{Bmatrix} \quad ; \quad \mathbf{e}_y \mathrel{\hat{=}} \begin{Bmatrix}0\\1\\0\end{Bmatrix} \quad ; \quad \mathbf{e}_z = \mathbf{n} \mathrel{\hat{=}} \begin{Bmatrix}0\\0\\1\end{Bmatrix} \quad ; \quad \widetilde{\mathbf{k}} \mathrel{\hat{=}} [Z]^T\begin{Bmatrix}0\\0\\1\end{Bmatrix} \tag{3.76}$$

$$\widetilde{\mathbf{K}} \mathrel{\hat{=}} [Z]^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} [Z] \quad ; \quad \mathbf{h} \mathrel{\hat{=}} \begin{Bmatrix} h_x \\ h_y \\ h_z \end{Bmatrix} \quad ; \quad \mathbf{H} \mathrel{\hat{=}} \begin{bmatrix} 1-h_x^2 & -h_x h_y & -h_x h_z \\ -h_y h_x & 1-h_y^2 & -h_y h_z \\ -h_z h_x & -h_z h_y & 1-h_z^2 \end{bmatrix}$$

$$\boldsymbol{\mathcal{E}} = \mathbf{e}_x \otimes \mathbf{e}_y \otimes \mathbf{n} + \mathbf{e}_y \otimes \mathbf{n} \otimes \mathbf{e}_x + \mathbf{n} \otimes \mathbf{e}_x \otimes \mathbf{e}_y - \mathbf{e}_y \otimes \mathbf{e}_x \otimes \mathbf{n} - \mathbf{e}_x \otimes \mathbf{n} \otimes \mathbf{e}_y - \mathbf{n} \otimes \mathbf{e}_y \otimes \mathbf{e}_x$$

$$\mathbf{N} = \mathbf{e}_x \otimes \mathbf{e}_y + \mathbf{e}_y \otimes \mathbf{e}_x \mathrel{\hat{=}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ; \quad \mathbf{E} = \boldsymbol{\mathcal{E}}\mathbf{n} = \mathbf{e}_x \otimes \mathbf{e}_y - \mathbf{e}_y \otimes \mathbf{e}_x \mathrel{\hat{=}} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and for the deformation [3.52,3.55]

$$\mathbf{u} \mathrel{\hat{=}} \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} \quad ; \quad \boldsymbol{\omega} \mathrel{\hat{=}} \begin{Bmatrix} \omega_x \\ \omega_y \\ 0 \end{Bmatrix} \quad ; \quad \widetilde{\boldsymbol{\gamma}} \mathrel{\hat{=}} \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & 0 \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ; \quad \mathbf{n}_S \mathrel{\hat{=}} \begin{Bmatrix} n_{Sx} \\ n_{Sy} \\ n_{Sz} \end{Bmatrix}$$

$$(\boldsymbol{\nabla}_n \otimes \mathbf{u}) \simeq \widetilde{\boldsymbol{\gamma}} + \Omega \mathbf{E} + \boldsymbol{\omega} \otimes \mathbf{n} \mathrel{\hat{=}} \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} + \Omega & \omega_x \\ \frac{1}{2}\gamma_{xy} - \Omega & \varepsilon_y & \omega_y \\ 0 & 0 & 0 \end{bmatrix} \tag{3.77}$$

$$\boldsymbol{\omega}_S = \chi \mathbf{n}_S = \Omega \mathbf{n} + \mathbf{E}\boldsymbol{\omega} \mathrel{\hat{=}} \begin{Bmatrix} 0 \\ 0 \\ \Omega \end{Bmatrix} + \begin{Bmatrix} \omega_y \\ -\omega_x \\ 0 \end{Bmatrix} = \begin{Bmatrix} \omega_y \\ -\omega_x \\ \Omega \end{Bmatrix} = \chi \begin{Bmatrix} n_{Sx} \\ n_{Sy} \\ n_{Sz} \end{Bmatrix} = \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_z \end{Bmatrix}$$

Equation (3.65) then reads with equations (3.48), (3.74) and (3.75)

$$\widetilde{\mathbf{K}}\widehat{\mathbf{Q}}\widehat{\mathbf{M}}^T \mathbf{M}\widetilde{\mathbf{w}} \mathrel{\hat{=}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -e_{x3}/e_{z3} & -e_{y3}/e_{z3} & 0 \end{bmatrix} \begin{Bmatrix} \tilde{\mathrm{w}}_x \\ \tilde{\mathrm{w}}_y \\ 0 \end{Bmatrix} + \ldots \tag{3.78}$$

with

$$\begin{Bmatrix} \tilde{\mathrm{w}}_x \\ \tilde{\mathrm{w}}_y \\ 0 \end{Bmatrix} = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} + \Omega & \omega_x \\ \frac{1}{2}\gamma_{xy} - \Omega & \varepsilon_y & \omega_y \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} e_{x3} - h_x \\ e_{y3} - h_y \\ e_{z3} - h_z \end{Bmatrix} - \frac{1}{L_S} \begin{bmatrix} 1-h_x^2 & -h_x h_y & -h_x h_z \\ -h_y h_x & 1-h_y^2 & -h_y h_z \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix}$$

The two parameters σ_1 and σ_2 are calculated from equations (3.65) and (3.78) as follows

$$\begin{aligned} \sigma_1 &= -\widetilde{\mathbf{m}}_1 \cdot \widetilde{\mathbf{K}}\widehat{\mathbf{Q}}\widehat{\mathbf{M}}^T \mathbf{M}\widetilde{\mathbf{w}} \\ \sigma_2 &= -\widetilde{\mathbf{m}}_2 \cdot \widetilde{\mathbf{K}}\widehat{\mathbf{Q}}\widehat{\mathbf{M}}^T \mathbf{M}\widetilde{\mathbf{w}} \end{aligned} \quad ; \quad \widetilde{\mathbf{m}}_1 \mathrel{\hat{=}} [Z]^T \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \quad ; \quad \widetilde{\mathbf{m}}_2 \mathrel{\hat{=}} [Z]^T \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \tag{3.79}$$

and can be introduced in equation (3.71) for fringe recovery together with the auxiliary relation for the lateral components u_1 and u_2 of the displacement

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} e_{x1} & e_{y1} & e_{z1} \\ e_{x2} & e_{y2} & e_{z2} \\ e_{x3} & e_{y3} & e_{z3} \end{bmatrix} \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} \tag{3.80}$$

3.7.2 Numerical verification

The following quantitative experiment confirms the theory. The object shown in figure 3.8 allows to perform deformations with large displacements and moderate rotations. Figures 3.9 to 3.11 show decreasing fringe spacing and contrast while increasing the object deformation. If not precised, all numerical values are written in the system $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$.

- Geometry of the holographic setup:

$$p \simeq 427.87 \text{ mm} \quad ; \quad \widetilde{q} \simeq 822.35 \text{ mm} \quad ; \quad L_S \simeq 387.58 \text{ mm}$$

$$\widetilde{\mathbf{c}} \,\hat{=}\, \begin{Bmatrix} 0.760 \\ 0.134 \\ 0.636 \end{Bmatrix} \quad ; \quad \mathbf{h} \,\hat{=}\, \begin{Bmatrix} 0.537 \\ 0.013 \\ -0.844 \end{Bmatrix} \quad ; \quad \widehat{\mathbf{n}} \,\hat{=}\, \begin{Bmatrix} -0.492 \\ 0.174 \\ -0.853 \end{Bmatrix}$$

- Observing system ($\widetilde{\mathbf{k}}$ is chosen parallel to the optical axis):

$$\widetilde{\mathbf{m}}_1 \,\hat{=}\, \begin{Bmatrix} 0.966 \\ -0.051 \\ -0.254 \end{Bmatrix} \quad ; \quad \widetilde{\mathbf{m}}_2 \,\hat{=}\, \begin{Bmatrix} 0.000 \\ 0.980 \\ -0.198 \end{Bmatrix} \quad ; \quad \widetilde{\mathbf{k}} \,\hat{=}\, \begin{Bmatrix} 0.259 \\ 0.192 \\ 0.947 \end{Bmatrix}$$

- Deformation of the object (Fig. 3.12):

$$\mathbf{u} \,\hat{=}\, \begin{Bmatrix} 1.073 \text{ mm} \\ 0.116 \text{ mm} \\ 0.045 \text{ mm} \end{Bmatrix} \quad ; \quad \chi = 20.31\ .10^{-4} = 0.12^\circ \quad ; \quad \mathbf{n}_S \,\hat{=}\, \begin{Bmatrix} -0.000172 \\ 0.171832 \\ -0.985126 \end{Bmatrix}$$

$$\boldsymbol{\omega}_S \,\hat{=}\, \begin{Bmatrix} 0.00\ .10^{-4} \\ 3.49\ .10^{-4} \\ -20.01\ .10^{-4} \end{Bmatrix} \quad ; \quad \boldsymbol{\omega} \,\hat{=}\, \begin{Bmatrix} -3.49\ .10^{-4} \\ 0.00\ .10^{-4} \\ 0 \end{Bmatrix} \quad ; \quad \Omega\mathbf{n} \,\hat{=}\, \begin{Bmatrix} 0 \\ 0 \\ -20.01\ .10^{-4} \end{Bmatrix}$$

$$\widetilde{\gamma} \simeq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ; \quad \widetilde{\mathbf{K}}\widehat{\mathbf{Q}}\widehat{\mathbf{M}}^T\mathbf{M}\widetilde{\mathbf{w}} \,\hat{=}\, \begin{Bmatrix} -30.04\ .10^{-4} \\ -8.38\ .10^{-4} \\ 9.93\ .10^{-4} \end{Bmatrix} \quad ; \quad \begin{matrix} \sigma_1 = 31.11\ .10^{-4} \\ \sigma_2 = 10.18\ .10^{-4} \end{matrix}$$

Note that in the system $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, the displacement $\mathbf{u}$ reads $u_1 = 1.019$ mm, $u_2 = 0.105$ mm and $u_3 = 0.343$ mm.

- Needed modification for fringe recovery (Fig. 3.13):

$$\mathbf{t} \,\hat{=}\, \begin{Bmatrix} 2.004 \text{ mm} \\ 0.213 \text{ mm} \\ -1.672 \text{ mm} \end{Bmatrix} \quad ; \quad \hat{\chi} = 92.87\ .10^{-4} = 0.53^\circ \quad ; \quad \widehat{\mathbf{n}}_H \,\hat{=}\, \begin{Bmatrix} 0.00 \\ 1.00 \\ 0.00 \end{Bmatrix}$$

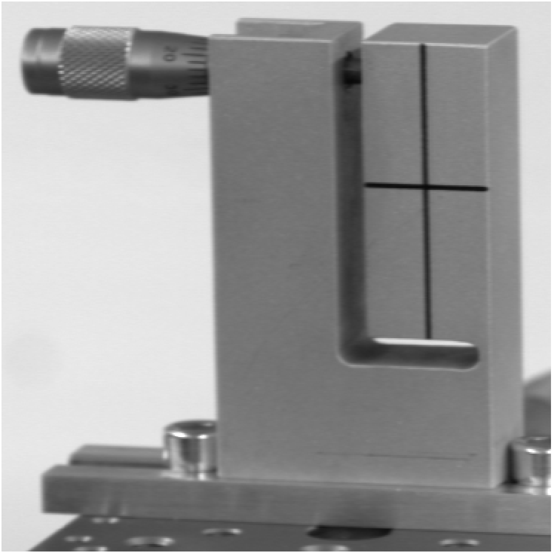

Fig. 3.8 Real object. The center of the cross represents the point P on the object surface at recording. The nearly horizontal line gives the direction of $\mathbf{e}_1$ in the object plane of the optical system (this means that $\mathbf{e}_2$ is not parallel to the vertical line).

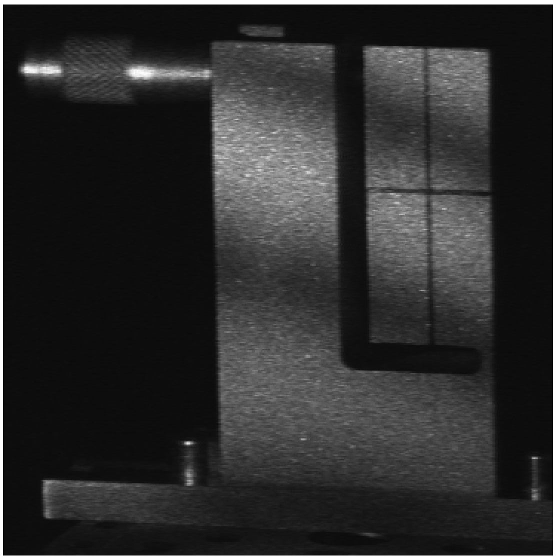

Fig. 3.9 Real-time reconstruction with no deformation and no modification. The camera is focused on the object. Because of repositioning errors, some interference fringes appear.

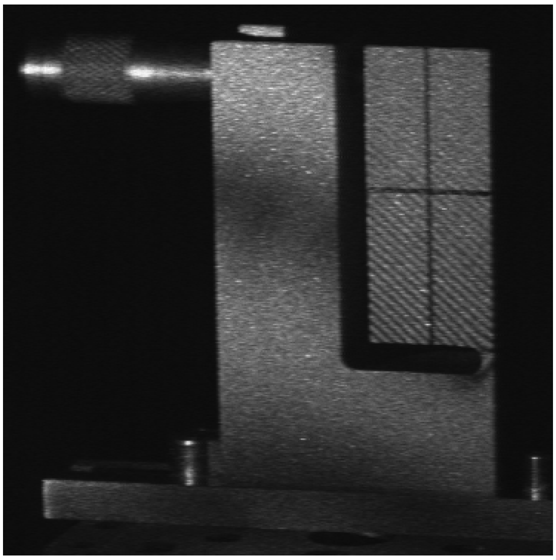

Fig. 3.10 Real-time reconstruction with a moderate deformation and no modification. The camera is focused on the object. The interference fringes disappear slowly.

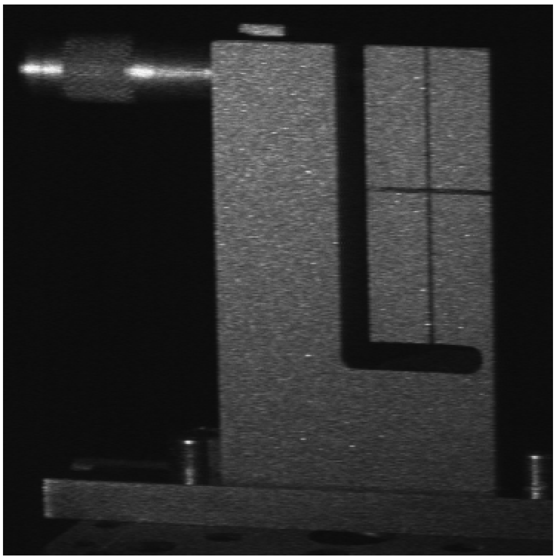

Fig. 3.11 Real-time reconstruction with a large deformation and no modification. No interference fringes can be observed. Deformation measurements are impossible.

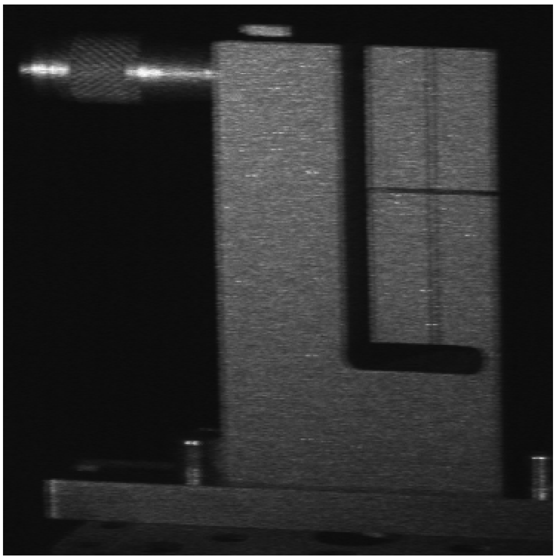

Fig. 3.12 Real-time reconstruction with a large deformation so that the relative lateral displacement of the images becomes visible and can be measured by image processing (see the two vertical lines around P). No modification is performed and no interference fringes can be observed.

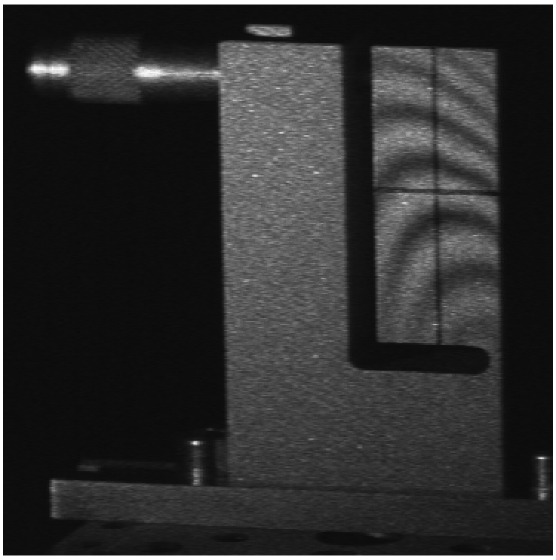

Fig. 3.13 *Fringe recovery*. Real-time reconstruction with the same large deformation as in figure 3.12, but with a suitable modification of the hologram. The images are superposed and the interference fringes spaced and contrasted. The deformation can be measured by taking into account the modification values. The center of the cross represents now the point $\widetilde{\mathrm{P}}$.

Summary:

Figure 3.10 shows the following moderate deformation of the object:

$u_x = 0.037$ mm, $u_y = 0.012$ mm, $u_z = 0.000$ mm,
$\chi = -15.00\ .10^{-4} = -0.09°, \mathbf{n}_S = (0, 0, 1)$.

Figure 3.11 shows the following large deformation of the object:

$u_x = 0.152$ mm, $u_y = 0.116$ mm, $u_z = 0.045$ mm,
$\chi = 20.31\ .10^{-4} = 0.12°, \mathbf{n}_S = (-0.000172, 0.171832, -0.985126)$.

Figure 3.12 shows the following large deformation of the object:

$u_x = 1.073$ mm, $u_y = 0.116$ mm, $u_z = 0.045$ mm,
$\chi = 20.31\ .10^{-4} = 0.12°, \mathbf{n}_S = (-0.000172, 0.171832, -0.985126)$.
$u_1 = 1.019$ mm, $u_2 = 0.105$ mm, $u_3 = 0.343$ mm.

Figure 3.13 shows the fringe recovery with the following modification of the hologram:

$u_x = 1.073$ mm, $u_y = 0.116$ mm, $u_z = 0.045$ mm,
$\chi = 20.31\ .10^{-4} = 0.12°, \mathbf{n}_S = (-0.000172, 0.171832, -0.985126)$.
$t_x = 2.004$ mm, $t_y = 0.213$ mm, $t_z = -1.672$ mm,
$\hat{\chi} = 92.87\ .10^{-4} = 0.53°, \widehat{\mathbf{n}}_H = (0, 1, 0)$.

4. Calibration of projection moiré pattern

4.1 Introduction

Before to perform any deformation measurement of opaque objects by means of holographic interferometry as described in the previous section, it is necessary to first determine the shape of the object surface in the 3-dimensional space. For plane surfaces, this process is obviously very trivial. For curved surfaces however, we need an optical method which allows accurate quantitative acquisition of the whole surface shape. This can be achieved by applying the projection moiré technique, which for example allows measuring the direction of the unit normal in each point P on the surface and thus enables the calculation of the corresponding normal and oblique projections. In this section, we assume that the reader already knows the basic concepts of projection moiré and show how to apply the intrinsic tensor calculus to this topic.

The shape of an opaque curved object surface in space can be quantitatively described using the projection moiré technique [4.3–4.20]. The purpose of this section is to present the general tensor equations of projection moiré for all geometrical cases. Emphasis is put on relative moiré, which is used in most experiments, and on difference moiré, which is generally used to calibrate optical systems. The concept of the sensitivity vector, which comes from holographic interferometry, is introduced. The obtained theoretical tensor equations are used to describe how an optical setup can be correctly calibrated without using other optical methods.

In the last section, we describe how to calibrate an optical setup and perform a quantitative experiment. Using a computer-based image processing system, an experimental verification of the theoretical equations is performed. Simultaneously, we gain evidence of a few nonlinear effects and show which parameters of the setup are of importance and should be carefully controlled.

4.2 Principle of projection moiré

The introduction of computer-based image processing systems have enabled a rapid development of the applications that use the projection moiré technique, which as such is not new. By means of a light source S (or projector), a grating $\widehat{\mathrm{G}}$ is projected onto the surface G of an object (Fig. 4.1). The observation of the projected grating $\widehat{\mathrm{G}}$ on the object through another grating $\overset{\frown}{\mathrm{G}}$ from the point R (or camera) enables seeing moiré fringes if the projections of the two gratings $\widehat{\mathrm{G}}$ and $\overset{\frown}{\mathrm{G}}$ onto the object surface are similar. The whole information concerning the shape of the object surface is contained in these fringes.

