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Antenna subtraction for the production of heavy particles at hadron colliders

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ABSTRACT: The antenna subtraction method developed originally for the computation of higher order corrections to jet observables from a colourless initial state is extended for hadron collider processes involving a pair of massive particles and jets in the final state at the next-to-leading order (NLO) level. Due to the presence of coloured initial states, the subtraction terms need to be divided into three categories (final-final, initial-final and initial-initial). In this paper, we outline their construction and derive the necessary ingredients: phase space factorisation, antenna functions and also integrated antennae, including the effects of massive final states in all of those building parts. As a first application, we explicitly construct the colour-ordered real radiation and the corresponding antenna subtraction terms required at NLO for the production of a top quark pair and for the production of a top quark pair in association with a hard jet. The latter constitutes an essential ingredient for the computation of the hadronic production of a top-antitop pair at NNLO.

KEYWORDS: Jets, NLO Computations, Hadronic Colliders, QCD

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1 Introduction

At LHC, physics beyond the Standard Model will almost inevitably manifest itself by the creation of massive particles which decay instantaneously into multiparticle final states. Searches for supersymmetric particles often involve final states with four or more jets. Top quarks [1, 2] are measured through their decay into a bottom quark and a subsequently decaying W -boson, yielding up to six-jet final states for top quark pair production. Meaningful searches for these signals require not only a very good anticipation of the expected signal, but also of all standard model backgrounds yielding identical final state signatures. Since leading-order calculations are affected by large uncertainties in their normalisation and their kinematical dependence, it appears almost mandatory to include NLO corrections. For a long time, these corrections were available only for at most three final state particles. Up to recently, the principle obstacle to NLO calculations of final states with higher multiplicities were the one-loop virtual corrections to multi-parton scattering amplitudes. Application of standard quantum field theory methods to those produced extremely large and numerically unstable expressions. In the recent past, several new algorithms have emerged to circumvent these numerical instabilities [3–8]. The recently released packages CutTools [9], BlackHat [10], Golem [11], Rocket [12] and Samurai [13] provide automated implementations of these new methods. They were already applied to in a number of pioneering calculations [14–25].

With a mass $m_t = 173 \pm 1.3$ GeV, the top quark is the heaviest particle produced at colliders and due to its very large mass it decays before it hadronises. By studying its properties in detail, it is hoped to elucidate the origin of particle masses and the mechanism of electroweak symmetry breaking. Since its discovery at the Fermilab Tevatron, a number of its properties (mass, couplings) have been determined to an accuracy of ten to twenty per cent. With the large number of top quark pairs expected to be produced at the LHC, the study of its properties will become precision physics. This large production rate will

allow precise measurements of their properties and their production cross sections with an expected experimental accuracy of five per cent.

Current theoretical predictions for the top quark pair production cross section include NLO corrections [24–27] and next-to-leading-logarithmic resummation (NLL) [28–30]. More recently even the NNLL resummation effects have been completed in [31]. These predictions lead to a theoretical uncertainty of the order of ten per cent. The same precision is available for single top quark production [32], top-pair-plus-jets production [23, 33, 34] and for top-pair-plus-bottom-pair production [21, 22].

The top quark appears as virtual particle in hadron collider processes and due to the small ratio between the top quark width and its mass, it is possible to factor the cross section of processes involving top quarks into the product of the production cross section for on-shell top quarks and the top quark decay width. Most of the calculations mentioned above are performed for on-shell top quark pair production. Only most recently, the decay of the top quark has been included in NLO calculations [24, 25, 35] leading to a similar theoretical accuracy.

Even for on-shell top quark pair production, a full fixed order calculation of the total top-antitop rate at NNLO, required to match the experimental accuracy, is missing. NNLO calculations involving massive quarks require the same ingredients as their massless counterparts. Three classes of contributions enter: double real, mixed real-virtual and two-loop type virtual contributions. However, the quark mass introduces one additional scale into the calculation. Especially the two-loop virtual corrections become therefore more involved than in the massless case. On the other hand, the treatment of the real radiation $n + 2$ parton processes is expected to be easier, since the heavy quark mass acts as an extra infrared regulator, thus eliminating part of the singularity structure.

Recent progresses has been accomplished concerning the two-loop contributions. Part of these two-loop virtual corrections are built with products of one-loop virtual amplitudes. Those corrections have been computed in [37–39]. Concerning the two-loop virtual corrections build with product of two-loop and tree-level amplitudes, the situation is different. The two-loop virtual corrections for the processes $q\bar{q} \rightarrow t\bar{t}$ and $gg \rightarrow t\bar{t}$ are not fully available at present. A purely numerical evaluation of the quark-initiated process [40] could be partly confirmed by analytical results [41–43], which were most recently extended also to the gluon-induced subprocess.

Real radiation corrections involving heavy quarks have up to now been investigated only at NLO. Most recently a semi-numerical scheme to evaluate the double real radiation corrections at NNLO has been introduced and applied to top quark pair production [44–46].