Assuming two sinusoidal gratings, the transmittance functions $\widehat{T}$ of $\widehat{\mathrm{G}}$ and $\overset{\frown}{T}$ of $\overset{\frown}{\mathrm{G}}$ are written

$$\widehat{T} = \frac{1}{2}[1 - \cos(2\pi \widehat{D})] \qquad ; \qquad \overset{\frown}{T} = \frac{1}{2}[1 - \cos(2\pi \overset{\frown}{D})] \tag{4.1}$$

where $\widehat{D}$ is the line order of grating $\widehat{\mathrm{G}}$ and $\overset{\frown}{D}$ that of $\overset{\frown}{\mathrm{G}}$. Maximal transmittance (i.e. a white line) is reached for $\widehat{T} = 1$ or $\overset{\frown}{T} = 1$ respectively, and no transmittance (e.g. a black line) for $\widehat{T} = 0$ or $\overset{\frown}{T} = 0$ (Fig. 4.2). The intensity distribution I_P over the object shape depends on the intensity I_S of the projector and the transmittance $\widehat{T}$ of the grating $\widehat{\mathrm{G}}$. In a similar way, the intensity I_R received by the camera depends on the intensity I_P over the object and the transmittance $\widehat{T}$ of the grating $\widehat{\mathrm{G}}$.

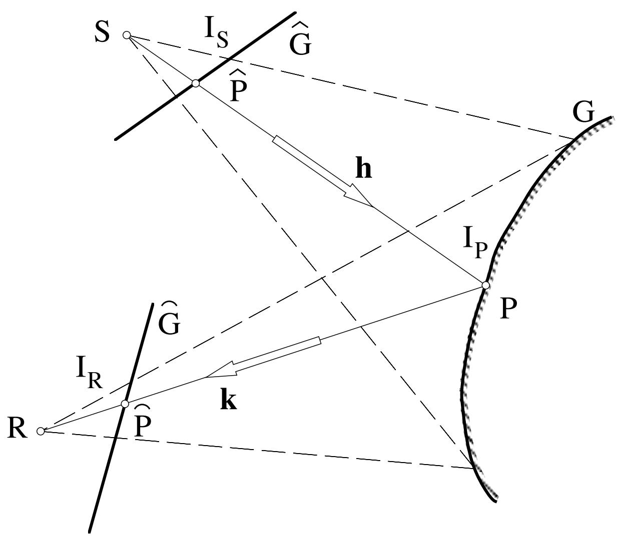

Fig. 4.1: Principle of projection moiré

Assuming a uniform intensity, we have

$$
\begin{aligned}
I_P &= \widehat{T} I_S = I_S[1 - \cos(2\pi\widehat{D})]/2 \\
I_R &= \widehat{T} I_P = \widehat{T}\widehat{T} I_S = \frac{I_S}{4}[1 - \cos(2\pi\widehat{D}) - \cos(2\pi\widehat{D})] + \frac{I_S}{8}\cos(2\pi D_m) + \frac{I_S}{8}\cos(2\pi D_M)
\end{aligned}
\tag{4.2}
$$

In the above expression for the intensity I_R, the term before the last term represents an *invisible* high frequency moiré with fringe order $D_m = \widehat{D} + \widehat{D}$ and the last term a *visible* low frequency moiré with fringe order $D_M = \widehat{D} - \widehat{D}$ (Fig. 4.3). Here, we are only interested in the visible moiré D_M which contains the needed information on the shape of the object surface.

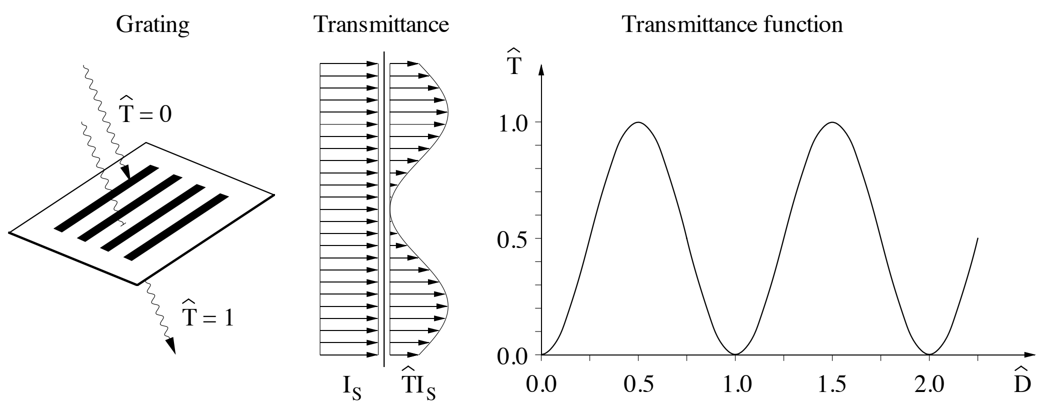

Fig. 4.2: Transmittance function of a sinusoidal grating

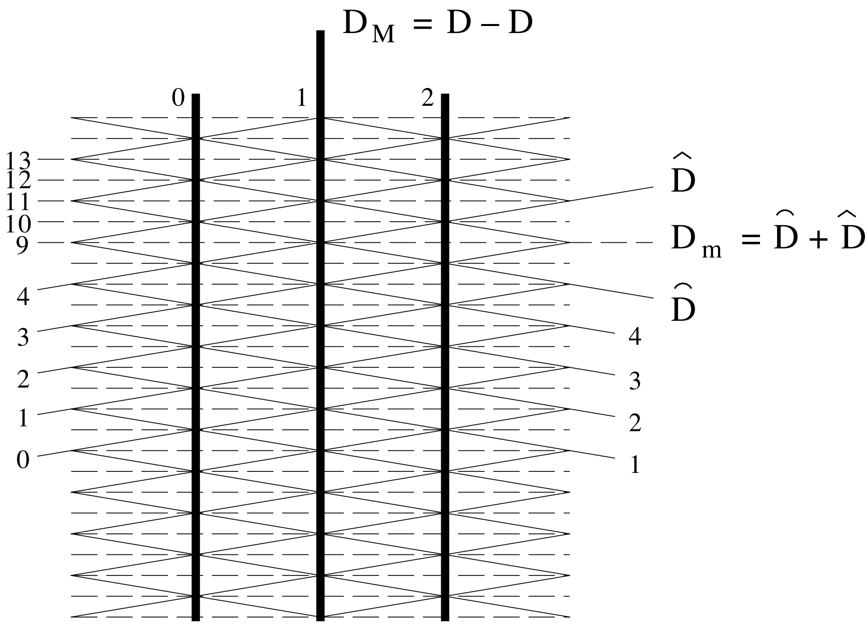

Fig. 4.3: Formation of moiré fringes

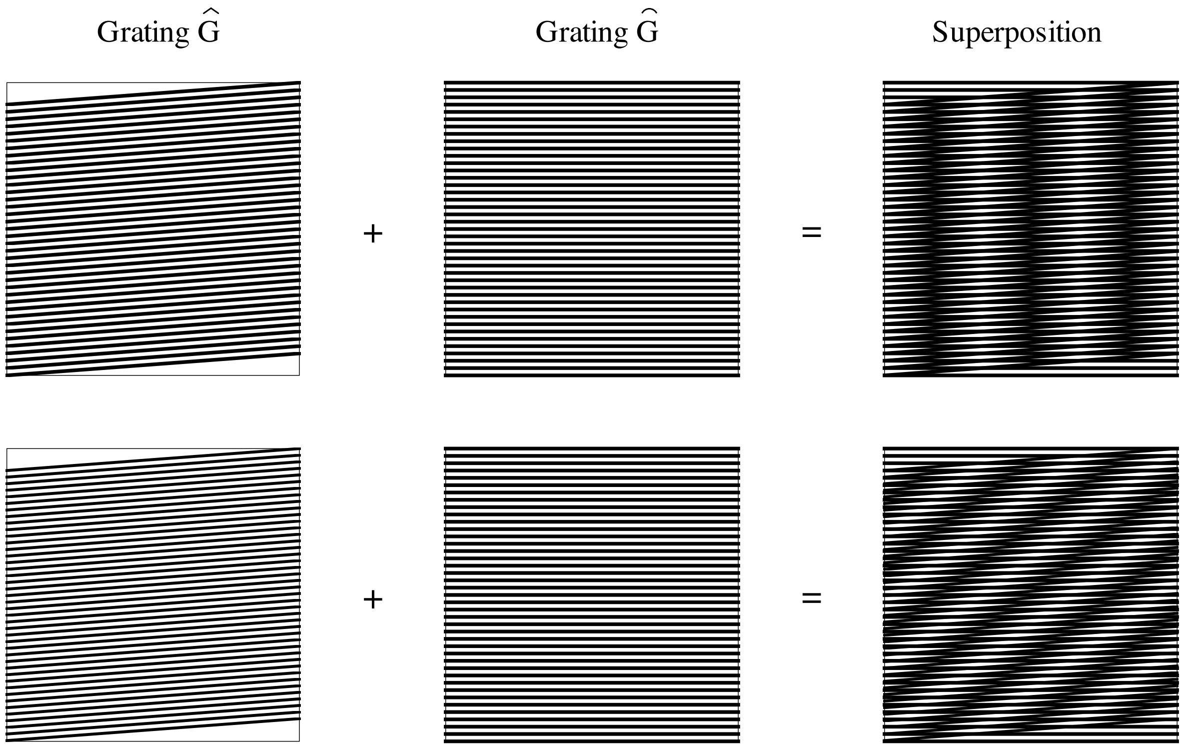

Fig. 4.4: Examples of moiré fringes formation in the case of two linear gratings

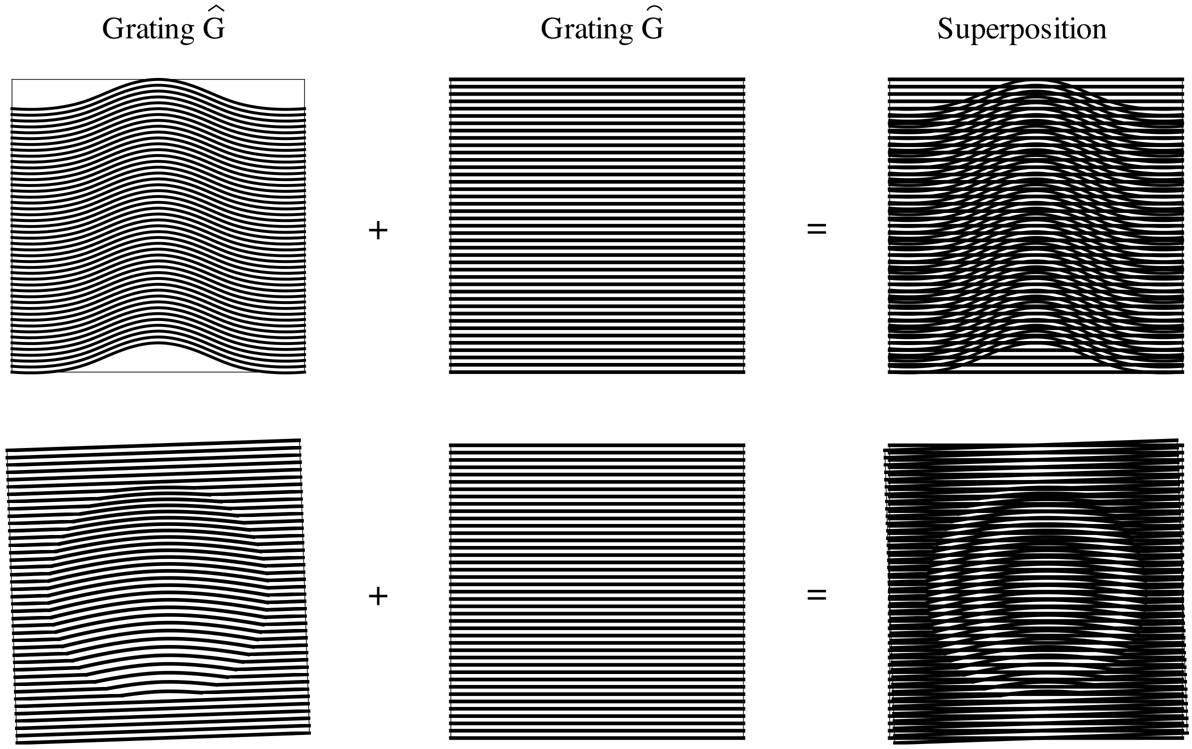

Fig. 4.5: Examples of moiré fringes formation in the case of one nonlinear grating

4.3 Optical model

The optical system of the camera (like that of the projector) can be described in the ideal case by the following geometrical relations (Fig. 4.6)

$$\frac{1}{f} = \frac{1}{s} + \frac{1}{s^*} = \frac{1}{L_e} - \frac{1}{L_e^*} \tag{4.3}$$

where f is the focal length, where s and s^* are respectively the distances from the principal points H and H* to the "object" and "image" points $\check{\mathrm{K}}$ and $\check{\mathrm{K}}^*$, and where L_e and L_e^* are respectively the distances from the principal points H and H* to the collineation centers R and R*. For the camera, the projection centers R and R* are respectively located in the entrance pupil (aperture stop) and in the exit pupil [4.1–4.5]. With $\xi/\xi^* = (s-f)/f$, we may define in this optical model a virtual collinear image K_v on the object side associated to the point K and its image K* such that

$$\xi_v = \xi^* \quad ; \quad \frac{\xi_v}{s_v - L_e} = \frac{\xi}{s - L_e} \qquad \Rightarrow \qquad s_v = f\left(\frac{s - L_e}{s - f}\right) + L_e \tag{4.4}$$

where ξ, ξ^* and ξ_v are respectively positive distances in the object, image and virtual image planes of the optical system.

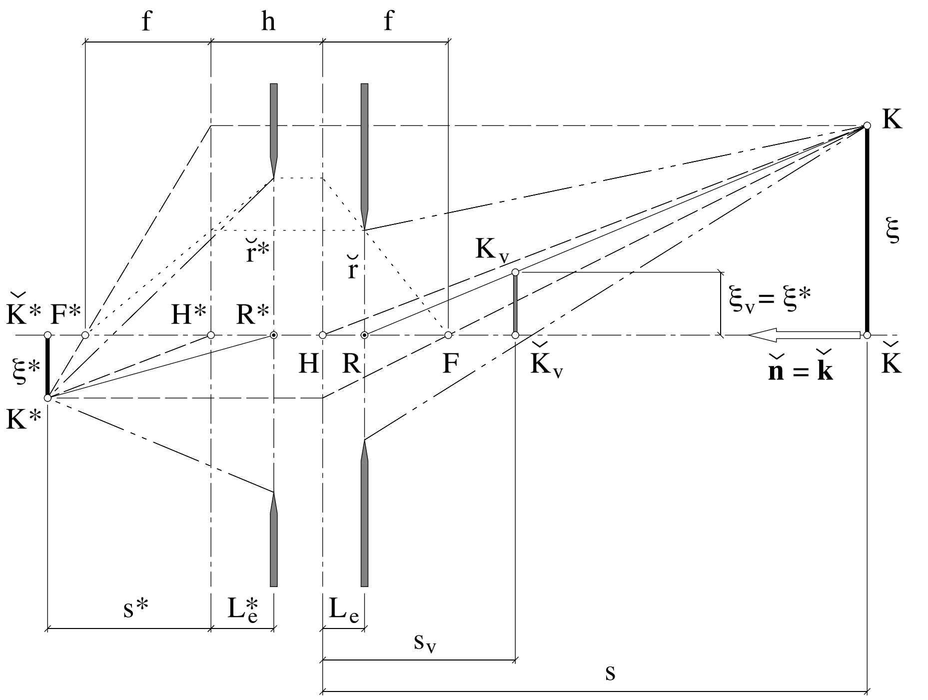

Fig. 4.6: Model of the optical system

4.4 Line order of the camera grating

The line order $\widehat{D}$ of the camera grating is first defined on the camera grid-plane in point $\widehat{\mathrm{P}}$ (Fig. 4.7). In case of a grating of equidistant straight line, the expression for the dimensionless scalar $\widehat{D}$ reads

$$\widehat{D} = \widehat{D}(\widehat{\mathbf{r}}) = \frac{1}{\widehat{\lambda}}\widehat{\mathbf{g}} \cdot \widehat{\mathbf{N}}(\widehat{\mathbf{r}} - \widehat{\mathbf{r}}_c) \qquad ; \qquad \widehat{\mathbf{N}} = \mathbf{I} - \widehat{\mathbf{n}} \otimes \widehat{\mathbf{n}} \tag{4.5}$$

where the vector $\widehat{\mathbf{n}}$ is the unit normal to the grid-plane and the tensor $\widehat{\mathbf{N}}$ the corresponding normal projector. The constant scalar value $\widehat{\lambda}$ represents the line spacing between two neighboring lines on the camera grid-plane. The caracteristic unit vector $\widehat{\mathbf{g}} \equiv \widehat{\mathbf{N}}\widehat{\mathbf{g}}$ is situated in the camera grid-plane ($\widehat{\mathbf{g}} \perp \widehat{\mathbf{n}}$) and is perpendicular to the grating lines. The variable vector $\widehat{\mathbf{r}}$ is a vector coordinate giving the position of point $\widehat{\mathrm{P}}$ on the camera grid-plane, and $\widehat{\mathbf{r}}_c$ represents the vector coordinate of some reference point on the grid-plane such that $\widehat{D}(\widehat{\mathbf{r}}_c) = 0$.

The line order $\widehat{D}$ of the camera grating can be extended in the space $\mathbb{R}^3$ by central projection relatively to the collineation center R (Fig. 4.8), its value remaining the same on the straight line passing through the points R, $\widehat{\mathrm{P}}$, P and K. With the vector coordinate $\mathbf{r} = \widehat{\mathbf{r}} - p\mathbf{k}$ of the point P in space, we have

$$\widehat{D} = \widehat{D}(\mathbf{r}) = \frac{1}{\widehat{\lambda}}\widehat{\mathbf{g}} \cdot \widehat{\mathbf{M}}_k^T(\mathbf{r} - \widehat{\mathbf{r}}_c) \qquad ; \qquad \widehat{\mathbf{M}}_k = \mathbf{I} - \frac{\widehat{\mathbf{n}} \otimes \mathbf{k}}{\widehat{\mathbf{n}} \cdot \mathbf{k}} \tag{4.5'}$$

where $\widehat{\mathbf{M}}_k$ is an oblique projector projecting along the direction $\widehat{\mathbf{n}}$ onto a plane normal to $\mathbf{k}$. With $\widehat{\mathbf{M}}_k^T\mathbf{k} = 0$ and $\widehat{\mathbf{M}}_k^T\widehat{\mathbf{N}} = \widehat{\mathbf{N}}$, we can demonstrate the following

$$\widehat{D} = \widehat{D}(\mathbf{r}) = \frac{1}{\widehat{\lambda}}\widehat{\mathbf{g}} \cdot \widehat{\mathbf{M}}_k^T(\mathbf{r} - \widehat{\mathbf{r}}_c) = \frac{1}{\widehat{\lambda}}\widehat{\mathbf{g}} \cdot \widehat{\mathbf{M}}_k^T(\widehat{\mathbf{r}} - p\mathbf{k} - \widehat{\mathbf{r}}_c) = \frac{1}{\widehat{\lambda}}\widehat{\mathbf{g}} \cdot \widehat{\mathbf{M}}_k^T(\widehat{\mathbf{r}} - \widehat{\mathbf{r}}_c)$$
$$= \frac{1}{\widehat{\lambda}}\widehat{\mathbf{g}} \cdot \widehat{\mathbf{M}}_k^T\widehat{\mathbf{N}}(\widehat{\mathbf{r}} - \widehat{\mathbf{r}}_c) = \frac{1}{\widehat{\lambda}}\widehat{\mathbf{g}} \cdot \widehat{\mathbf{N}}(\widehat{\mathbf{r}} - \widehat{\mathbf{r}}_c) = \widehat{D}(\widehat{\mathbf{r}}) \qquad \square \text{ qed}$$

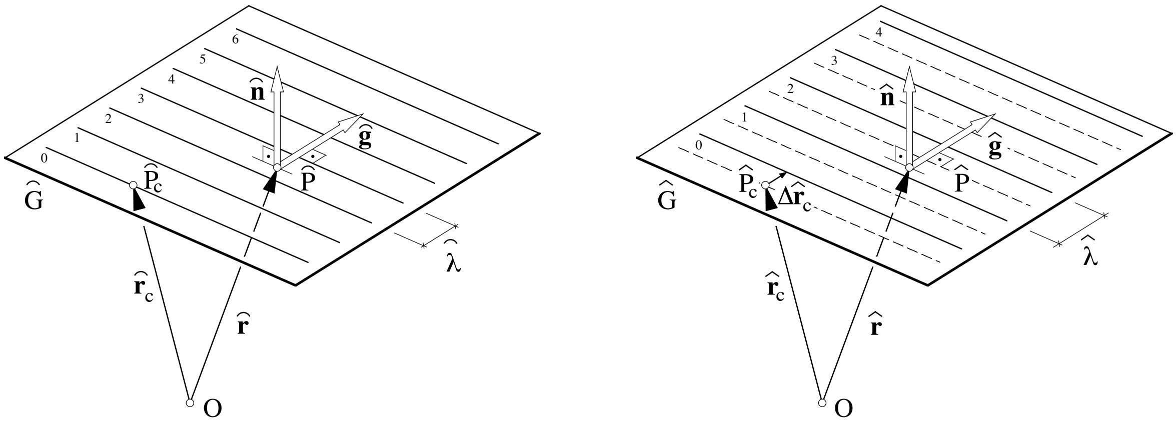

Fig. 4.7: Line orders of the camera grating and of the projector grating

4.5 Line order of the projector grating

The line order $\widehat{D}$ of the projector grating is also first defined on the projector grid-plane in point $\widehat{\mathrm{P}}$ (Fig. 4.7). In case of a grating of equidistant straight line, the expression of the dimensionless scalar $\widehat{D}$ reads

$$\widehat{D} = \widehat{D}(\widehat{\mathbf{r}}) = \frac{1}{\widehat{\lambda}}\widehat{\mathbf{g}} \cdot \widehat{\mathbf{N}}(\widehat{\mathbf{r}} - \widehat{\mathbf{r}}_c) - \Psi \quad ; \quad \Psi = \frac{1}{\widehat{\lambda}}\widehat{\mathbf{g}} \cdot \Delta\widehat{\mathbf{r}}_c \quad ; \quad \Delta\widehat{\mathbf{r}}_c \equiv \widehat{\mathbf{N}}\Delta\widehat{\mathbf{r}}_c = \Psi\widehat{\lambda}\widehat{\mathbf{g}} \tag{4.6}$$

where the vector $\widehat{\mathbf{n}}$ the unit normal to the grid-plane and the tensor $\widehat{\mathbf{N}} = \mathbf{I} - \widehat{\mathbf{n}} \otimes \widehat{\mathbf{n}}$ the corresponding normal projector. The constant scalar value $\widehat{\lambda}$ represents the line spacing between two neighboring lines on the projector grid-plane. The caracteristic unit vector $\widehat{\mathbf{g}} \equiv \widehat{\mathbf{N}}\widehat{\mathbf{g}}$ is situated in the projector grid-plane ($\widehat{\mathbf{g}} \perp \widehat{\mathbf{n}}$) and is perpendicular to the grating lines. The variable vector $\widehat{\mathbf{r}}$ is a vector coordinate giving the position of point $\widehat{\mathrm{P}}$ on the projector grid-plane, and $\widehat{\mathbf{r}}_c$ represents the vector coordinate of some reference point on the grid-plane such that $\widehat{D}(\widehat{\mathbf{r}}_c) = -\Psi$. The scalar increment Ψ plays an important role in the *phase shifting* method to get a phase image [4.6,4.7] and the vector increment $\Delta\widehat{\mathbf{r}}_c = \Psi\widehat{\lambda}\widehat{\mathbf{g}}$ describes a uniform in-plane translation of the grating on the grid-plane, the translation being in the direction of $\widehat{\mathbf{g}}$ for Ψ positive. The line order $\widehat{D}$ of the projector grating can also be extended in the space $\mathbb{R}^3$ by central projection relatively to the collineation center S (Fig. 4.8), its value remaining the same on the straight line passing through the points S, $\widehat{\mathrm{P}}$, P and Q. With the vector coordinate $\mathbf{r} = \widehat{\mathbf{r}} + p_S\mathbf{h}$ of the point P in space, we have

$$\widehat{D} = \widehat{D}(\mathbf{r}) = \frac{1}{\widehat{\lambda}}\widehat{\mathbf{g}} \cdot \widehat{\mathbf{M}}_h^T(\mathbf{r} - \widehat{\mathbf{r}}_c) \qquad ; \qquad \widehat{\mathbf{M}}_h = \mathbf{I} - \frac{\widehat{\mathbf{n}} \otimes \mathbf{h}}{\widehat{\mathbf{n}} \cdot \mathbf{h}} \tag{4.6'}$$

where $\widehat{\mathbf{M}}_h$ is an oblique projector projecting along the direction $\widehat{\mathbf{n}}$ onto a plane normal to $\mathbf{h}$. With $\widehat{\mathbf{M}}_h^T\mathbf{h} = 0$ and $\widehat{\mathbf{M}}_h^T\widehat{\mathbf{N}} = \widehat{\mathbf{N}}$, we can demonstrate the following

$$\begin{aligned}\widehat{D} = \widehat{D}(\mathbf{r}) &= \frac{1}{\widehat{\lambda}}\widehat{\mathbf{g}}\cdot\widehat{\mathbf{M}}_h^T(\mathbf{r}-\widehat{\mathbf{r}}_c) = \frac{1}{\widehat{\lambda}}\widehat{\mathbf{g}}\cdot\widehat{\mathbf{M}}_h^T(\widehat{\mathbf{r}}+p_S\mathbf{h}-\widehat{\mathbf{r}}_c) = \frac{1}{\widehat{\lambda}}\widehat{\mathbf{g}}\cdot\widehat{\mathbf{M}}_h^T(\widehat{\mathbf{r}}-\widehat{\mathbf{r}}_c)\\ &= \frac{1}{\widehat{\lambda}}\widehat{\mathbf{g}}\cdot\widehat{\mathbf{M}}_h^T\widehat{\mathbf{N}}(\widehat{\mathbf{r}}-\widehat{\mathbf{r}}_c) = \frac{1}{\widehat{\lambda}}\widehat{\mathbf{g}}\cdot\widehat{\mathbf{N}}(\widehat{\mathbf{r}}-\widehat{\mathbf{r}}_c) = \widehat{D}(\widehat{\mathbf{r}}) \qquad \square \text{ qed}\end{aligned}$$

4.6 Moiré fringe order

Considering the formation of moiré fringes in space with the two fixed collineation centers R and S and their corresponding fixed grating $\overset{\frown}{\mathrm{G}}$ and $\widehat{\mathrm{G}}$, the vector variables $\overset{\frown}{\mathbf{r}}$ and $\widehat{\mathbf{r}}$ are not independent (Fig. 4.8). Both are functions of the vector coordinate $\mathbf{r}$ of some point P in the space $\mathbb{R}^3$, which means $\mathbf{r} \rightarrow \overset{\frown}{\mathbf{r}} = \overset{\frown}{\mathbf{r}}(\mathbf{r})$ and $\mathbf{r} \rightarrow \widehat{\mathbf{r}} = \widehat{\mathbf{r}}(\mathbf{r})$. The moiré fringe order D_M in point P is then written as follows

$$D_M = D_M(\mathbf{r}) = D_M(\overset{\frown}{\mathbf{r}},\widehat{\mathbf{r}}) = \overset{\frown}{D} - \widehat{D} = \frac{1}{\overset{\frown}{\lambda}}\overset{\frown}{\mathbf{g}}\cdot\overset{\frown}{\mathbf{N}}(\overset{\frown}{\mathbf{r}}-\overset{\frown}{\mathbf{r}}_c) - \frac{1}{\widehat{\lambda}}\widehat{\mathbf{g}}\cdot\widehat{\mathbf{N}}(\widehat{\mathbf{r}}-\widehat{\mathbf{r}}_c) + \Psi \tag{4.7}$$

Note that a *moiré surface* in the space $\mathbb{R}^3$ can be described by a set of vectors $\mathbf{r} = \mathbf{r}_M$ for which the moiré fringe order D_M is constant, which means $\mathbf{r}_M \rightarrow D_M(\mathbf{r}_M) = ¢$. On another hand, we may choose for our purposes the point P on the object surface G and look at the behaviour of D_M. This is useful for the study of the object shape. Then, for a given fixed object surface G, the function D_M can also by definition be extended in the space $\mathbb{R}^3$ by central projection relatively to the collineation center R, its value remaining the same on the straight line passing through R, $\overset{\frown}{\mathrm{P}}$, P and K. For another point P_0 on the object surface, we have

$$D_{M0} = D_M(\mathbf{r}_0) = D_M(\overset{\frown}{\mathbf{r}}_0,\widehat{\mathbf{r}}_0) = \overset{\frown}{D}_0 - \widehat{D}_0 = \frac{1}{\overset{\frown}{\lambda}}\overset{\frown}{\mathbf{g}}\cdot\overset{\frown}{\mathbf{N}}(\overset{\frown}{\mathbf{r}}_0-\overset{\frown}{\mathbf{r}}_c) - \frac{1}{\widehat{\lambda}}\widehat{\mathbf{g}}\cdot\widehat{\mathbf{N}}(\widehat{\mathbf{r}}_0-\widehat{\mathbf{r}}_c) + \Psi_0 \tag{4.8}$$

where $\overset{\frown}{D}_0 = \overset{\frown}{D}(\overset{\frown}{\mathbf{r}}_0) = \overset{\frown}{\mathbf{g}}\cdot\overset{\frown}{\mathbf{N}}(\overset{\frown}{\mathbf{r}}_0-\overset{\frown}{\mathbf{r}}_c)/\overset{\frown}{\lambda}$ and $\widehat{D}_0 = \widehat{D}(\widehat{\mathbf{r}}_0) = [\widehat{\mathbf{g}}\cdot\widehat{\mathbf{N}}(\widehat{\mathbf{r}}_0-\widehat{\mathbf{r}}_c)/\widehat{\lambda}] - \Psi_0$ are respectively the line order of the camera grating in $\overset{\frown}{\mathrm{P}}_0$ and that of the projector grating in $\widehat{\mathrm{P}}_0$. Practically, Ψ and Ψ_0 are the same in most of the cases. The value D_{M0} is used later as reference for the other values of D_M.

4.7 General expression of relative moiré

We can now define a relative moiré value $\mathit{\Delta} D_M$ in some point P in the space $\mathbb{R}^3$ by taking the value of the moiré fringe order in a point P_0 as reference (Fig. 4.8). We write with $\Psi = \Psi_0$

$$\begin{aligned}\mathit{\Delta} D_M = D_M - D_{M0} &= D_M(\mathbf{r}) - D_M(\mathbf{r}_0) = (\overset{\frown}{D} - \overset{\frown}{D}_0) - (\widehat{D} - \widehat{D}_0)\\ &= \frac{1}{\overset{\frown}{\lambda}}\overset{\frown}{\mathbf{g}}\cdot\overset{\frown}{\mathbf{N}}(\overset{\frown}{\mathbf{r}}-\overset{\frown}{\mathbf{r}}_0) - \frac{1}{\widehat{\lambda}}\widehat{\mathbf{g}}\cdot\widehat{\mathbf{N}}(\widehat{\mathbf{r}}-\widehat{\mathbf{r}}_0) = \frac{1}{\overset{\frown}{\lambda}}\overset{\frown}{\mathbf{g}}\cdot\Delta\overset{\frown}{\mathbf{r}} - \frac{1}{\widehat{\lambda}}\widehat{\mathbf{g}}\cdot\Delta\widehat{\mathbf{r}}\end{aligned} \tag{4.9}$$

where $\Delta\overset{\frown}{\mathbf{r}} \equiv \overset{\frown}{\mathbf{N}}(\overset{\frown}{\mathbf{r}}-\overset{\frown}{\mathbf{r}}_0)$ and $\Delta\widehat{\mathbf{r}} \equiv \widehat{\mathbf{N}}(\widehat{\mathbf{r}}-\widehat{\mathbf{r}}_0)$ are, respectively, the vector increments on the camera and projector grid-planes. The dimensionless scalar $\mathit{\Delta} D_M$ is simply called *relative moiré* in point P.

Because $D_{M0} = ¢$, the moiré surfaces are still described by a set of vectors $\mathbf{r} = \mathbf{r}_M$ for which $\Delta D_M = ¢$. Equation (4.9) is general and is valid for all geometrical configurations of the optical setup. For a given fixed object surface G, the relative moiré ΔD_M can also be extended in space like the moiré fringe order D_M, its value remaining the same on the straight line passing through R, $\widehat{\mathrm{P}}$, P and K.

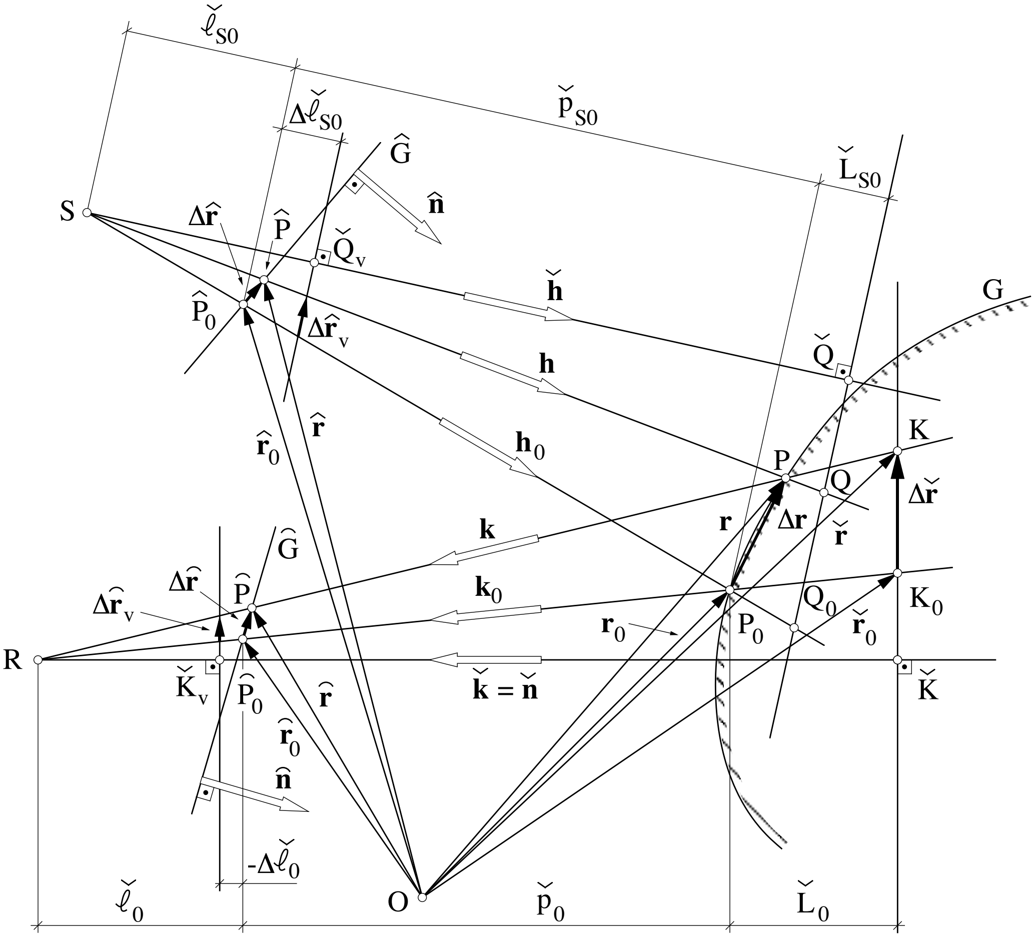

Fig. 4.8: Model of a general geometrical moiré setup

In order to write equation (4.9) explicitly, we introduce the exact *affine connections* making the bridge among the vectors $\Delta\mathbf{r}$, $\Delta\widehat{\mathbf{r}}$, $\Delta\hat{\mathbf{r}}$, $\Delta\widehat{\mathbf{r}}_v$, $\Delta\hat{\mathbf{r}}_v$, $\Delta\check{\mathbf{r}}$ and $\Delta\check{\mathbf{r}}_v$. These affine connections involve normal and oblique projectors in $\mathbb{R}^3$ similar to those previously introduced. Figure 4.8 shows that $\Delta\mathbf{r} = \mathbf{r} - \mathbf{r}_0$ is the vector going from P_0 to P in the 3-dimensional space, that $\Delta\widehat{\mathbf{r}}_v$ and $\Delta\hat{\mathbf{r}}_v$ are, respectively, the virtual collinear images of $\Delta\widehat{\mathbf{r}}$ and $\Delta\hat{\mathbf{r}}$ in the virtual image planes of the camera and the projector, that $\Delta\check{\mathbf{r}}_v = \Delta\widehat{\mathbf{r}}_v$ is the virtual collinear image of $\Delta\check{\mathbf{r}}$ in the virtual image plane of the camera, and that $\Delta\check{\mathbf{r}}$ is the collinear image of $\Delta\mathbf{r}$ in the object plane of the optical system of the camera. With the unit vectors $\check{\mathbf{k}}$ and $\check{\mathbf{h}}$, we first write the two normal projectors $\check{\mathbf{K}} = \mathbf{I} - \check{\mathbf{k}} \otimes \check{\mathbf{k}}$ and $\check{\mathbf{H}} = \mathbf{I} - \check{\mathbf{h}} \otimes \check{\mathbf{h}}$, with $\check{\mathbf{K}}$ projecting onto a plane perpendicular to the optical axis of the camera, and with $\check{\mathbf{H}}$ projecting onto a plane perpendicular to the optical axis of the projector. We then write the following oblique projectors

$$\check{\mathbf{M}}_k = \mathbf{I} - \frac{\check{\mathbf{k}} \otimes \mathbf{k}}{\check{\mathbf{k}} \cdot \mathbf{k}} \quad ; \quad \check{\mathbf{M}}_h = \mathbf{I} - \frac{\check{\mathbf{h}} \otimes \mathbf{h}}{\check{\mathbf{h}} \cdot \mathbf{h}} \quad ; \quad \check{\mathbf{M}} = \mathbf{I} - \frac{\check{\mathbf{h}} \otimes \check{\mathbf{k}}}{\check{\mathbf{h}} \cdot \check{\mathbf{k}}} \tag{4.10}$$

with $\check{\mathbf{M}}_k$ projecting along $\check{\mathbf{k}}$ onto a plane normal to $\mathbf{k}$, with $\check{\mathbf{M}}_h$ projecting along $\check{\mathbf{h}}$ onto a plane normal to $\mathbf{h}$, and with $\check{\mathbf{M}}$ projecting along $\check{\mathbf{h}}$ onto a plane normal to $\check{\mathbf{k}}$. Remember that this description is only valid if the vector is applied "on the right" of the corresponding oblique projector. The two directions $\mathbf{k}$ and $\mathbf{h}$ are commonly called *observing* and *illuminating* direction. Thus, the *exact* affine connections read

$$\Delta\overset{\frown}{\mathbf{r}} \equiv \overset{\frown}{\mathbf{N}}\Delta\overset{\frown}{\mathbf{r}} = \frac{\ell_0}{\ell_0+p_0}\overset{\frown}{\mathbf{M}}{}_k^T\Delta\mathbf{r} = \frac{\ell_0}{\ell_0+p_0+L_0}\overset{\frown}{\mathbf{M}}{}_k^T\Delta\check{\mathbf{r}} = \frac{\ell_0}{\ell_0+\Delta\ell_0}\overset{\frown}{\mathbf{M}}{}_k^T\Delta\overset{\frown}{\mathbf{r}}_v \tag{4.11}$$

$$\Delta\hat{\mathbf{r}} \equiv \hat{\mathbf{N}}\Delta\hat{\mathbf{r}} = \frac{\ell_{S0}}{\ell_{S0}+p_{S0}}\widehat{\mathbf{M}}_h^T\Delta\mathbf{r} = \frac{\ell_{S0}}{\ell_{S0}+\Delta\ell_{S0}}\widehat{\mathbf{M}}_h^T\Delta\hat{\mathbf{r}}_v \tag{4.12}$$

$$\Delta\overset{\frown}{\mathbf{r}}_v \equiv \Delta\check{\mathbf{r}}_v \equiv \check{\mathbf{K}}\Delta\overset{\frown}{\mathbf{r}}_v = \frac{\ell_0+\Delta\ell_0}{\ell_0}\check{\mathbf{M}}_k^T\Delta\overset{\frown}{\mathbf{r}} = \frac{\ell_0+\Delta\ell_0}{\ell_0+p_0}\check{\mathbf{M}}_k^T\Delta\mathbf{r} = \frac{\ell_0+\Delta\ell_0}{\ell_0+p_0+L_0}\Delta\check{\mathbf{r}} \tag{4.13}$$

$$\Delta\hat{\mathbf{r}}_v \equiv \check{\mathbf{H}}\Delta\hat{\mathbf{r}}_v = \frac{\ell_{S0}+\Delta\ell_{S0}}{\ell_{S0}}\check{\mathbf{M}}_h^T\Delta\hat{\mathbf{r}} = \frac{\ell_{S0}+\Delta\ell_{S0}}{\ell_{S0}+p_{S0}}\check{\mathbf{M}}_h^T\Delta\mathbf{r} \tag{4.14}$$

$$\check{\mathbf{M}}_k^T\Delta\mathbf{r} = \frac{\ell_0+p_0}{\ell_0+p_0+L_0}\Delta\check{\mathbf{r}} = \frac{\ell_0+p_0}{\ell_0+\Delta\ell_0}\Delta\check{\mathbf{r}}_v \qquad ; \qquad \Delta\check{\mathbf{r}} \equiv \check{\mathbf{K}}\Delta\check{\mathbf{r}} \tag{4.15}$$

with

$$\check{\ell}_0 = \ell_0\mathbf{k}_0\cdot\check{\mathbf{k}} \quad ; \quad \Delta\check{\ell}_0 = \Delta\ell_0\mathbf{k}_0\cdot\check{\mathbf{k}} \quad ; \quad \check{p}_0 = p_0\mathbf{k}_0\cdot\check{\mathbf{k}} \quad ; \quad \check{L}_0 = L_0\mathbf{k}_0\cdot\check{\mathbf{k}}$$
$$\check{\ell}_{S0} = \ell_{S0}\mathbf{h}_0\cdot\check{\mathbf{h}} \quad ; \quad \Delta\check{\ell}_{S0} = \Delta\ell_{S0}\mathbf{h}_0\cdot\check{\mathbf{h}} \quad ; \quad \check{p}_{S0} = p_{S0}\mathbf{h}_0\cdot\check{\mathbf{h}} \quad ; \quad \check{L}_{S0} = L_{S0}\mathbf{h}_0\cdot\check{\mathbf{h}}$$