In this paper, towards calculating the double real contributions to top-antitop production in hadronic collisions at NNLO, we extend the antenna subtraction formalism to be applicable to the production of massive heavy quark pairs in the presence of coloured initial states and present results for the real and subtraction contributions for $t\bar{t}$ and $t\bar{t}$ +jet production at NLO.

Generally, for hard scattering observables, the inclusive cross section with two incoming hadrons H_1, H_2 can be written as

$$d\sigma = \sum_{a,b} \int \frac{d\xi_1}{\xi_1} \frac{d\xi_2}{\xi_2} f_{a/1}(\xi_1) f_{b/2}(\xi_2) d\hat{\sigma}_{ab}(\xi_1 H_1, \xi_2 H_2), \quad (1.1)$$

where ξ_1 and ξ_2 are the momentum fractions of the partons of species a and b in both incoming hadrons, f being the corresponding parton distribution functions and $d\hat{\sigma}_{ab}(\xi_1 H_1, \xi_2 H_2)$ is the parton-level scattering cross section for incoming partons a and b .

The partonic cross section $d\hat{\sigma}_{ab}$ has a perturbative expansion in the strong coupling α_s such that theoretical predictions for a hadronic process at a given order in α_s are obtained when all partonic channels contributing to that order to the partonic cross section are summed and convoluted with the appropriate parton distribution functions as in eq. (1.1).

In general, beyond the leading order, each partonic channel contains both ultraviolet and infrared (soft and collinear) divergences. The ultraviolet poles are removed by renormalisation in each channel. Collinear poles originating from the radiation of initial state partons are cancelled by mass factorisation counterterms and absorbed in the parton distribution functions. The remaining soft and collinear poles cancel among each other when all partonic contributions are summed over [47, 48]. As these observables depend in a non trivial manner on the experimental criteria needed to define them, they can only be calculated numerically. The computation of hadronic observables including higher order corrections therefore requires a systematic procedure to cancel infrared singularities among different partonic channels before any numerical computation of the observable can be performed.

For the task of next-to-leading order (NLO) calculations [49, 50], the infrared divergencies present in real radiation contributions can be systematically extracted by process-independent procedures, called subtraction methods.

More specifically, let us consider the hadronic production of m -jets at NLO. A theoretical prediction for this observable is obtained by summing the following massless partonic contributions: At LO, the tree-level contribution contains m partons in the final state which build m jets.

At NLO, the differential cross section for the production of m -jets, $d\hat{\sigma}_{\text{NLO}}$ may symbolically be written as,

$$d\hat{\sigma}_{\text{NLO}} = \int_{d\Phi_{m+1}} d\hat{\sigma}_{\text{NLO}}^R J_m^{(m+1)} + \int_{d\Phi_m} (d\hat{\sigma}_{\text{NLO}}^V + d\hat{\sigma}_{\text{NLO}}^{\text{MF}}) J_m^{(m)} \quad (1.2)$$

where $\int_{d\Phi_N}$ corresponds to the integration over the N parton phase space. The cross section is built with the real radiation cross section contribution $d\hat{\sigma}^R$ which has $(m+1)$ massless partons in the final state, the one-loop cross section $d\hat{\sigma}^V$ and the mass factorisation counterterm $d\hat{\sigma}_{\text{NLO}}^{\text{MF}}$ which have both m partons in the final state. A jet algorithm is applied separately on each of these contributions to ensure that out of n partons, m jets are built in the final state. Symbolically this recombination procedure is denoted by $J_m^{(n)}$.

The purpose of any subtraction method is to provide a subtraction term $d\hat{\sigma}_{\text{NLO}}^S$ which has the same singular behaviour as the real radiation squared matrix element and is sufficiently simple to be integrated analytically over a factorised form of the $(m+1)$ -phase space.

Using a subtraction method, the NLO partonic cross section given in eq. (1.2) may then be written as,

$$\begin{aligned}
 d\hat{\sigma}_{\text{NLO}} = & \int_{d\Phi_{m+1}} (d\hat{\sigma}_{\text{NLO}}^R - d\hat{\sigma}_{\text{NLO}}^S) J_m^{(m+1)} \\
 & + \int_{d\Phi_m} \left(\int_1 d\hat{\sigma}_{\text{NLO}}^S + d\hat{\sigma}_{\text{NLO}}^V + d\hat{\sigma}_{\text{NLO}}^{\text{MF}} \right) J_m^{(m)}. \tag{1.3}
 \end{aligned}$$

With this, the first integral is finite and can be integrated numerically in four dimensions. The integrated form of the subtraction term has the same number of final-state partons as the virtual contributions and the mass factorisation counterterms. It can therefore be combined with those thereby canceling analytically the explicit infrared divergences. The second integral in eq. (1.3) is therefore finite as well.

The actual form of the subtraction term $d\hat{\sigma}_{\text{NLO}}^S$ depends on the subtraction formalism used. The approximations of the matrix-element in the unresolved limits being non unique, several successful subtraction formulations have been proposed in the literature [51–55]. The dipole formalism of Catani and Seymour [51] and the FKS [52] of Frixione, Kunszt and Signer have been implemented in an automated way in [56–61]. In its original formulation, the formalism of Catani-Seymour [51] deals with massless partons in final and/or initial state at NLO.