Assuming P and P_0 on the object surface, the vector $\Delta\mathbf{r}$ is of particular interest because it quantitatively describes the surface shape. A normal decomposition of $\Delta\mathbf{r}$ into *interior* and *exterior* parts relatively to the direction $\check{\mathbf{k}} = \check{\mathbf{n}}$ of the optical axis of our observing system gives

$$\Delta\mathbf{r} = \mathbf{I}\Delta\mathbf{r} = (\check{\mathbf{K}} + \check{\mathbf{k}}\otimes\check{\mathbf{k}})\Delta\mathbf{r} = \check{\mathbf{K}}\Delta\mathbf{r} + (\check{\mathbf{k}}\cdot\Delta\mathbf{r})\check{\mathbf{k}} = \check{\mathbf{K}}\Delta\mathbf{r} + z\check{\mathbf{k}} \tag{4.16}$$

with

$$\check{\mathbf{K}}\Delta\mathbf{r} = x\check{\mathbf{e}} + y\check{\mathbf{e}}_\perp \quad ; \quad z = \check{\mathbf{k}}\cdot\Delta\mathbf{r} \qquad ; \qquad \begin{cases} \check{\mathbf{e}} \equiv \check{\mathbf{K}}\check{\mathbf{e}} \quad ; \quad \check{\mathbf{e}}_\perp \equiv \check{\mathbf{K}}\check{\mathbf{e}}_\perp \quad ; \quad \check{\mathbf{e}}\times\check{\mathbf{e}}_\perp = \check{\mathbf{k}} \\ \check{\mathbf{e}}\cdot\check{\mathbf{e}} = \check{\mathbf{e}}_\perp\cdot\check{\mathbf{e}}_\perp = 1 \quad ; \quad \check{\mathbf{e}}\cdot\check{\mathbf{e}}_\perp = 0 \end{cases}$$

where $\check{\mathbf{e}}$ and $\check{\mathbf{e}}_\perp$ are unit vectors situated in the plane normal to the direction $\check{\mathbf{k}}$ and where x, y, z are the cartesian components of point P relatively to P_0. An oblique decomposition of $\Delta\mathbf{r}$ relatively to the direction $\mathbf{k}$ and the plane normal to $\check{\mathbf{k}}$ gives with $z = \zeta\mathbf{k}\cdot\check{\mathbf{k}}$

$$\Delta\mathbf{r} = \mathbf{I}\Delta\mathbf{r} = \left(\check{\mathbf{M}}_k^T + \frac{\mathbf{k}\otimes\check{\mathbf{k}}}{\mathbf{k}\cdot\check{\mathbf{k}}}\right)\Delta\mathbf{r} = \check{\mathbf{M}}_k^T\Delta\mathbf{r} + \left(\frac{\check{\mathbf{k}}\cdot\Delta\mathbf{r}}{\mathbf{k}\cdot\check{\mathbf{k}}}\right)\mathbf{k} = \check{\mathbf{M}}_k^T\Delta\mathbf{r} + \zeta\mathbf{k} \tag{4.17}$$

With equations (4.12) and (4.13), the general equation for relative moiré (4.9) reads explicitly

$$\Delta D_M = \frac{1}{\overset{\frown}{\lambda}}\overset{\frown}{\mathbf{g}}\cdot\Delta\overset{\frown}{\mathbf{r}} - \frac{1}{\hat{\lambda}}\hat{\mathbf{g}}\cdot\Delta\hat{\mathbf{r}} = \Delta\mathbf{r}\cdot\left(\frac{\ell_0}{\overset{\frown}{\lambda}(\ell_0+p_0)}\overset{\frown}{\mathbf{M}}_k\overset{\frown}{\mathbf{g}} - \frac{\ell_{S0}}{\widehat{\lambda}(\ell_{S0}+p_{S0})}\widehat{\mathbf{M}}_h\hat{\mathbf{g}}\right) = \Delta\mathbf{r}\cdot\mathbf{g}(\mathbf{k},\mathbf{h}) \tag{4.18}$$

The meaning of the above equation (4.18) becomes clear by considering the scalar product of the vector $\Delta\mathbf{r} = \mathbf{r} - \mathbf{r}_0$ with the vector $\mathbf{g} = \mathbf{g}(\mathbf{k}, \mathbf{h})$. The so-called *shape* vector $\Delta\mathbf{r}$ describes the object shape in the 3-dimensional space and the *sensitivity* vector $\mathbf{g}$ depends only on the observing direction $\mathbf{k}$ and on the illuminating direction $\mathbf{h}$ for a given optical setup. Note that this definition is similar to that of the fringe order $D = \mathbf{u} \cdot \mathbf{g}$ in standard holographic and speckle interferometry. The shape vector $\Delta\mathbf{r}$ and the sensitivity vector $\mathbf{g}$ respectively correspond to the displacement vector $\mathbf{u}$ and to the sensitivity vector $\mathbf{g}$ of holographic interferometry (only their definitions differ). The moiré surfaces in space are then described by a set of vectors $\Delta\mathbf{r} = \Delta\mathbf{r}_M = \mathbf{r}_M - \mathbf{r}_0$ for which $\mathit{\Delta} D_M = \not{c}$. Because the sensitivity vector $\mathbf{g}(\mathbf{k}, \mathbf{h})$ is not constant, we must pay a particular attention to the fact that the moiré surfaces in space may generally be curved.

4.8 General expression of difference moiré

4.8.1 Theoretical calibration of a moiré setup

To illustrate the calibration procedure in a projection moiré experiment, we consider the general setup of figure 4.9, where, without restricting the generality, the two grid-planes are perpendicular to their respective optical axes, which means $\widehat{\mathbf{n}} = -\check{\mathbf{k}}$, $\widehat{\mathbf{n}} = \check{\mathbf{h}}$ and where $\mathit{\Delta}\ell_0 = \mathit{\Delta}\ell_{S0} = 0$. In place of the object, we consider a calibration plane G of unit normal $\mathbf{n}$ which can be moved in any directions to a new position $\widetilde{\mathrm{G}}$ (Fig. 4.9). In a real experiment, neither the positon of the centers of projection (R and S) nor the distance to the object surface are known exactly. We can therefore choose in the 3-dimensional space some arbitrary reference point P_0 (not necessarily on the calibration plane) that plays a central role in the calibration process. All the geometrical quantities needed to calculate the calibration factors of the moiré setup are expressed relatively to this point. This approach is very useful to practically get accurate calibration constants and correction terms from the exact theoretical equations. Considering both points P and P_0 on the calibration plane in its original position G, we have according to equation (4.18)

$$\mathit{\Delta} D_M = \Delta\mathbf{r} \cdot \mathbf{g}(\mathbf{k}, \mathbf{h}) = \Delta\mathbf{r} \cdot \left(\frac{\ell_0}{\widehat{\lambda}(\ell_0 + p_0)} \widehat{\mathbf{M}}_k \widehat{\mathbf{g}} - \frac{\ell_{S0}}{\widehat{\lambda}(\ell_{S0} + p_{S0})} \widehat{\mathbf{M}}_h \widehat{\mathbf{g}} \right) \tag{4.19}$$

Moving the calibration plane G by a known amount in translation and rotation, the moiré fringe order in each point as observed by the camera changes. Considering a point $\widetilde{\mathrm{P}}$ on the calibration plane $\widetilde{\mathrm{G}}$ in its new position (unit normal $\widetilde{\mathbf{n}}$), the general expression for the relative moiré relatively to the same reference point P_0 reads

$$\mathit{\Delta}\widetilde{D}_M = \Delta\widetilde{\mathbf{r}} \cdot \widetilde{\mathbf{g}}(\widetilde{\mathbf{k}}, \widetilde{\mathbf{h}}) = (\Delta\mathbf{r} + \eta\mathbf{k}) \cdot \left(\frac{\ell_0}{\widehat{\lambda}(\ell_0 + p_0)} \widehat{\widetilde{\mathbf{M}}}_k \widehat{\mathbf{g}} - \frac{\ell_{S0}}{\widehat{\lambda}(\ell_{S0} + p_{S0})} \widehat{\widetilde{\mathbf{M}}}_h \widehat{\mathbf{g}} \right) \tag{4.20}$$

with $\mathit{\Delta}\widetilde{D}_M = \mathit{\Delta} D_M + \delta\widetilde{D}_M$, $\Delta\widetilde{\mathbf{r}} = \Delta\mathbf{r} + \eta\mathbf{k}$ and the two oblique projectors $\widehat{\widetilde{\mathbf{M}}}_k = \mathbf{I} - [\widehat{\mathbf{n}} \otimes \widetilde{\mathbf{k}} / (\widehat{\mathbf{n}} \cdot \widetilde{\mathbf{k}})]$ and $\widehat{\widetilde{\mathbf{M}}}_h = \mathbf{I} - [\widehat{\mathbf{n}} \otimes \widetilde{\mathbf{h}} / (\widehat{\mathbf{n}} \cdot \widetilde{\mathbf{h}})]$. By choosing point $\widetilde{\mathrm{P}}$ on the same observing direction as point P, we have $\widetilde{\mathbf{k}} = \mathbf{k}$ and $\widehat{\widetilde{\mathbf{M}}}_k = \widehat{\mathbf{M}} = \widehat{\mathbf{M}}_k$. Consequently, the so-called *difference moiré* $\delta\widetilde{D}_M = \mathit{\Delta}\widetilde{D}_M - \mathit{\Delta} D_M$ gives the difference in fringe order between both configurations $\widetilde{\mathrm{G}}$ and G of the calibration plane as viewed by a single pixel of the CCD array of the camera. With $\widehat{\mathbf{M}}_k^T \mathbf{k} = 0$ and the definition $\lambda = \widehat{\lambda}(\ell_{S0} + p_{S0}) / \ell_{S0}$, the difference moiré $\delta\widetilde{D}_M$ reads

$$\delta\widetilde{D}_M = \frac{1}{\lambda} \Delta\mathbf{r} \cdot \left(\widehat{\mathbf{M}}_h \widehat{\mathbf{g}} - \widehat{\widetilde{\mathbf{M}}}_h \widehat{\mathbf{g}} \right) - \frac{\eta}{\lambda} \mathbf{k} \cdot \widehat{\widetilde{\mathbf{M}}}_h \widehat{\mathbf{g}} \tag{4.21}$$

and only depends on the projector grating (the camera grating is however still needed to view moiré fringes because of our low pixel resolution). Note that in the particular case of collimated illuminating light, which means parallel projection $\widetilde{\mathbf{h}} = \mathbf{h}$, we have $\widehat{\widetilde{\mathbf{M}}}_h = \widehat{\mathbf{M}}_h$ and only the last term of equation (4.21) gives a contribution. In the general case, which does not depend on $\widehat{\mathbf{g}}/\widehat{\lambda}$, we can use the vector definitions

$$\begin{aligned} \underline{\mathrm{SP}} &= (\ell_S + p_S)\mathbf{h} = (\ell_{S0} + p_{S0})\mathbf{h}_0 + \Delta\mathbf{r} \\ \underline{\mathrm{S\widetilde{P}}} &= (\widetilde{\ell}_S + \widetilde{p}_S)\widetilde{\mathbf{h}} = (\ell_S + p_S)\mathbf{h} + \eta\mathbf{k} = (\ell_{S0} + p_{S0})\mathbf{h}_0 + \Delta\mathbf{r} + \eta\mathbf{k} \end{aligned} \tag{4.22}$$

to write the oblique projectors as follows

$$\widehat{\mathbf{M}}_h = \mathbf{I} - \frac{\widehat{\mathbf{n}} \otimes [(\ell_{S0} + p_{S0})\mathbf{h}_0 + \Delta\mathbf{r}]}{\widehat{\mathbf{n}} \cdot [(\ell_{S0} + p_{S0})\mathbf{h}_0 + \Delta\mathbf{r}]} \quad ; \quad \widehat{\widetilde{\mathbf{M}}}_h = \mathbf{I} - \frac{\widehat{\mathbf{n}} \otimes [(\ell_{S0} + p_{S0})\mathbf{h}_0 + \Delta\mathbf{r} + \eta\mathbf{k}]}{\widehat{\mathbf{n}} \cdot [(\ell_{S0} + p_{S0})\mathbf{h}_0 + \Delta\mathbf{r} + \eta\mathbf{k}]} \tag{4.23}$$

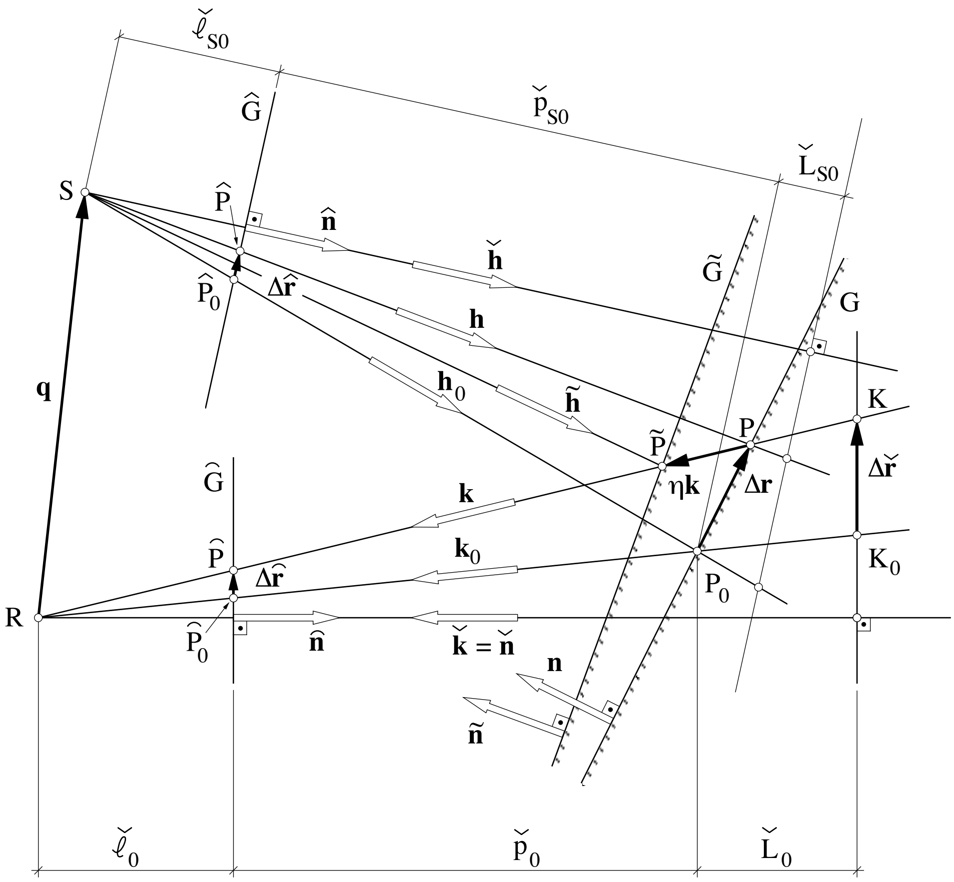

Fig. 4.9: Calibration process of a general moiré setup

Equation (4.21) can now be written in a slightly different form. We get after some calculations

$$\delta\widetilde{D}_M = -\frac{1}{\lambda}(\eta\mathbf{k} \cdot \widehat{\mathbf{M}}_h \widehat{\mathbf{g}}) \frac{(\ell_{S0} + p_{S0})\mathbf{h}_0 \cdot \widehat{\mathbf{n}}}{[(\ell_{S0} + p_{S0})\mathbf{h}_0 + \Delta\mathbf{r} + \eta\mathbf{k}] \cdot \widehat{\mathbf{n}}} \tag{4.24}$$

where $(\ell_{S0} + p_{S0})\mathbf{h}_0 \cdot \widehat{\mathbf{n}} = \check{\ell}_{S0} + \check{p}_{S0} = ¢$. The above expression for $\delta\widetilde{D}_M$ obviously only depends on the vectors $\Delta\mathbf{r}$ and $\eta\mathbf{k}$, all other quantities remaining constant during the calibration process. The vector $\Delta\mathbf{r}$, which does not depend on the translation and rotation of the calibration plane, describes the behavior of the difference moiré $\delta\widetilde{D}_M$ across the field of view of our observing system while traveling on the calibration plane and going from one pixel to another on the CCD-array. Considering only one pixel at a time, the vector $\eta\mathbf{k}$, which depends on the translation and rotation of the calibration plane and also on the vector $\Delta\mathbf{r}$, describes the behavior of the difference moiré $\delta\widetilde{D}_M$ along the observing direction $\mathbf{k}$ while moving the calibration plane.

Note that equation (4.24) remains valid if the reference point P_0 is not situated on the calibration plane (Fig. 4.10). In this case, we must write

$$\Delta\mathbf{r} = \mathbf{v} + \mathbf{v}_0 \quad ; \quad \begin{array}{l} \mathbf{v} = \mathbf{N}\mathbf{v} = \mathbf{M}_0^T\Delta\mathbf{r} \\ \mathbf{v}_0 = \left(\dfrac{\mathbf{n}\cdot\Delta\mathbf{r}}{\mathbf{n}\cdot\mathbf{k}_0}\right)\mathbf{k}_0 = v_0\mathbf{k}_0 = ¢ \end{array} \quad ; \quad \begin{array}{l} \mathbf{N} = \mathbf{I} - \mathbf{n}\otimes\mathbf{n} \\ \mathbf{M}_0 = \mathbf{I} - \dfrac{\mathbf{n}\otimes\mathbf{k}_0}{\mathbf{n}\cdot\mathbf{k}_0} \end{array} \qquad (4.25)$$

where $\mathbf{v}$ is a vector lying on the calibration plane, $\mathbf{v}_0$ a corresponding constant offset vector parallel to the direction $\mathbf{k}_0$, $\mathbf{N}$ the (well-known) normal projector onto the calibration plane G and $\mathbf{M}_0$ an oblique projector. This case will not be treated here.

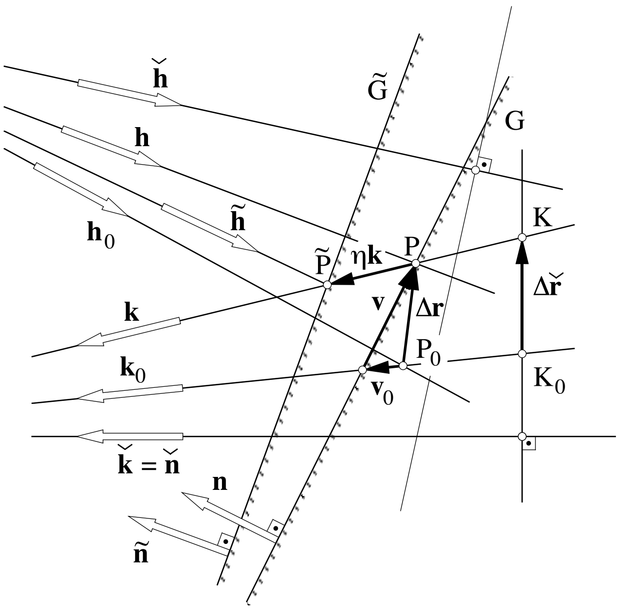

Fig. 4.10: General case where P_0 is not lying on the calibration plane

In our case, without restricting the generality, the point P_0 is choosen for practical reasons on the calibration plane G in its initial position. With the vector definitions

$$\underline{\mathrm{KR}} = (\ell + p + L)\mathbf{k} = (\ell_0 + p_0 + L_0)\mathbf{k}_0 - \Delta\check{\mathbf{r}} \quad ; \quad \Delta\check{\mathbf{r}} = \check{\mathbf{N}}\Delta\check{\mathbf{r}} = \check{x}\check{\mathbf{e}} + \check{y}\check{\mathbf{e}}_\perp \qquad (4.26)$$

where $\Delta\check{\mathbf{r}}$ is a variable vector in the object plane of the observing system (also called lateral object plane vector) and $\check{\mathbf{N}} = \mathbf{I} - \check{\mathbf{n}}\otimes\check{\mathbf{n}}$ the corresponding normal projector, we introduce the oblique projector

$$\mathbf{M} = \mathbf{I} - \frac{\mathbf{n}\otimes\mathbf{k}}{\mathbf{n}\cdot\mathbf{k}} = \mathbf{I} - \frac{\mathbf{n}\otimes[(\ell_0 + p_0 + L_0)\mathbf{k}_0 - \Delta\check{\mathbf{r}}]}{\mathbf{n}\cdot[(\ell_0 + p_0 + L_0)\mathbf{k}_0 - \Delta\check{\mathbf{r}}]} \qquad (4.27)$$

which allows us to write the following exact affine connection

$$\Delta\mathbf{r} = \mathbf{N}\Delta\mathbf{r} = \frac{\ell_0+p_0}{\ell_0+p_0+L_0}\mathbf{M}^T\Delta\check{\mathbf{r}} = \frac{\ell_0+p_0}{\ell_0+p_0+L_0}\left(\mathbf{I} - \frac{[(\ell_0+p_0+L_0)\mathbf{k}_0 - \Delta\check{\mathbf{r}}]\otimes\mathbf{n}}{[(\ell_0+p_0+L_0)\mathbf{k}_0 - \Delta\check{\mathbf{r}}]\cdot\mathbf{n}}\right)\Delta\check{\mathbf{r}} \quad (4.28)$$

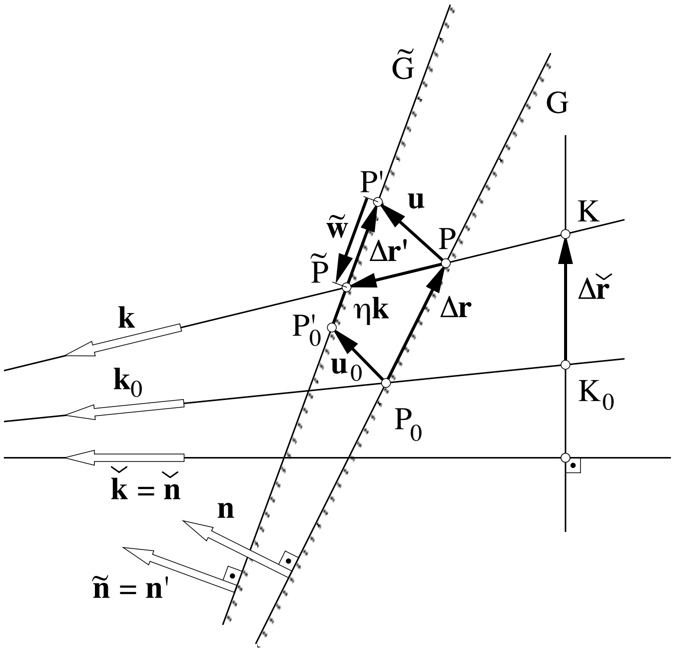

Fig. 4.11: Motion of the calibration plane

In order to write equation (4.24) explicitly, we first write the kinematic relations of the motion of the calibration plane. To bring the calibration plane from its initial configuration G to its final configuration $\mathrm{G}' = \widetilde{\mathrm{G}}$, we can perform a rotation followed by a translation (Lagrangean representation) or a translation followed by a rotation (Eulerian representation). For a point P, we have

$$\mathrm{P} \longrightarrow \mathrm{P}' \quad : \quad \mathbf{r}' = \mathbf{r} + \mathbf{u} \quad ; \quad \mathbf{n}' = \widetilde{\mathbf{n}} = \mathbf{Q}\mathbf{n} \quad (4.29)$$

with $\mathbf{r} = \mathbf{r}_0 + \Delta\mathbf{r}$, $\mathbf{u} = \mathbf{u}(\mathbf{r}) = \mathbf{u}(\Delta\mathbf{r})$ and $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$. The constant orthogonal tensor $\mathbf{Q}$ describes the rotation of the whole calibration plane G, the vector $\mathbf{u}$ represents the displacement of point P and $\mathbf{r}'$ the vector coordinate of point P′ on $\widetilde{\mathrm{G}}$. Generally, both points P′ and $\widetilde{\mathrm{P}}$ do not coincide, but their associated unit normals $\mathbf{n}'$ and $\widetilde{\mathbf{n}}$ are the same. Assuming that the displacement $\mathbf{u}_0$ of the reference point P_0 is known, we have

$$\mathrm{P}_0 \longrightarrow \mathrm{P}_0' \quad : \quad \mathbf{r}_0' = \mathbf{r}_0 + \mathbf{u}_0 \quad ; \quad \Delta\mathbf{r}' = \mathbf{r}' - \mathbf{r}_0' = \mathbf{Q}\Delta\mathbf{r} \quad (4.30)$$

and get explicitly

$$\mathbf{u} = \mathbf{u}_0 + (\mathbf{Q} - \mathbf{I})\Delta\mathbf{r} \quad (4.31)$$

Thus, the motion of the calibration plane from G to $\widetilde{\mathrm{G}}$ can be either decomposed in a rotation $\mathbf{Q}$ around an axis of direction $\mathbf{n}_\Delta$ going through point P_0 followed by a translation $\mathbf{u}_0$ from P_0 to point P_0', or in a translation $\mathbf{u}_0$ from P_0 to point P_0' followed by a rotation $\mathbf{Q}$ around an axis of same direction $\mathbf{n}_\Delta$ but going through point P_0'. By introducing a right-handed cartesian vector base $(\mathbf{e}, \mathbf{e}_\perp, \mathbf{n})$ on G such that

$$\begin{gathered}\mathbf{e} \equiv \mathbf{N}\mathbf{e} \quad ; \quad \mathbf{e}_\perp \equiv \mathbf{N}\mathbf{e}_\perp \quad ; \quad \mathbf{e} \times \mathbf{e}_\perp = \mathbf{n} \\ \mathbf{e}\cdot\mathbf{e} = \mathbf{e}_\perp\cdot\mathbf{e}_\perp = \mathbf{n}\cdot\mathbf{n} = 1 \quad ; \quad \mathbf{e}\cdot\mathbf{e}_\perp = \mathbf{e}_\perp\cdot\mathbf{n} = \mathbf{n}\cdot\mathbf{e} = 0\end{gathered} \tag{4.32}$$

the rotation tensor $\mathbf{Q}$ can be explicitly written as follows

$$\mathbf{Q} = \mathbf{e}'\otimes\mathbf{e} + \mathbf{e}_\perp'\otimes\mathbf{e}_\perp + \mathbf{n}'\otimes\mathbf{n} = \mathbf{N}_\Delta\cos\chi - \mathbf{E}_\Delta\sin\chi + \mathbf{n}_\Delta\otimes\mathbf{n}_\Delta \tag{4.33}$$

where

$$\begin{gathered}\mathbf{e}' \equiv \widetilde{\mathbf{N}}\mathbf{e}' = \mathbf{Q}\mathbf{e} \quad ; \quad \mathbf{e}_\perp' \equiv \widetilde{\mathbf{N}}\mathbf{e}_\perp' = \mathbf{Q}\mathbf{e}_\perp \quad ; \quad \mathbf{n}' \equiv \widetilde{\mathbf{n}} = \mathbf{Q}\mathbf{n} \\ \widetilde{\mathbf{N}} = \mathbf{I} - \widetilde{\mathbf{n}}\otimes\widetilde{\mathbf{n}} \quad ; \quad \mathbf{N}' = \mathbf{I} - \mathbf{n}'\otimes\mathbf{n}' \quad ; \quad \mathbf{N}' = \widetilde{\mathbf{N}} \\ \mathbf{N}_\Delta = \mathbf{I} - \mathbf{n}_\Delta\otimes\mathbf{n}_\Delta \quad ; \quad \mathbf{E}_\Delta = \boldsymbol{\mathcal{E}}\mathbf{n}_\Delta \\ \boldsymbol{\mathcal{E}} = \mathbf{e}\otimes\mathbf{e}_\perp\otimes\mathbf{n} - \mathbf{e}\otimes\mathbf{n}\otimes\mathbf{e}_\perp + \mathbf{e}_\perp\otimes\mathbf{n}\otimes\mathbf{e} - \mathbf{e}_\perp\otimes\mathbf{e}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{e}\otimes\mathbf{e}_\perp - \mathbf{n}\otimes\mathbf{e}_\perp\otimes\mathbf{e}\end{gathered} \tag{4.34}$$

and where χ is the rotation angle around the axis Δ of direction $\mathbf{n}_\Delta$. Both projectors $\widetilde{\mathbf{N}}$ and $\mathbf{N}'$ are identical and project onto the calibration plane $\widetilde{\mathrm{G}}$ in its final position. The expression (4.33) for $\mathbf{Q}$ can be demonstrated by rigidly connecting some point P_3 in the 3-dimensional space with the calibration plane G as follows

$$\mathbf{r}_3 = \mathbf{r}_0 + \Delta\mathbf{r} + z\mathbf{n} \quad ; \quad \Delta\mathbf{r} = x\mathbf{e} + y\mathbf{e}_\perp \tag{4.35}$$

where $\mathbf{r}_3$ is the vector coordinate of point P_3 and x, y the cartesian components of $\Delta\mathbf{r}$ on G. The motion of the calibration plane implies a motion of point P_3 to a new position P_3', which reads

$$\mathbf{r}_3' = \mathbf{r}_0' + \Delta\mathbf{r}' + z\mathbf{n}' = \mathbf{r}_3 + \mathbf{u}_3 \quad ; \quad \Delta\mathbf{r}' = x\mathbf{e}' + y\mathbf{e}_\perp' \tag{4.36}$$

where $\mathbf{r}_3'$ is the vector coordinate of point P_3' and $\mathbf{u}_3$ the displacement of P_3 to P_3'. Subtracting equation (4.35) from (4.36), we get

$$\begin{aligned}\mathbf{u}_3 = \mathbf{r}_3' - \mathbf{r}_3 &= \mathbf{r}_0' - \mathbf{r}_0 + \Delta\mathbf{r}' - \Delta\mathbf{r} + z(\mathbf{n}' - \mathbf{n}) \\ &= \mathbf{u}_0 + x(\mathbf{e}' - \mathbf{e}) + y(\mathbf{e}_\perp' - \mathbf{e}_\perp) + z(\mathbf{n}' - \mathbf{n})\end{aligned} \tag{4.37}$$

With the identity $\mathbf{I} = \mathbf{e}\otimes\mathbf{e} + \mathbf{e}_\perp\otimes\mathbf{e}_\perp + \mathbf{n}\otimes\mathbf{n}$ and the 3-dimensional derivative operator

$$\boldsymbol{\nabla} = \mathbf{e}\frac{\partial}{\partial x} + \mathbf{e}_\perp\frac{\partial}{\partial y} + \mathbf{n}\frac{\partial}{\partial z} \tag{4.38}$$

the deformation gradient of the 3-dimensional space around G reads

$$\begin{aligned}\mathbf{F} = \mathbf{Q}\mathbf{U} &= \mathbf{I} + (\boldsymbol{\nabla}\otimes\mathbf{u}_3)^T \\ &= \mathbf{I} + (\mathbf{e}' - \mathbf{e})\otimes\mathbf{e} + (\mathbf{e}_\perp' - \mathbf{e}_\perp)\otimes\mathbf{e}_\perp + (\mathbf{n}' - \mathbf{n})\otimes\mathbf{n} \\ &= \mathbf{e}'\otimes\mathbf{e} + \mathbf{e}_\perp'\otimes\mathbf{e}_\perp + \mathbf{n}'\otimes\mathbf{n}\end{aligned} \tag{4.39}$$

With $\mathbf{F}^T\mathbf{F} = \mathbf{U}^2 = \mathbf{I}$ and $\mathbf{U} = \mathbf{I}$, we have

$$\mathbf{F} = \mathbf{Q} = \mathbf{e}' \otimes \mathbf{e} + \mathbf{e}'_\perp \otimes \mathbf{e}_\perp + \mathbf{n}' \otimes \mathbf{n} \tag{4.40}$$

Proof:

$$\begin{aligned}
\mathbf{F}^T\mathbf{F} &= \mathbf{U}\mathbf{Q}^T\mathbf{Q}\mathbf{U} = \mathbf{U}^2 \\
&= (\mathbf{e} \otimes \mathbf{e}' + \mathbf{e}_\perp \otimes \mathbf{e}'_\perp + \mathbf{n} \otimes \mathbf{n}')(\mathbf{e}' \otimes \mathbf{e} + \mathbf{e}'_\perp \otimes \mathbf{e}_\perp + \mathbf{n}' \otimes \mathbf{n}) \\
&= \mathbf{e} \otimes \mathbf{e} + \mathbf{e}_\perp \otimes \mathbf{e}_\perp + \mathbf{n} \otimes \mathbf{n} = \mathbf{I}
\end{aligned} \tag{4.41}$$

$$\begin{aligned}
\mathbf{F}^T\mathbf{Q} &= \mathbf{U}\mathbf{Q}^T\mathbf{Q} = \mathbf{U} \\
&= (\mathbf{e} \otimes \mathbf{e}' + \mathbf{e}_\perp \otimes \mathbf{e}'_\perp + \mathbf{n} \otimes \mathbf{n}')\mathbf{Q} = \mathbf{e} \otimes \mathbf{Q}^T\mathbf{e}' + \mathbf{e}_\perp \otimes \mathbf{Q}^T\mathbf{e}'_\perp + \mathbf{n} \otimes \mathbf{Q}^T\mathbf{n}' \\
&= \mathbf{e} \otimes \mathbf{e} + \mathbf{e}_\perp \otimes \mathbf{e}_\perp + \mathbf{n} \otimes \mathbf{n} = \mathbf{I} \qquad \square \text{ qed}
\end{aligned}$$

where $\mathbf{e}' \cdot \mathbf{e}'_\perp = \mathbf{e} \cdot \mathbf{Q}^T\mathbf{Q}\mathbf{e}_\perp = \mathbf{e} \cdot \mathbf{e}_\perp = 0$, $\mathbf{e}'_\perp \cdot \mathbf{n}' = \mathbf{e}_\perp \cdot \mathbf{Q}^T\mathbf{Q}\mathbf{n} = \mathbf{e}_\perp \cdot \mathbf{n} = 0$ and $\mathbf{n}' \cdot \mathbf{e}' = \mathbf{n} \cdot \mathbf{Q}^T\mathbf{Q}\mathbf{e} = \mathbf{n} \cdot \mathbf{e} = 0$.

According to Lagrange, the motion of the calibration plane G can be described by a rotation

$$\mathbf{Q} = \mathbf{Q}_i \mathbf{Q}_p \tag{4.42}$$

└ $\mathbf{Q}_p$ gives no contribution
└ $\mathbf{Q}_i$ gives a contribution

followed by a translation

$$\mathbf{u}_0 = \mathbf{I}\mathbf{u}_0 = \left(\widetilde{\mathbf{M}} + \frac{\mathbf{n} \otimes \widetilde{\mathbf{n}}}{\mathbf{n} \cdot \widetilde{\mathbf{n}}}\right)\mathbf{u}_0 = \widetilde{\mathbf{M}}\mathbf{u}_0 + \left(\frac{\mathbf{u}_0 \cdot \widetilde{\mathbf{n}}}{\mathbf{n} \cdot \widetilde{\mathbf{n}}}\right)\mathbf{n} = \widetilde{\mathbf{M}}\mathbf{u}_0 + z_0\mathbf{n} \tag{4.43}$$

└ $z_0\mathbf{n}$ gives a contribution
└ $\widetilde{\mathbf{M}}\mathbf{u}_0$ gives no contribution

where $\mathbf{Q}_p$ is the in-plane rotation of G around the axis Δ_p of direction $\mathbf{n}$ going through point P_0, $\mathbf{Q}_i$ the out-of-plane rotation around an axis Δ_i of direction $\mathbf{n}_i = \mathbf{N}\mathbf{n}_i$ lying on G and going through the same point P_0, $z_0\mathbf{n} = [\mathbf{u}_0 \cdot \widetilde{\mathbf{n}}/(\mathbf{n} \cdot \widetilde{\mathbf{n}})]\mathbf{n}$ the translation normal to the calibration plane G, $\widetilde{\mathbf{M}}\mathbf{u}_0$ the translation parallel to the calibration plane $\widetilde{\mathrm{G}}$ and $\widetilde{\mathbf{M}} = \mathbf{I} - [\mathbf{n} \otimes \widetilde{\mathbf{n}}/(\mathbf{n} \cdot \widetilde{\mathbf{n}})]$ the corresponding oblique projector, which projects along the direction $\mathbf{n}$ onto a plane normal to $\widetilde{\mathbf{n}}$. Obviously, both in-plane rotation and translation of the calibration plane do not change the value of $\delta\widetilde{D}_M$ and consequently give no contribution to the calibration process.

According to Euler, the motion of the calibration plane G can be described by a translation

$$\mathbf{u}_0 = \mathbf{I}\mathbf{u}_0 = (\widetilde{\mathbf{N}} + \widetilde{\mathbf{n}} \otimes \widetilde{\mathbf{n}})\mathbf{u}_0 = \widetilde{\mathbf{N}}\mathbf{u}_0 + (\mathbf{u}_0 \cdot \widetilde{\mathbf{n}})\widetilde{\mathbf{n}} = \widetilde{\mathbf{N}}\mathbf{u}_0 + \widetilde{z}_0\widetilde{\mathbf{n}} \tag{4.44}$$

└ $\widetilde{z}_0\widetilde{\mathbf{n}}$ gives a contribution
└ $\widetilde{\mathbf{N}}\mathbf{u}_0$ gives no contribution

followed by a rotation

$$\mathbf{Q} = \mathbf{Q}'_p \mathbf{Q}'_i \tag{4.45}$$

└ $\mathbf{Q}'_i$ gives a contribution
└ $\mathbf{Q}'_p$ gives no contribution

where $\widetilde{z}_0\widetilde{\mathbf{n}} = (\mathbf{u}_0 \cdot \widetilde{\mathbf{n}})\widetilde{\mathbf{n}}$ is the translation normal to the calibration plane $\widetilde{\mathrm{G}}$, $\widetilde{\mathbf{N}}\mathbf{u}_0$ the translation parallel to the calibration plane $\widetilde{\mathrm{G}}$, $\mathbf{Q}'_i$ the out-of-plane rotation around an axis Δ'_i of direction $\mathbf{n}'_i = \mathbf{n}_i$ going through point P'_0 and $\mathbf{Q}'_p$ the in-plane rotation of $\widetilde{\mathrm{G}}$ around the axis Δ'_p of direction $\mathbf{n}' = \widetilde{\mathbf{n}}$ going through the same point P'_0. Obviously, both in-plane rotation and translation of the calibration plane do not change the value of $\delta\widetilde{D}_M$ and consequently give no contribution to the calibration process. With $\mathbf{e}'_p = \mathbf{Q}_p\mathbf{e}$, $\mathbf{e}'_{\perp p} = \mathbf{Q}_p\mathbf{e}_{\perp p}$, $\mathbf{n} = \mathbf{Q}_p\mathbf{n}$, $\mathbf{N}' = \widetilde{\mathbf{N}}$ and $\mathbf{E}' = \widetilde{\mathbf{E}}$, we write explicitly

$$\begin{aligned}\mathbf{Q}_p &= \mathbf{N}\cos\chi_p - \mathbf{E}\sin\chi_p + \mathbf{n}\otimes\mathbf{n} \\ &= \mathbf{e}'_p\otimes\mathbf{e} + \mathbf{e}'_{\perp p}\otimes\mathbf{e}_\perp + \mathbf{n}\otimes\mathbf{n}\end{aligned} \quad ; \quad \mathbf{Q}'_p = \mathbf{N}'\cos\chi_p - \mathbf{E}'\sin\chi_p + \mathbf{n}'\otimes\mathbf{n}' \qquad (4.46)$$

where $\chi_p = \chi'_p$ is the rotation angle around the unit normal $\mathbf{n}$ (or $\mathbf{n}'$). It follows that $\mathbf{Q}'_i = \mathbf{Q}_i$. Proof:

$$\begin{aligned}\mathbf{Q}_i &= \mathbf{Q}\mathbf{Q}_p^T = (\mathbf{e}'\otimes\mathbf{e} + \mathbf{e}'_\perp\otimes\mathbf{e}_\perp + \mathbf{n}'\otimes\mathbf{n})(\mathbf{N}\cos\chi_p + \mathbf{E}\sin\chi_p + \mathbf{n}\otimes\mathbf{n}) \\ &= \mathbf{e}'\otimes\mathbf{e}\cos\chi_p + \mathbf{e}'_\perp\otimes\mathbf{e}_\perp\cos\chi_p + \mathbf{e}'\otimes\mathbf{e}_\perp\sin\chi_p - \mathbf{e}'_\perp\otimes\mathbf{e}\sin\chi_p + \mathbf{n}'\otimes\mathbf{n}\end{aligned} \qquad (4.47)$$

$$\begin{aligned}\mathbf{Q}'_i &= \mathbf{Q}'^T_p\mathbf{Q} = (\mathbf{N}'\cos\chi_p + \mathbf{E}'\sin\chi_p + \mathbf{n}'\otimes\mathbf{n}')(\mathbf{e}'\otimes\mathbf{e} + \mathbf{e}'_\perp\otimes\mathbf{e}_\perp + \mathbf{n}'\otimes\mathbf{n}) \\ &= \mathbf{e}'\otimes\mathbf{e}\cos\chi_p + \mathbf{e}'_\perp\otimes\mathbf{e}_\perp\cos\chi_p - \mathbf{e}'_\perp\otimes\mathbf{e}\sin\chi_p + \mathbf{e}'\otimes\mathbf{e}_\perp\sin\chi_p + \mathbf{n}'\otimes\mathbf{n} = \mathbf{Q}_i\end{aligned} \qquad \square \text{ qed}$$

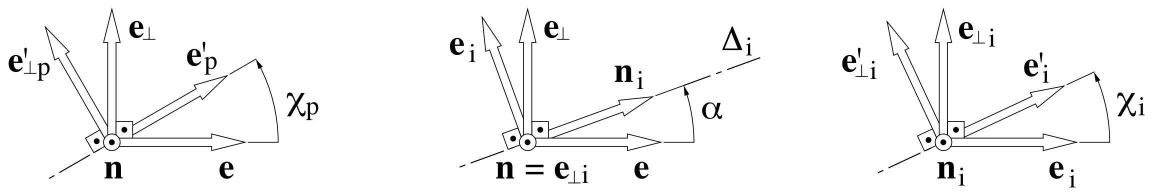

Fig. 4.12: Unit base vectors and rotation angles

Let us now introduce the orthogonal tensor $\mathbf{Q}_\alpha$ such that

$$\mathbf{Q}_\alpha = \mathbf{n}_i\otimes\mathbf{e} + \mathbf{e}_i\otimes\mathbf{e}_\perp + \mathbf{e}_{\perp i}\otimes\mathbf{n} = \mathbf{N}\cos\alpha - \mathbf{E}\sin\alpha + \mathbf{n}\otimes\mathbf{n} \qquad (4.48)$$

which describes a rotation of angle α around the unit normal $\mathbf{n}$ (Fig. 4.12). The direction $\mathbf{n}_i$ of the rotation axis Δ_i together with the associated unit vectors $\mathbf{e}_i$ and $\mathbf{e}_{\perp i}$ read explicitly

$$\begin{aligned}\mathbf{n}_i &= \mathbf{Q}_\alpha\mathbf{e} \\ &= \mathbf{e}\cos\alpha + \mathbf{e}_\perp\sin\alpha\end{aligned} \quad ; \quad \begin{aligned}\mathbf{e}_i &= \mathbf{Q}_\alpha\mathbf{e}_\perp \\ &= -\mathbf{e}\sin\alpha + \mathbf{e}_\perp\cos\alpha\end{aligned} \quad ; \quad \begin{aligned}\mathbf{e}_{\perp i} &= \mathbf{Q}_\alpha\mathbf{n} \\ &= \mathbf{n}\end{aligned} \qquad (4.49)$$

with

$$\mathbf{e}_i \equiv \mathbf{N}_i\mathbf{e}_i \quad ; \quad \mathbf{e}_{\perp i} \equiv \mathbf{N}_i\mathbf{e}_{\perp i} = \mathbf{n} \quad ; \quad \mathbf{e}_i\times\mathbf{e}_{\perp i} = \mathbf{n}_i \quad ; \quad \mathbf{e}_i\cdot\mathbf{e}_{\perp i} = 0 \qquad (4.50)$$

The out-of-plane rotation $\mathbf{Q}_i$, which rigidly rotate the system $(\mathbf{e}_i, \mathbf{e}_{\perp i}, \mathbf{n}_i)$ onto $(\mathbf{e}'_i, \mathbf{e}'_{\perp i}, \mathbf{n}'_i)$, or better to say $(\mathbf{e}_i, \mathbf{n}, \mathbf{n}_i)$ onto $(\mathbf{e}'_i, \mathbf{n}', \mathbf{n}_i)$, such that

$$\begin{aligned}\mathbf{e}'_i &= \mathbf{Q}_i\mathbf{e}_i \\ &= \mathbf{e}_i\cos\chi_i + \mathbf{e}_{\perp i}\sin\chi_i\end{aligned} \quad ; \quad \begin{aligned}\mathbf{e}'_{\perp i} &= \mathbf{Q}_i\mathbf{e}_{\perp i} = \mathbf{n}' = \widetilde{\mathbf{n}} \\ &= -\mathbf{e}_i\sin\chi_i + \mathbf{e}_{\perp i}\cos\chi_i\end{aligned} \quad ; \quad \begin{aligned}\mathbf{n}'_i &= \mathbf{Q}_i\mathbf{n}_i \\ &= \mathbf{n}_i\end{aligned} \qquad (4.51)$$

is completely defined by the exact rotation vector $\boldsymbol{\omega}_i \equiv \mathbf{N}\boldsymbol{\omega}_i = \chi_i \mathbf{n}_i$, where χ_i is the out-of-plane rotation angle around the rotation axis Δ_i of unit direction $\mathbf{n}_i$. With

$$\begin{aligned}\mathbf{N}_i &= \mathbf{e}_i \otimes \mathbf{e}_i + \mathbf{e}_{\perp i} \otimes \mathbf{e}_{\perp i} = \mathbf{I} - \mathbf{n}_i \otimes \mathbf{n}_i = \mathbf{Q}_\alpha (\mathbf{I} - \mathbf{e} \otimes \mathbf{e}) \mathbf{Q}_\alpha^T \\ \mathbf{E}_i &= \mathbf{e}_i \otimes \mathbf{e}_{\perp i} - \mathbf{e}_{\perp i} \otimes \mathbf{e}_i = \boldsymbol{\mathcal{E}} \mathbf{n}_i = \boldsymbol{\mathcal{E}} \mathbf{Q}_\alpha \mathbf{e} = \mathbf{Q}_\alpha (\mathbf{e}_\perp \otimes \mathbf{n} - \mathbf{n} \otimes \mathbf{e}_\perp) \mathbf{Q}_\alpha^T\end{aligned} \tag{4.52}$$

we have

$$\begin{aligned}\mathbf{Q}_i &= \mathbf{e}'_i \otimes \mathbf{e}_i + \mathbf{e}'_{\perp i} \otimes \mathbf{e}_{\perp i} + \mathbf{n}'_i \otimes \mathbf{n}_i = \mathbf{N}_i \cos\chi_i - \mathbf{E}_i \sin\chi_i + \mathbf{n}_i \otimes \mathbf{n}_i \\ &= \mathbf{Q}_\alpha [(\mathbf{I} - \mathbf{e} \otimes \mathbf{e}) \cos\chi_i - (\mathbf{e}_\perp \otimes \mathbf{n} - \mathbf{n} \otimes \mathbf{e}_\perp) \sin\chi_i + \mathbf{e} \otimes \mathbf{e}] \mathbf{Q}_\alpha^T\end{aligned} \tag{4.53}$$

The exact displacement vector $\mathbf{u}$ can be connected to the apparent displacement vector $\eta\mathbf{k}$ lying on the observing direction $\mathbf{k}$ by using the 2-dimensional unknown vector $\widetilde{\mathbf{w}}$ connecting P' with $\widetilde{\mathrm{P}}$ on the calibration plane $\widetilde{\mathrm{G}}$. We write

$$\eta\mathbf{k} = \mathbf{u} + \widetilde{\mathbf{w}} \qquad ; \qquad \widetilde{\mathbf{w}} \equiv \widetilde{\mathbf{N}}\widetilde{\mathbf{w}} \perp \widetilde{\mathbf{n}} \tag{4.54}$$

By contracting $\eta\mathbf{k}$ with the unit normal $\widetilde{\mathbf{n}}$, the vector $\widetilde{\mathbf{w}}$ can be eliminated and we get with equation (4.31)

$$\eta\mathbf{k} \cdot \widetilde{\mathbf{n}} = \mathbf{u} \cdot \widetilde{\mathbf{n}} = \mathbf{u} \cdot \mathbf{Q}\mathbf{n} = \mathbf{u}_0 \cdot \mathbf{Q}\mathbf{n} + \Delta\mathbf{r} \cdot (\mathbf{I} - \mathbf{Q})\mathbf{n} = (\mathbf{u}_0 - \Delta\mathbf{r}) \cdot \mathbf{Q}\mathbf{n} \tag{4.55}$$

With the vector definitions

$$\underline{\mathrm{P_0R}} = (\ell_0 + p_0)\mathbf{k}_0 \qquad ; \qquad \underline{\mathrm{PR}} = (\ell + p)\mathbf{k} = (\ell_0 + p_0)\mathbf{k}_0 - \Delta\mathbf{r} \tag{4.56}$$

the apparent displacement reads explicitly

$$\eta\mathbf{k} = \left(\frac{\eta\mathbf{k} \cdot \widetilde{\mathbf{n}}}{(\ell{+}p)\mathbf{k} \cdot \widetilde{\mathbf{n}}}\right)(\ell{+}p)\mathbf{k} = \left(\frac{\mathbf{u}_0 \cdot \mathbf{Q}\mathbf{n} - \Delta\mathbf{r} \cdot \mathbf{Q}\mathbf{n}}{[(\ell_0{+}p_0)\mathbf{k}_0 - \Delta\mathbf{r}] \cdot \mathbf{Q}\mathbf{n}}\right)[(\ell_0{+}p_0)\mathbf{k}_0 - \Delta\mathbf{r}] \tag{4.57}$$

or with equation (4.28)

$$\eta\mathbf{k} = \left(\frac{\eta\mathbf{k} \cdot \widetilde{\mathbf{n}}}{(\ell{+}p{+}L)\mathbf{k} \cdot \widetilde{\mathbf{n}}}\right)(\ell{+}p{+}L)\mathbf{k} = \left(\frac{\mathbf{u}_0 \cdot \mathbf{Q}\mathbf{n} - \dfrac{\ell_0{+}p_0}{\ell_0{+}p_0{+}L_0}\Delta\check{\mathbf{r}} \cdot \mathbf{M}\mathbf{Q}\mathbf{n}}{[(\ell_0{+}p_0{+}L_0)\mathbf{k}_0 - \Delta\check{\mathbf{r}}] \cdot \mathbf{Q}\mathbf{n}}\right)[(\ell_0{+}p_0{+}L_0)\mathbf{k}_0 - \Delta\check{\mathbf{r}}] \tag{4.58}$$

Thus, with equations (4.28), (4.58) and

$$\begin{aligned}\mathbf{Q}\mathbf{n} &= \mathbf{Q}_i\mathbf{Q}_p\mathbf{n} = \mathbf{Q}_i\mathbf{n} = \widetilde{\mathbf{n}} = \mathbf{n}' = \mathbf{e}\sin\alpha\sin\chi_i - \mathbf{e}_\perp\cos\alpha\sin\chi_i + \mathbf{n}\cos\chi_i \\ \mathbf{u}_0 \cdot \mathbf{Q}\mathbf{n} &= \mathbf{u}_0 \cdot \widetilde{\mathbf{n}} = \widetilde{z}_0 = (\widetilde{\mathbf{M}\mathbf{u}_0 + z_0\mathbf{n}}) \cdot \widetilde{\mathbf{n}} = z_0\mathbf{n} \cdot \widetilde{\mathbf{n}} = z_0\mathbf{n} \cdot \mathbf{Q}\mathbf{n} = z_0\mathbf{n} \cdot \mathbf{Q}_i\mathbf{n} = z_0\cos\chi_i\end{aligned} \tag{4.59}$$

the variation of $\delta\widetilde{D}_M$ along the observing direction and across the field of view is exclusively described by the following five independent scalar parameters

$$\begin{aligned} \check{x}, \check{y} &\quad : \quad \text{contained in the lateral object plane vector } \Delta\check{\mathbf{r}} \\ \alpha, \chi_i &\quad : \quad \text{contained in the rotation tensor } \mathbf{Q} \\ z_0 &\quad : \quad \text{contained in the translation vector } \mathbf{u}_0 \end{aligned} \tag{4.60}$$

Before explicitly writing equation (4.24) for $\delta\widetilde{D}_M$ as function of these five independent parameters, we first have to introduce some definitions and to perform some calculations:

Geometrical constants:

$$\lambda = \frac{\widehat{\lambda}(\ell_{S0}+p_{S0})}{\ell_{S0}} = ¢ \quad ; \quad \frac{A_0}{\lambda} = \frac{\ell_{S0}\mathbf{h}_0\cdot\widehat{\mathbf{n}}}{\widehat{\lambda}} = \frac{\check{\ell}_{S0}}{\widehat{\lambda}} = ¢ \quad ; \quad C_0 = \frac{\ell_0+p_0}{\ell_0+p_0+L_0} = ¢ \tag{4.61}$$

$$\begin{array}{lll} A_0 = (\ell_{S0}+p_{S0})\mathbf{h}_0\cdot\widehat{\mathbf{n}} = ¢ & & \\ A_1 = (\ell_{S0}+p_{S0})\mathbf{h}_0\cdot\widehat{\mathbf{g}} = ¢ & B_1 = \check{\mathbf{e}}\cdot\mathbf{n} = ¢ & C_1 = \check{\mathbf{e}}_\perp\cdot\mathbf{n} = ¢ \\ A_2 = (\ell_0+p_0+L_0)\mathbf{k}_0\cdot\widehat{\mathbf{n}} = ¢ & B_2 = \check{\mathbf{e}}\cdot\widehat{\mathbf{n}} = ¢ & C_2 = \check{\mathbf{e}}_\perp\cdot\widehat{\mathbf{n}} = ¢ \\ A_3 = (\ell_0+p_0+L_0)\mathbf{k}_0\cdot\widehat{\mathbf{g}} = ¢ \quad ; & B_3 = \check{\mathbf{e}}\cdot\widehat{\mathbf{g}} = ¢ \quad ; & C_3 = \check{\mathbf{e}}_\perp\cdot\widehat{\mathbf{g}} = ¢ \\ A_4 = (\ell_0+p_0+L_0)\mathbf{k}_0\cdot\mathbf{n} = ¢ & B_4 = \check{\mathbf{e}}\cdot\mathbf{e} = ¢ & C_4 = \check{\mathbf{e}}_\perp\cdot\mathbf{e} = ¢ \\ A_5 = (\ell_0+p_0+L_0)\mathbf{k}_0\cdot\mathbf{e} = ¢ & B_5 = \check{\mathbf{e}}\cdot\mathbf{e}_\perp = ¢ & C_5 = \check{\mathbf{e}}_\perp\cdot\mathbf{e}_\perp = ¢ \\ A_6 = (\ell_0+p_0+L_0)\mathbf{k}_0\cdot\mathbf{e}_\perp = ¢ & & \end{array}$$

The constants A_i have the dimension of a length and the constants B_j and C_j have no dimension.

Functions:

$$\begin{array}{lcl} f_0 = f_0(\chi_i) = \cos\chi_i & ; & F_1 = F_1(\alpha) = B_4\sin\alpha - B_5\cos\alpha \\ G_1 = G_1(\alpha) = A_5\sin\alpha - A_6\cos\alpha & & H_1 = H_1(\alpha) = C_4\sin\alpha - C_5\cos\alpha \end{array} \tag{4.62}$$

$$\begin{aligned} f_1 &= f_1(\alpha,\chi_i) = B_4\sin\alpha\sin\chi_i - B_5\cos\alpha\sin\chi_i + B_1\cos\chi_i = F_1\sin\chi_i + B_1\cos\chi_i \\ h_1 &= h_1(\alpha,\chi_i) = C_4\sin\alpha\sin\chi_i - C_5\cos\alpha\sin\chi_i + C_1\cos\chi_i = H_1\sin\chi_i + C_1\cos\chi_i \\ g_1 &= g_1(\alpha,\chi_i) = A_5\sin\alpha\sin\chi_i - A_6\cos\alpha\sin\chi_i + A_4\cos\chi_i = G_1\sin\chi_i + A_4\cos\chi_i \end{aligned}$$

Intermediate calculation:

$$\eta\mathbf{k}\cdot\widehat{\mathbf{M}}_h\widehat{\mathbf{g}} = \eta\mathbf{k}\cdot\widehat{\mathbf{g}} - \frac{\eta\mathbf{k}\cdot\widehat{\mathbf{n}}(A_1+\Delta\mathbf{r}\cdot\widehat{\mathbf{g}})}{A_0+\Delta\mathbf{r}\cdot\widehat{\mathbf{n}}} = \frac{(A_0+\Delta\mathbf{r}\cdot\widehat{\mathbf{n}})\eta\mathbf{k}\cdot\widehat{\mathbf{g}} - (A_1+\Delta\mathbf{r}\cdot\widehat{\mathbf{g}})\eta\mathbf{k}\cdot\widehat{\mathbf{n}}}{A_0+\Delta\mathbf{r}\cdot\widehat{\mathbf{n}}}$$

$$\Delta\mathbf{r}\cdot\widehat{\mathbf{n}} = C_0\left(\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{n}} - \frac{(A_2-\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{n}})\Delta\check{\mathbf{r}}\cdot\mathbf{n}}{A_4-\Delta\check{\mathbf{r}}\cdot\mathbf{n}}\right) = C_0\left(\frac{A_4\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{n}} - A_2\Delta\check{\mathbf{r}}\cdot\mathbf{n}}{A_4-\Delta\check{\mathbf{r}}\cdot\mathbf{n}}\right)$$

$$\Delta\mathbf{r}\cdot\widehat{\mathbf{g}} = C_0\left(\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{g}} - \frac{(A_3-\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{g}})\Delta\check{\mathbf{r}}\cdot\mathbf{n}}{A_4-\Delta\check{\mathbf{r}}\cdot\mathbf{n}}\right) = C_0\left(\frac{A_4\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{g}} - A_3\Delta\check{\mathbf{r}}\cdot\mathbf{n}}{A_4-\Delta\check{\mathbf{r}}\cdot\mathbf{n}}\right)$$

$$A_0+\Delta\mathbf{r}\cdot\widehat{\mathbf{n}} = A_0 + C_0\left(\frac{A_4\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{n}} - A_2\Delta\check{\mathbf{r}}\cdot\mathbf{n}}{A_4-\Delta\check{\mathbf{r}}\cdot\mathbf{n}}\right) = \frac{A_0A_4 - (A_0+C_0A_2)\Delta\check{\mathbf{r}}\cdot\mathbf{n} + C_0A_4\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{n}}}{A_4-\Delta\check{\mathbf{r}}\cdot\mathbf{n}}$$

$$A_1+\Delta\mathbf{r}\cdot\widehat{\mathbf{g}} = A_1 + C_0\left(\frac{A_4\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{g}} - A_3\Delta\check{\mathbf{r}}\cdot\mathbf{n}}{A_4-\Delta\check{\mathbf{r}}\cdot\mathbf{n}}\right) = \frac{A_1A_4 - (A_1+C_0A_3)\Delta\check{\mathbf{r}}\cdot\mathbf{n} + C_0A_4\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{g}}}{A_4-\Delta\check{\mathbf{r}}\cdot\mathbf{n}}$$

$$\eta\mathbf{k}\cdot\widehat{\mathbf{n}} = \left(\frac{\mathbf{u}_0\cdot\mathbf{Q}\mathbf{n} - C_0\Delta\check{\mathbf{r}}\cdot\mathbf{M}\mathbf{Q}\mathbf{n}}{(\ell_0+p_0+L_0)\mathbf{k}_0\cdot\mathbf{Q}\mathbf{n} - \Delta\check{\mathbf{r}}\cdot\mathbf{Q}\mathbf{n}}\right)(A_2 - \Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{n}})$$

$$
\begin{aligned}
\eta\mathbf{k}\cdot\widehat{\mathbf{g}} &= \left(\frac{\mathbf{u}_0\cdot\mathbf{Qn} - C_0\Delta\check{\mathbf{r}}\cdot\mathbf{MQn}}{(\ell_0+p_0+L_0)\mathbf{k}_0\cdot\mathbf{Qn} - \Delta\check{\mathbf{r}}\cdot\mathbf{Qn}}\right)(A_3 - \Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{g}}) \\
\Delta\check{\mathbf{r}}\cdot\mathbf{MQn} &= \Delta\check{\mathbf{r}}\cdot\mathbf{Qn} - \frac{\Delta\check{\mathbf{r}}\cdot\mathbf{n}[(\ell_0+p_0+L_0)\mathbf{k}_0\cdot\mathbf{Qn} - \Delta\check{\mathbf{r}}\cdot\mathbf{Qn}]}{A_4 - \Delta\check{\mathbf{r}}\cdot\mathbf{n}} \\
&= \frac{A_4\Delta\check{\mathbf{r}}\cdot\mathbf{Qn} - [(\ell_0+p_0+L_0)\mathbf{k}_0\cdot\mathbf{Qn}]\Delta\check{\mathbf{r}}\cdot\mathbf{n}}{A_4 - \Delta\check{\mathbf{r}}\cdot\mathbf{n}} \\
\mathbf{u}_0\cdot\mathbf{Qn} - C_0\Delta\check{\mathbf{r}}\cdot\mathbf{MQn} &= \mathbf{u}_0\cdot\mathbf{Qn} - C_0\left(\frac{A_4\Delta\check{\mathbf{r}}\cdot\mathbf{Qn} - [(\ell_0+p_0+L_0)\mathbf{k}_0\cdot\mathbf{Qn}]\Delta\check{\mathbf{r}}\cdot\mathbf{n}}{A_4 - \Delta\check{\mathbf{r}}\cdot\mathbf{n}}\right) \\
&= \frac{(\mathbf{u}_0\cdot\mathbf{Qn})(A_4-\Delta\check{\mathbf{r}}\cdot\mathbf{n}) - C_0A_4\Delta\check{\mathbf{r}}\cdot\mathbf{Qn} + C_0[(\ell_0+p_0+L_0)\mathbf{k}_0\cdot\mathbf{Qn}]\Delta\check{\mathbf{r}}\cdot\mathbf{n}}{A_4 - \Delta\check{\mathbf{r}}\cdot\mathbf{n}} \\
\mathbf{Qn} &= \mathbf{e}\sin\alpha\sin\chi_i - \mathbf{e}_\perp\cos\alpha\sin\chi_i + \mathbf{n}\cos\chi_i \quad ; \quad \mathbf{u}_0\cdot\mathbf{Qn} = z_0\mathbf{n}\cdot\mathbf{Q}_i\mathbf{n} = z_0\cos\chi_i = f_0 z_0 \\
\Delta\check{\mathbf{r}}\cdot\mathbf{n} &= B_1\check{x} + C_1\check{y} \quad ; \quad \Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{n}} = B_2\check{x} + C_2\check{y} \quad ; \quad \Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{g}} = B_3\check{x} + C_3\check{y} \\
\Delta\check{\mathbf{r}}\cdot\mathbf{Qn} &= (B_4\sin\alpha\sin\chi_i - B_5\cos\alpha\sin\chi_i + B_1\cos\chi_i)\check{x} \\
&\quad + (C_4\sin\alpha\sin\chi_i - C_5\cos\alpha\sin\chi_i + C_1\cos\chi_i)\check{y} \\
&= f_1\check{x} + h_1\check{y} \\
(\ell_0+p_0+L_0)\mathbf{k}_0\cdot\mathbf{Qn} &= A_5\sin\alpha\sin\chi_i - A_6\cos\alpha\sin\chi_i + A_4\cos\chi_i = g_1 \\
(\ell_0+p_0+L_0)\mathbf{k}_0\cdot\mathbf{Qn} - \Delta\check{\mathbf{r}}\cdot\mathbf{Qn} &= A_5\sin\alpha\sin\chi_i - A_6\cos\alpha\sin\chi_i + A_4\cos\chi_i \\
&\quad - (B_4\sin\alpha\sin\chi_i - B_5\cos\alpha\sin\chi_i + B_1\cos\chi_i)\check{x} \\
&\quad - (C_4\sin\alpha\sin\chi_i - C_5\cos\alpha\sin\chi_i + C_1\cos\chi_i)\check{y} \\
&= g_1 - f_1\check{x} - h_1\check{y}
\end{aligned}
\tag{4.63}
$$

Introducing (4.61) to (4.63) into equation (4.24) for $\delta\widetilde{D}_M$ gives

$$
\begin{aligned}
\delta\widetilde{D}_M &= -\frac{A_0}{\lambda}\left(\frac{(\eta\mathbf{k}\cdot\widehat{\mathbf{M}}_h\widehat{\mathbf{g}})}{A_0+\Delta\mathbf{r}\cdot\widehat{\mathbf{n}}+\eta\mathbf{k}\cdot\widehat{\mathbf{n}}}\right) = -\frac{A_0}{\lambda}\left[\frac{(A_0+\Delta\mathbf{r}\cdot\widehat{\mathbf{n}})\eta\mathbf{k}\cdot\widehat{\mathbf{g}} - (A_1+\Delta\mathbf{r}\cdot\widehat{\mathbf{g}})\eta\mathbf{k}\cdot\widehat{\mathbf{n}}}{(A_0+\Delta\mathbf{r}\cdot\widehat{\mathbf{n}})(A_0+\Delta\mathbf{r}\cdot\widehat{\mathbf{n}}+\eta\mathbf{k}\cdot\widehat{\mathbf{n}})}\right] \\
&= -\frac{A_0}{\lambda}(A_4-\Delta\check{\mathbf{r}}\cdot\mathbf{n})\left(\frac{[A_0A_4-(A_0+C_0A_2)\Delta\check{\mathbf{r}}\cdot\mathbf{n}+C_0A_4\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{n}}]\eta\mathbf{k}\cdot\widehat{\mathbf{g}}}{[A_0A_4-(A_0+C_0A_2)\Delta\check{\mathbf{r}}\cdot\mathbf{n}+C_0A_4\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{n}}]}\cdots\right. \\
&\qquad \left.\cdots\frac{-[A_1A_4-(A_1+C_0A_3)\Delta\check{\mathbf{r}}\cdot\mathbf{n}+C_0A_4\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{g}}]\eta\mathbf{k}\cdot\widehat{\mathbf{n}}}{[A_0A_4-(A_0+C_0A_2)\Delta\check{\mathbf{r}}\cdot\mathbf{n}+C_0A_4\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{n}}+(A_4-\Delta\check{\mathbf{r}}\cdot\mathbf{n})\eta\mathbf{k}\cdot\widehat{\mathbf{n}}]}\right) \\
&= -\frac{A_0}{\lambda}\left[\frac{(\mathbf{u}_0\cdot\mathbf{Qn}-C_0\Delta\check{\mathbf{r}}\cdot\mathbf{MQn})(A_4-\Delta\check{\mathbf{r}}\cdot\mathbf{n})}{A_0A_4-(A_0+C_0A_2)\Delta\check{\mathbf{r}}\cdot\mathbf{n}+C_0A_4\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{n}}}\right]\left[\frac{[A_0A_4-(A_0+C_0A_2)\Delta\check{\mathbf{r}}\cdot\mathbf{n}}{[(\ell_0+p_0+L_0)\mathbf{k}_0\cdot\mathbf{Qn}}\cdots\right. \\
&\qquad \cdots\frac{+C_0A_4\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{n}}](A_3-\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{g}}) - [A_1A_4-(A_1+C_0A_3)\Delta\check{\mathbf{r}}\cdot\mathbf{n}}{-\Delta\check{\mathbf{r}}\cdot\mathbf{Qn}][A_0A_4-(A_0+C_0A_2)\Delta\check{\mathbf{r}}\cdot\mathbf{n}+C_0A_4\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{n}}]}\cdots \\
&\qquad \left.\cdots\frac{+C_0A_4\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{g}}](A_2-\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{n}})}{+(\mathbf{u}_0\cdot\mathbf{Qn}-C_0\Delta\check{\mathbf{r}}\cdot\mathbf{MQn})(A_4-\Delta\check{\mathbf{r}}\cdot\mathbf{n})(A_2-\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{n}})}\right] \\
&= -\frac{A_0}{\lambda}\left[\frac{(\mathbf{u}_0\cdot\mathbf{Qn})(A_4-\Delta\check{\mathbf{r}}\cdot\mathbf{n}) - C_0A_4\Delta\check{\mathbf{r}}\cdot\mathbf{Qn} + C_0(\ell_0+p_0+L_0)(\mathbf{k}_0\cdot\mathbf{Qn})\Delta\check{\mathbf{r}}\cdot\mathbf{n}}{A_0A_4-(A_0+C_0A_2)\Delta\check{\mathbf{r}}\cdot\mathbf{n}+C_0A_4\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{n}}}\right] \\
&\quad \left[\frac{[A_0A_4-(A_0+C_0A_2)\Delta\check{\mathbf{r}}\cdot\mathbf{n}+C_0A_4\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{n}}](A_3-\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{g}})}{[(\ell_0+p_0+L_0)\mathbf{k}_0\cdot\mathbf{Qn}-\Delta\check{\mathbf{r}}\cdot\mathbf{Qn}][A_0A_4-(A_0+C_0A_2)\Delta\check{\mathbf{r}}\cdot\mathbf{n}+C_0A_4\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{n}}]}\cdots\right.
\end{aligned}
$$

$$\cdots \frac{-[A_1A_4 - (A_1 + C_0A_3)\Delta\check{\mathbf{r}}\cdot\mathbf{n}}{+[(\mathbf{u}_0\cdot\mathbf{Qn})(A_4 - \Delta\check{\mathbf{r}}\cdot\mathbf{n}) - C_0A_4\Delta\check{\mathbf{r}}\cdot\mathbf{Qn}} \cdots$$

$$\cdots \left.\frac{+C_0A_4\Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{g}}](A_2 - \Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{n}})}{+C_0(\ell_0 + p_0 + L_0)(\mathbf{k}_0\cdot\mathbf{Qn})\Delta\check{\mathbf{r}}\cdot\mathbf{n}](A_2 - \Delta\check{\mathbf{r}}\cdot\widehat{\mathbf{n}})}\right]$$

$$= \frac{A_0}{\lambda}\left[\frac{A_1A_2 - A_0A_3 + [A_0B_3 - A_1B_2 + C_0(A_2B_3 - A_3B_2)]\check{x} + [A_0C_3 - A_1C_2}{A_0A_4 - [A_0B_1 + C_0(A_2B_1 - A_4B_2)]\check{x} - [A_0C_1} \cdots\right.$$

$$\cdots \left.\frac{+C_0(A_2C_3 - A_3C_2)]\check{y}}{+C_0(A_2C_1 - A_4C_2)]\check{y}}\right]\left[\frac{A_4f_0z_0 - [B_1f_0z_0}{A_0g_1 + A_2f_0z_0 - [A_0f_1 + B_2f_0z_0} \cdots\right.$$

$$\cdots \left.\frac{+C_0(A_4f_1 - B_1g_1)]\check{x} - [C_1f_0z_0 + C_0(A_4h_1 - C_1g_1)]\check{y}}{+C_0(A_2f_1 - B_2g_1)]\check{x} - [A_0h_1 + C_2f_0z_0 + C_0(A_2h_1 - C_2g_1)]\check{y}}\right]$$

$$= \left[\frac{A_{11} + B_{11}\check{x} + C_{11}\check{y}}{1 + B_{12}\check{x} + C_{12}\check{y}}\right]\left[\frac{(A_4 - B_1\check{x} - C_1\check{y})z_0\cos\chi_i}{(A_2 - B_2\check{x} - C_2\check{y})z_0\cos\chi_i + [A_0A_4 + (-A_0B_1} \cdots\right.$$

$$\cdots \frac{+[(C_0[A_4B_5 - A_6B_1]\check{x}}{+C_0[-A_2B_1 + A_4B_2])\check{x} + (-A_0C_1 + C_0[-A_2C_1 + A_4C_2])\check{y}]\cos\chi_i} \cdots$$

$$\cdots \frac{+C_0[A_4C_5 - A_6C_1]\check{y})\cos\alpha}{+([-A_0A_6 + (A_0B_5 + C_0[A_2B_5 - A_6B_2])\check{x} + (A_0C_5 + C_0[A_2C_5} \cdots$$

$$\cdots \frac{+(C_0[A_5B_1 - A_4B_4]\check{x}}{-A_6C_2])\check{y}]\cos\alpha + [A_0A_5 + (-A_0B_4 + C_0[A_5B_2 - A_2B_4])\check{x}} \cdots$$

$$\cdots \left.\frac{+C_0[A_5C_1 - A_4C_4]\check{y})\sin\alpha]\sin\chi_i}{+(-A_0C_4 + C_0[A_5C_2 - A_2C_4])\check{y}]\sin\alpha)\sin\chi_i}\right]$$

$$= \left[\frac{A_{11} + B_{11}\check{x} + C_{11}\check{y}}{1 + B_{12}\check{x} + C_{12}\check{y}}\right]\left[\frac{A_4z_0\cos\chi_i + [-B_1z_0\cos\chi_i}{(A_0A_4 + A_2z_0)\cos\chi_i + A_0(-A_6\cos\alpha} \cdots\right.$$

$$\cdots \frac{+(C_0[A_4B_5 - A_6B_1]\cos\alpha}{+A_5\sin\alpha)\sin\chi_i + [(-A_0B_1 + C_0[A_4B_2 - A_2B_1] - B_2z_0)\cos\chi_i} \cdots$$

$$\cdots \frac{+C_0[A_5B_1 - A_4B_4]\sin\alpha)\sin\chi_i]\check{x}}{+((A_0B_5 + C_0[A_2B_5 - A_6B_2])\cos\alpha + (-A_0B_4 + C_0[A_5B_2} \cdots$$

$$\cdots \frac{+[-C_1z_0\cos\chi_i}{-A_2B_4])\sin\alpha)\sin\chi_i]\check{x} + [(-A_0C_1 + C_0[A_4C_2 - A_2C_1]} \cdots$$

$$\cdots \frac{+(C_0[A_4C_5 - A_6C_1]\cos\alpha}{-C_2z_0)\cos\chi_i + ((A_0C_5 + C_0[A_2C_5 - A_6C_2])\cos\alpha} \cdots$$

$$\cdots \left.\frac{+C_0[A_5C_1 - A_4C_4]\sin\alpha)\sin\chi_i]\check{y}}{+(-A_0C_4 + C_0[A_5C_2 - A_2C_4])\sin\alpha)\sin\chi_i]\check{y}}\right] \qquad (4.64)$$

With the constants

$$A_{11} = \frac{A_1A_2 - A_0A_3}{\lambda A_4} \quad ; \quad A_{12} = -\frac{A_6}{A_4} \quad ; \quad A_{13} = \frac{A_5}{A_4}$$

$$B_{11} = \frac{A_0B_3 - A_1B_2 + C_0(A_2B_3 - A_3B_2)}{\lambda A_4} \quad ; \quad B_{12} = \frac{-A_0B_1 + C_0(A_4B_2 - A_2B_1)}{A_0A_4}$$

$$B_{13} = \frac{A_0B_5 + C_0(A_2B_5 - A_6B_2)}{A_0A_4} \quad ; \quad B_{14} = \frac{-A_0B_4 + C_0(A_5B_2 - A_2B_4)}{A_0A_4}$$

$$B_{15} = \frac{C_0(A_4B_5 - A_6B_1)}{A_0A_4} \quad ; \quad B_{16} = \frac{C_0(A_5B_1 - A_4B_4)}{A_0A_4}$$
$$C_{11} = \frac{A_0C_3 - A_1C_2 + C_0(A_2C_3 - A_3C_2)}{\lambda A_4} \quad ; \quad C_{12} = \frac{-A_0C_1 + C_0(A_4C_2 - A_2C_1)}{A_0A_4}$$
$$C_{13} = \frac{A_0C_5 + C_0(A_2C_5 - A_6C_2)}{A_0A_4} \quad ; \quad C_{14} = \frac{-A_0C_4 + C_0(A_5C_2 - A_2C_4)}{A_0A_4}$$
$$C_{15} = \frac{C_0(A_4C_5 - A_6C_1)}{A_0A_4} \quad ; \quad C_{16} = \frac{C_0(A_5C_1 - A_4C_4)}{A_0A_4}$$
$$A_{21} = \frac{1}{A_0} \quad ; \quad A_{22} = \frac{A_2}{A_0A_4} \quad ; \quad B_{21} = -\frac{B_1}{A_0A_4} \tag{4.65}$$
$$B_{22} = -\frac{B_2}{A_0A_4} \quad ; \quad C_{21} = -\frac{C_1}{A_0A_4} \quad ; \quad C_{22} = -\frac{C_2}{A_0A_4}$$

and the functions

$$\begin{aligned}
g_{11} &= \frac{A_{21}z_0}{1 + A_{22}z_0 + (A_{12}\cos\alpha + A_{13}\sin\alpha)\,\mathrm{tg}\,\chi_i} \\
f_{11} &= \frac{B_{21}z_0 + (B_{15}\cos\alpha + B_{16}\sin\alpha)\,\mathrm{tg}\,\chi_i}{1 + A_{22}z_0 + (A_{12}\cos\alpha + A_{13}\sin\alpha)\,\mathrm{tg}\,\chi_i} \\
h_{11} &= \frac{C_{21}z_0 + (C_{15}\cos\alpha + C_{16}\sin\alpha)\,\mathrm{tg}\,\chi_i}{1 + A_{22}z_0 + (A_{12}\cos\alpha + A_{13}\sin\alpha)\,\mathrm{tg}\,\chi_i} \\
f_{12} &= \frac{B_{12} + B_{22}z_0 + (B_{13}\cos\alpha + B_{14}\sin\alpha)\,\mathrm{tg}\,\chi_i}{1 + A_{22}z_0 + (A_{12}\cos\alpha + A_{13}\sin\alpha)\,\mathrm{tg}\,\chi_i} \\
h_{12} &= \frac{C_{12} + C_{22}z_0 + (C_{13}\cos\alpha + C_{14}\sin\alpha)\,\mathrm{tg}\,\chi_i}{1 + A_{22}z_0 + (A_{12}\cos\alpha + A_{13}\sin\alpha)\,\mathrm{tg}\,\chi_i}
\end{aligned} \tag{4.66}$$

the equation (4.64) for $\delta\widetilde{D}_M$ reads:

$$\boxed{\delta\widetilde{D}_M = \left[\frac{A_{11} + B_{11}\check{x} + C_{11}\check{y}}{1 + B_{12}\check{x} + C_{12}\check{y}}\right]\left[\frac{g_{11} + f_{11}\check{x} + h_{11}\check{y}}{1 + f_{12}\check{x} + h_{12}\check{y}}\right]} \tag{4.67}$$

This equation describes the most general case of difference moiré and is of great utility for calibrating projection moiré systems for general geometries of optical setups. The real point position on the object plane of the optical system (represented by a pixel of a CCD camera) is now connected to the motion parameters z_0, α and χ_i of the calibration plane through the difference moiré value $\delta\widetilde{D}_M$. Therefore, the calibration plane can be used as reference plane for the point coordinates of the measured object shape. An example of calibration is shown in the next section.

4.8.2 Experimental calibration of a moiré setup

The purpose of this section is to demonstrate, with a practical example, that equation (4.67) works correctly. Without restricting the generality, by only performing a translation of the calibration plane (no rotation), we analytically compute a calibration process and compare the results with the corresponding quantitative experiment. In case of a translation of the calibration plane G with no rotation, we have

$$\mathbf{u} = \mathbf{u}_0 = z_0\mathbf{n} \quad ; \quad \mathbf{Q} = \mathbf{I} \tag{4.68}$$

which means that α is arbitrary and $\chi_i = 0$. In this case, equations (4.66) are written as follows

$$g_{11} = \frac{A_{21} z_0}{1 + A_{22} z_0} \quad ; \quad f_{11} = \frac{B_{21} z_0}{1 + A_{22} z_0} \quad ; \quad h_{11} = \frac{C_{21} z_0}{1 + A_{22} z_0}$$
$$f_{12} = \frac{B_{12} + B_{22} z_0}{1 + A_{22} z_0} \quad ; \quad h_{12} = \frac{C_{12} + C_{22} z_0}{1 + A_{22} z_0} \tag{4.69}$$

Thus, the equation for difference moiré (4.67) read

$$\delta \widetilde{D}_M = \left[\frac{A_{11} + B_{11} \check{x} + C_{11} \check{y}}{1 + B_{12} \check{x} + C_{12} \check{y}} \right] \left[\frac{(A_{21} + B_{21} \check{x} + C_{21} \check{y}) z_0}{1 + B_{12} \check{x} + C_{12} \check{y} + (A_{22} + B_{22} \check{x} + C_{22} \check{y}) z_0} \right] \tag{4.70}$$

and is valid in the general case of a translation of the calibration plane.

Analytical calibration

For better visuality, let us now assume in our optical setup that the four vectors $\widehat{\mathbf{g}}, \widehat{\mathbf{n}}, \mathbf{n}$ and $\widehat{\mathbf{n}}$ are coplanar and lie in the horizontal plane, the calibration plane beeing vertical. Let us also set both optical axes (of the camera and the projector) in the horizontal plane and intersect the calibration plane in the point P_0, which means that the object plane of the observing system is also vertical. For this particular setup, we have

$$\widehat{\mathbf{n}} = \check{\mathbf{h}} = \mathbf{h}_0 \quad ; \quad -\widehat{\mathbf{n}} = \check{\mathbf{n}} = \check{\mathbf{k}} = \mathbf{k}_0 \quad ; \quad \mathbf{h}_0 \cdot \widehat{\mathbf{g}} = 0 \quad ; \quad \mathbf{k}_0 \cdot \widehat{\mathbf{g}} = 0 \tag{4.71}$$

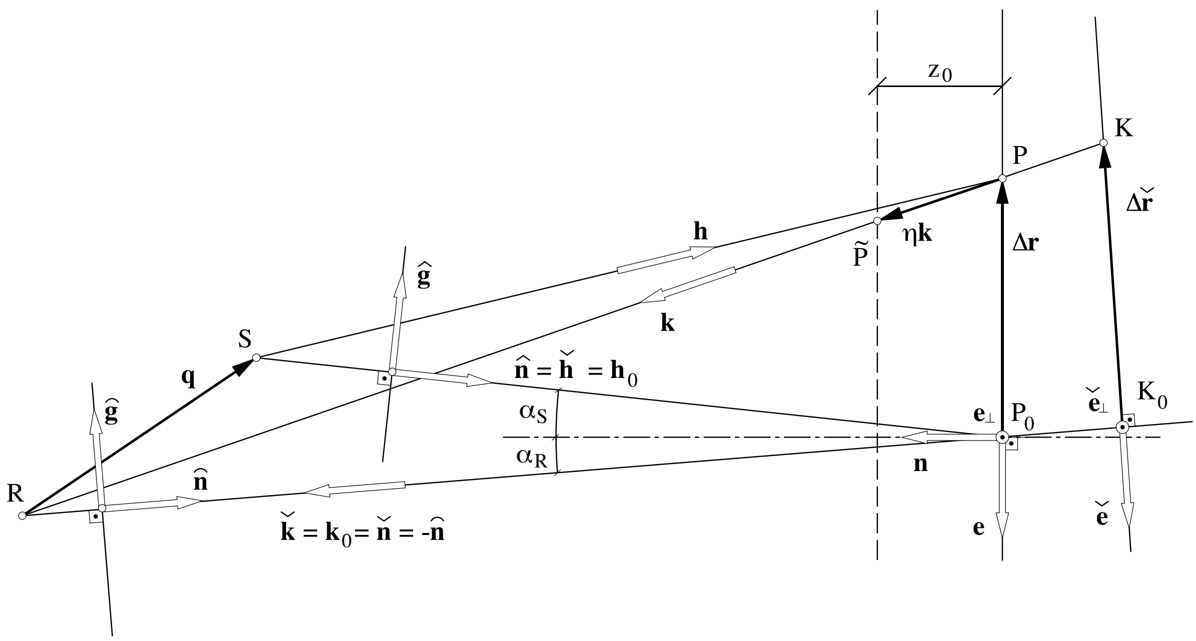

Fig. 4.13: Optical set-up for the experimental calibration

On the calibration plane and on the object plane of the camera, we respectively define two coordinate systems (P_0, x, y) and $(\mathrm{K}_0, \check{x}, \check{y})$, where both x and $\check{x}$-axes are horizontal and both y and $\check{y}$-axes are vertical (Fig. 4.13). With the corresponding unit direction vectors, the vectors $\Delta\mathbf{r}$ and $\Delta\check{\mathbf{r}}$ read

$$\Delta\mathbf{r} = x\mathbf{e} + y\mathbf{e}_\perp \quad ; \quad \Delta\check{\mathbf{r}} = \check{x}\check{\mathbf{e}} + \check{y}\check{\mathbf{e}}_\perp \tag{4.72}$$

Recalling the affine connection (4.28) between $\Delta\mathbf{r}$ and $\Delta\check{\mathbf{r}}$, we write

$$\begin{aligned}\Delta\mathbf{r} = x\mathbf{e} + y\mathbf{e}_\perp &= \frac{\ell_0+p_0}{\ell_0+p_0+L_0}\mathbf{M}^T\Delta\check{\mathbf{r}} \\ &= \frac{\ell_0+p_0}{\ell_0+p_0+L_0}\left(\mathbf{I} - \frac{\mathbf{k}\otimes\mathbf{n}}{\mathbf{k}\cdot\mathbf{n}}\right)(\check{x}\check{\mathbf{e}} + \check{y}\check{\mathbf{e}}_\perp) \\ &= \frac{\ell_0+p_0}{\ell_0+p_0+L_0}\left[\check{x}\check{\mathbf{e}} + \check{y}\check{\mathbf{e}}_\perp - \check{x}\left(\frac{\mathbf{n}\cdot\check{\mathbf{e}}}{\mathbf{k}\cdot\mathbf{n}}\right)\mathbf{k}\right]\end{aligned} \tag{4.73}$$

because $\mathbf{n}\cdot\check{\mathbf{e}}_\perp = 0$ and where $\mathbf{M} = \mathbf{I} - \mathbf{n}\otimes\mathbf{k}/(\mathbf{n}\cdot\mathbf{k})$. After contraction of equation (4.73) with $\mathbf{e}$ and $\mathbf{e}_\perp$, the components of $\Delta\mathbf{r}$ and $\Delta\check{\mathbf{r}}$ are related to each other as follows

$$\begin{aligned} x &= \Delta\mathbf{r}\cdot\mathbf{e} = C_0\left[\check{x}\cos\alpha_R + \check{x}\left(\frac{\mathbf{k}\cdot\mathbf{e}}{\mathbf{k}\cdot\mathbf{n}}\right)\sin\alpha_R\right] \\ y &= \Delta\mathbf{r}\cdot\mathbf{e}_\perp = C_0\left[\check{y} + \check{x}\left(\frac{\mathbf{k}\cdot\mathbf{e}_\perp}{\mathbf{k}\cdot\mathbf{n}}\right)\sin\alpha_R\right]\end{aligned} \tag{4.74}$$

where

$$\begin{aligned}\check{\mathbf{e}}\cdot\mathbf{e} &= \cos\alpha_R \\ \check{\mathbf{e}}\cdot\mathbf{n} &= -\sin\alpha_R\end{aligned} \quad ; \quad \begin{aligned}\check{\mathbf{e}}_\perp\cdot\mathbf{e} &= 0 \\ \check{\mathbf{e}}\cdot\mathbf{e}_\perp &= 0 \\ \check{\mathbf{e}}_\perp\cdot\mathbf{e}_\perp &= 1\end{aligned} \quad ; \quad C_0 = \frac{\ell_0+p_0}{\ell_0+p_0+L_0} \tag{4.75}$$

Considering the triangle PRP_0, we have

$$(\ell+p)\mathbf{k} = -\Delta\mathbf{r} + (\ell_0+p_0)\mathbf{k}_0 = -x\mathbf{e} - y\mathbf{e}_\perp + (\ell_0+p_0)\mathbf{k}_0 \tag{4.76}$$

which gives with $\mathbf{k}_0\cdot\mathbf{e} = \sin\alpha_R$, $\mathbf{k}_0\cdot\mathbf{e}_\perp = 0$ and $\mathbf{k}_0\cdot\mathbf{n} = \cos\alpha_R$

$$\begin{aligned}\frac{\mathbf{k}\cdot\mathbf{e}}{\mathbf{k}\cdot\mathbf{n}} &= \frac{(\ell+p)\mathbf{k}\cdot\mathbf{e}}{(\ell+p)\mathbf{k}\cdot\mathbf{n}} = \frac{-x + (\ell_0+p_0)\sin\alpha_R}{(\ell_0+p_0)\cos\alpha_R} \\ \frac{\mathbf{k}\cdot\mathbf{e}_\perp}{\mathbf{k}\cdot\mathbf{n}} &= \frac{(\ell+p)\mathbf{k}\cdot\mathbf{e}_\perp}{(\ell+p)\mathbf{k}\cdot\mathbf{n}} = \frac{-y}{(\ell_0+p_0)\cos\alpha_R}\end{aligned} \tag{4.77}$$

It follows that

$$\check{x} = \left[\frac{(\ell_0+p_0+L_0)\cos\alpha_R}{(\ell_0+p_0) - x\sin\alpha_R}\right]x \quad ; \quad \check{y} = \left[\frac{\ell_0+p_0+L_0}{(\ell_0+p_0) - x\sin\alpha_R}\right]y \tag{4.78}$$

The geometrical constants (4.61) read $\lambda = \widehat{\lambda}(\ell_{S0}+p_{S0})/\ell_{S0}$, $A_1 = A_6 = 0$, $B_5 = C_1 = C_2 = C_3 = C_4 = 0$ and

$$\begin{aligned} A_0 &= (\ell_{S0}+p_{S0}) \\ A_2 &= -(\ell_0+p_0+L_0)\cos(\alpha_R+\alpha_S) \\ A_3 &= -(\ell_0+p_0+L_0)\sin(\alpha_R+\alpha_S) \\ A_4 &= (\ell_0+p_0+L_0)\cos\alpha_R \\ A_5 &= (\ell_0+p_0+L_0)\sin\alpha_R \end{aligned} \quad ; \quad \begin{aligned} B_1 &= -\sin\alpha_R \\ B_2 &= \sin(\alpha_R+\alpha_S) \\ B_3 &= -\cos(\alpha_R+\alpha_S) \\ B_4 &= \cos\alpha_R \end{aligned} \quad ; \quad C_5 = 1 \tag{4.79}$$

The geometrical constant (4.65) then read $C_{11} = C_{12} = 0, C_{21} = C_{22} = 0$ and

$$\begin{aligned}
A_{11} &= \frac{\ell_{S0}\sin(\alpha_R+\alpha_S)}{\widehat{\lambda}\cos\alpha_R}\\
A_{21} &= \frac{1}{\ell_{S0}+p_{S0}}\\
A_{22} &= -\frac{\cos(\alpha_R+\alpha_S)}{(\ell_{S0}+p_{S0})\cos\alpha_R}\\
B_{11} &= \frac{\ell_{S0}[(\ell_0+p_0)-(\ell_{S0}+p_{S0})\cos(\alpha_R+\alpha_S)]}{\widehat{\lambda}(\ell_{S0}+p_{S0})(\ell_0+p_0+L_0)\cos\alpha_R}\\
B_{12} &= \frac{(\ell_0+p_0)\sin\alpha_S+(\ell_{S0}+p_{S0})\sin\alpha_R}{(\ell_{S0}+p_{S0})(\ell_0+p_0+L_0)\cos\alpha_R}\\
B_{21} &= \frac{\operatorname{tg}\alpha_R}{(\ell_{S0}+p_{S0})(\ell_0+p_0+L_0)}\\
B_{22} &= -\frac{\sin(\alpha_R+\alpha_S)}{(\ell_{S0}+p_{S0})(\ell_0+p_0+L_0)\cos\alpha_R}
\end{aligned} \tag{4.80}$$

The expression for the difference moiré $\delta\widetilde{D}_M$ only depends in our case of the translation of the calibration plane and of the horizontal position of point P, i.e. of point K. With equations (4.78), we have

$$\boxed{\begin{aligned}
\delta\widetilde{D}_M &= \left[\frac{A_{11}+B_{11}\check{x}}{1+B_{12}\check{x}}\right]\left[\frac{(A_{21}+B_{21}\check{x})z_0}{1+B_{12}\check{x}+(A_{22}+B_{22}\check{x})z_0}\right]\\
&= \left[\frac{A_{11}+R_{11}x}{1+R_{12}x}\right]\left[\frac{(A_{21}+R_{21}x)z_0}{1+R_{12}x+(A_{22}+R_{22}x)z_0}\right]
\end{aligned}} \tag{4.81}$$

where

$$\begin{aligned}
R_{11} &= \frac{\ell_{S0}[(\ell_0+p_0)\cos\alpha_R-(\ell_{S0}+p_{S0})\cos\alpha_S]}{\widehat{\lambda}(\ell_0+p_0)(\ell_{S0}+p_{S0})\cos\alpha_R}\\
R_{12} &= \frac{\sin\alpha_S}{\ell_{S0}+p_{S0}}\\
R_{21} &= 0\\
R_{22} &= -\frac{\sin\alpha_S}{(\ell_0+p_0)(\ell_{S0}+p_{S0})\cos\alpha_R}
\end{aligned} \tag{4.82}$$

As already mentioned, the exact expression (4.81) for the difference moiré $\delta\widetilde{D}_M$ describes the difference of the relative fringe order in some point K for a known translation z_0 of the calibration plane G. Let us also recall that K is the point in the object plane of the optical system associated to the corresponding pixel in the image plane of the camera. Considering a fixed pixel, i.e. a fixed point K, while moving the calibration plane by the translation amount z_0, the difference moiré $\delta\widetilde{D}_M$ gives the variation of the relative moiré fringe order in depth of field, which is the scope of the calibration.

Numerical calibration

Let us now quantitatively verify the theory with a measurement. In our experimental set-up, we used a white painted mirror as calibration plane, which was installed on a translation stage to move it normally to its surface. It was illuminated by a projector with a grating on a glass slide and observed through a CCD-interline transfer camera, which pixel columns served as reference grating. The experimental numerical values corresponding to the optical set-up of figure 4.13 needed to compute equation (4.81) are

	Projector	Camera
Distance P_0 to projection center	$\ell_{S0} + p_{S0} \simeq 304.0$ mm	$\ell_0 + p_0 \simeq 359.0$ mm
Distance grating to projection center*	$\ell_{S0} \simeq 73.97$ mm	$\ell_0 \simeq 29.92$ mm
Inclination of optical axis	$\alpha_S \simeq 5.15°$	$\alpha_R \simeq 6.70°$
Pitch of grating	$\widehat{\lambda} = 50\ \mu\text{m} = 0.050$ mm	$\widehat{\lambda} = 17\ \mu\text{m} = 0.017$ mm

* Respectively calculated with the magnification factors 4.11 and 12.0 on the optical axes

Introducing these numerical values in the equations (4.80) and (4.82) gives

$$\begin{matrix} A_{11} = 3.059\,.10^{2} \\ R_{11} = 7.340\,.10^{-1}\,\text{mm}^{-1} \end{matrix} \;;\; \begin{matrix} A_{21} = 3.289\,.10^{-3}\,\text{mm}^{-1} \\ R_{12} = 2.953\,.10^{-4}\,\text{mm}^{-1} \end{matrix} \;;\; \begin{matrix} A_{22} = -3.242\,.10^{-3}\,\text{mm}^{-1} \\ R_{22} = -8.281\,.10^{-7}\,\text{mm}^{-2} \end{matrix}$$

The experimental calibration consists in determining the fringe order change per unit length in depth of field for each pixel. Using a phase shifting device in the projector allows to measure the changes in terms of a *phase difference map*. The frames were digitized with 8 bit resolution, the fringe order corresponding to a phase change of 2π or 256 grey levels.

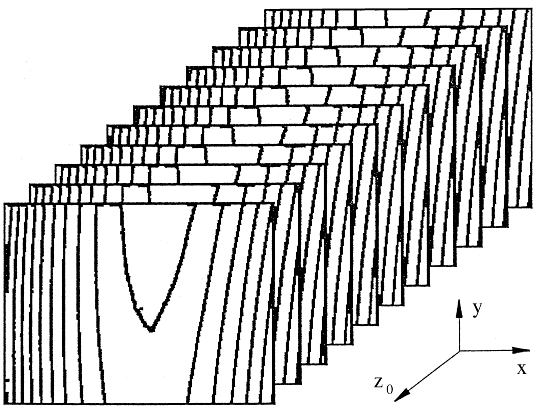

Fig. 4.14: Lines of equal moiré fringe order for a succession of ten translation steps

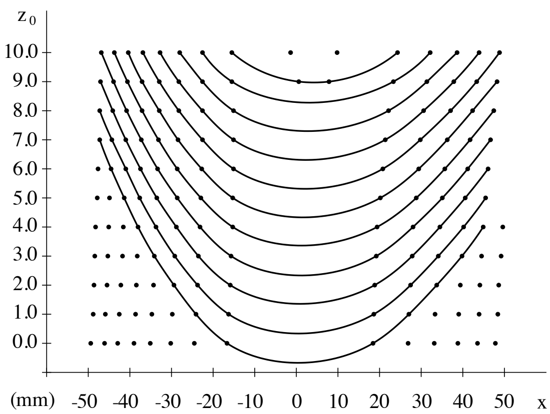

Fig. 4.15: Visualisation of the moiré surfaces

Figure 4.14 shows lines of equal moiré fringe order, i.e. equal phase, for a succession of 10 translation steps. Because the optic of the projector and that of the camera are not parallel, the density of moiré fringes varies across the calibration plane. For a step of $\Delta z_0 = 1.00$ mm, the pattern nearly repeats itself indicating a shift of approximatively one fringe order per mm. An horizontal cut (over the width) through the calibration planes of figure 4.14 shows the corresponding moiré surfaces intersecting the calibration plane in its different positions (Fig. 4.15). The phase differences were then obtained by digitally subtracting the phase map of the zero position ($z_0 = 0$) from the others. The result is a difference in fringe order shown in the comparative phase difference map of figure 4.16, with the difference moiré as function of the x-coordinate in mm. Figure 4.16 simultaneously represents our experimentally measured difference moiré and that theoretically computed with equation (4.81) for different values of z_0. This confirms that the experiment and the theory are in agreement. The experimental phase difference map contains the calibration factors over the width of the field of view for each translation step. Note that the difference moiré varies almost linearly in the x-direction by approximately 20% change over the width, whereas the change in the y-direction is negligible. It should also be mentionned that a small change of α_R and α_S first influences the value of $\delta\widetilde{D}_M$, whereas a small change of $\ell_0 + p_0$ and $\ell_{S0} + p_{S0}$ first influences the variation of $\delta\widetilde{D}_M$ across the field of view.

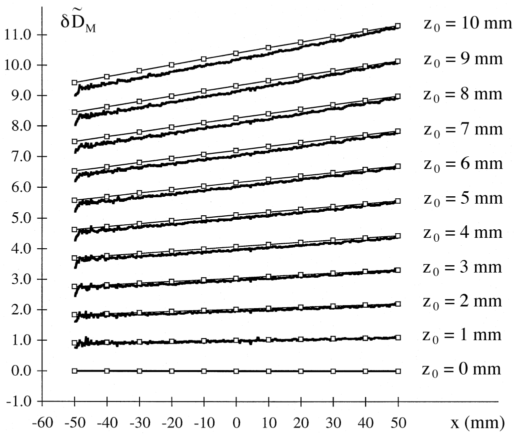

Fig. 4.16: Phase difference map
—— measured experimentally
-□- computed theoretically

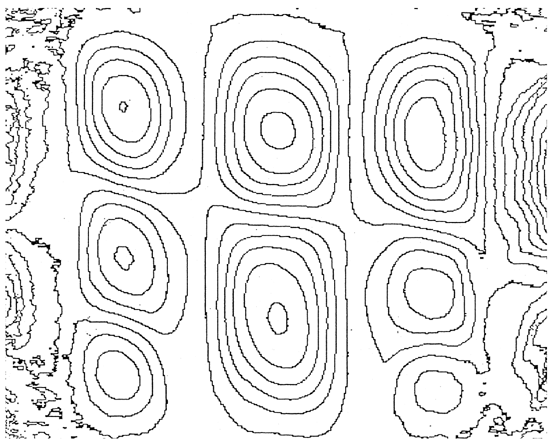

Fig. 4.17: Contour plot of a CFRP-panel under load

Practically, considering the same point P on the calibration plane in its initial position (i.e. $x = x_c = \not{c}$), which also means considering a <u>fixed pixel</u> on the CCD array or a fixed point K (i.e. $\check{x} = \check{x}_c = \not{c}$) on the object plane of the observing system, the calibration plane can be translated by different values z_{0i}, for which the corresponding difference moiré $\delta\widetilde{D}_{Mi}$ can be measured. Then, with equation (4.81), an equation system is built for each single pixel as follows

$$\delta\widetilde{D}_{Mi} = \left[\frac{A_{11} + R_{11}x_c}{1 + R_{12}x_c}\right]\left[\frac{(A_{21} + R_{21}x_c)z_{0i}}{1 + R_{12}x_c + (A_{22} + R_{22}x_c)z_{0i}}\right] = \frac{R_1 z_{0i}}{1 + R_2 z_{0i}} \tag{4.83}$$

where R_1 and R_2 are the geometrical constants associated to the considered pixel (other pixels get other values of R_1 and R_2). Obviously, it is not necessary to know the geometry of the optical setup to perform a moiré calibration. Both constants R_1 and R_2 can be determined by solving the equation system (4.83). This can be done by taking at least two set of values $\delta\widetilde{D}_{Mi}$ and z_{0i} or better by doing several measurements and applying the last square method.

In order to properly measure the object shape with projection moiré, the relative moiré ΔD_M corresponding to the calibration plane in its initial position must be subtracted from the new relative moiré

$\Delta\widetilde{D}_{MS}$ corresponding to the object shape. The result is a difference moiré $\delta\widetilde{D}_{MS} = \Delta\widetilde{D}_{MS} - \Delta D_M$ which must be compared with our experimental phase difference map $\delta\widetilde{D}_M$. If no experimental map is available, it is also possible to use in a first approximation a theoretically computed one if the geometry of the optical setup has been measured very carefully. Because R_1 and R_2 are known from the calibration, the object shape z_S corresponding to the difference moiré $\delta\widetilde{D}_{MS}$ is given by the following expression

$$z_S = \frac{\delta\widetilde{D}_{MS}}{R_1 - R_2\delta\widetilde{D}_{MS}} \tag{4.84}$$

For example, figure 4.17 shows the surface shape under load of a carbon fiber reinforced polymer (CFRP) panel reinforced with stringers on the back side. The phase pattern is pseudo-color processed to show a contour plot display with 0.5 mm displacement in depth of field between each level line.

In the more general case of general geometries of optical setups, but still in case of a translation of the calibration plane, we consider a fixed point K with constant coordinates $(\breve{x}_c, \breve{y}_c)$ on the object plane and write equation (4.70) as follows

$$\delta\widetilde{D}_{Mi} = \left[\frac{A_{11}+B_{11}\breve{x}_c+C_{11}\breve{y}_c}{1+B_{12}\breve{x}_c+C_{12}\breve{y}_c}\right]\left[\frac{(A_{21}+B_{21}\breve{x}_c+C_{21}\breve{y}_c)z_{0i}}{1+B_{12}\breve{x}_c+C_{12}\breve{y}_c+(A_{22}+B_{22}\breve{x}_c+C_{22}\breve{y}_c)z_{0i}}\right] = \frac{R_1 z_{0i}}{1+R_2 z_{0i}} \tag{4.85}$$

This shows that the moiré calibration can be performed independently for each pixel without needing to measure the geometry of the optical setup.

References

Chapter 1

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