An alternative subtraction formalism is given by the antenna subtraction method. This formalism [62–64] was originally derived for processes involving only (massless) final state partons in e^+e^- collisions. It has been applied in the computation of NNLO corrections to three-jet production in electron-positron annihilation [65–68] and related event shapes [69–73], which were subsequently used in precision determinations of the strong coupling constant [74–78].

For processes with initial-state partons and massless final states, the antenna subtraction formalism has been so far fully worked out only to NLO in [62, 63, 79]. It has been extended to NNLO for processes involving one initial state parton relevant for electron-proton scattering in [80] while an extension of the formalism to include two initial state hadrons at NNLO is under construction [81, 82].

Subtraction formalisms which deal with massive final state particles have been so far only developed up to the NLO level [83–86]. The kinematics is more involved due to the finite value of the parton masses. QCD radiation from massive particles can lead to soft divergencies but cannot lead to strict collinear divergencies, since the mass is acting as an infrared regulator. In a calculation of observables involving massive final state fermions, logarithmic terms of the form $\ln(Q^2/M^2)$, where M is the parton mass and Q , the typical scale of the hard scattering process can be generated. In kinematical configurations where $Q \gg M$, these logarithmically enhanced contributions can become large and can spoil the numerical convergence of the calculation. The cross section calculation of $t\bar{t}$ at LHC is an example where such enhanced logarithmic terms arise. These terms are related to a process-independent behaviour of the matrix elements; its singular behaviour in the massless limit ($M \rightarrow 0$). This singular behaviour is related to the *quasi-collinear* [83] limit of the matrix element. In the presence of massive particles, the factorisation properties of

matrix element and phase space in collinear and soft limits need to be generalised to take the mass effects into account.

The dipole formalism of Catani and Seymour [51] and the FKS formalism [52] have been extended to deal with massive particles in [83] for colourless and for coloured initial states up to the NLO. The subtraction terms constructed within this formalism account for the quasi-collinear limits of the matrix-element squared.

The antenna subtraction method has so far only been extended to deal with the production of massive fermions from a colourless initial state in [86]. It is the purpose of this paper to present an extension of this method to include radiation off final state massive fermions produced in hadronic processes. In this paper we aim to derive all necessary ingredients, massive antennae, phase space factorisation and finally integrated massive antennae for this extension. As a first application, we construct the subtraction terms $d\hat{\sigma}_{\text{NLO}}^S$ required for the hadronic production of a top quark pair in association with no or one hadronic jet.

These two processes, $pp \rightarrow t\bar{t}$ and $pp \rightarrow t\bar{t} + \text{jet}$ have been calculated in [28, 33] using the dipole subtraction method [83]. Our aim here is however not to redo this calculation using another subtraction scheme. Instead, we construct here the subtraction terms for $t\bar{t} + \text{jet}$ production in a colour-ordered form which is essential for the computation of top quark pair production without any jets at NNLO within the framework of the antenna formalism. Those subtraction terms can be used to capture all single unresolved radiation from the double real radiation contribution for the $t\bar{t}$ pair production at NNLO.

The plan of this paper is as follows. In section 2 we outline the construction of the subtraction terms for the hadronic production of a heavy quark pair in association with $(m-2)$ -jets at NLO. We present the form of the subtraction terms in all kinematical configurations with particular emphasis on the changes caused by the presence of massive final state particles in the expressions of the subtraction terms compared to those expressions in the massless case. Section 3 contains a list of all massive antenna functions required. In section 4 we tabulate all non-vanishing single unresolved limits of those massive antennae while in section 5 results for the integrated antennae are given. Section 6 presents a check on one of the integrated antenna. In section 7, for all partonic process involved, we present the colour ordered real contributions and their subtraction terms required to evaluate the hadronic production cross section of a $t\bar{t}$ pair and of $t\bar{t}$ pair and a jet at NLO. Finally section 8 contains our conclusions.

2 Antenna subtraction with massive final states

In this section, we present the general formalism necessary to evaluate the hadronic production of a pair of heavy quarks $Q\bar{Q}$ in association with $(m-2)$ jets at the next-to-leading order (NLO) in perturbative QCD.

2.1 Real radiation contributions to heavy quark pair production in association with jets

The leading order (LO) m -parton contribution to the hadronic production of a pair of heavy quarks $Q\bar{Q}$ in association with $(m - 2)$ jets may be written as,

$$d\hat{\sigma}_{\text{LO}}(p_1, p_2) = \mathcal{N} \sum_{m-2} d\Phi_m(k_Q, k_{\bar{Q}}, k_1, \dots, k_{m-2}; p_1, p_2) \times \frac{1}{S_{m-2}} |\mathcal{M}_m(k_Q, k_{\bar{Q}}, k_1, \dots, k_{m-2}; p_1, p_2)|^2 J_m^{(m)}(k_Q, k_{\bar{Q}}, k_1, \dots, k_{m-2}). \quad (2.1)$$

The momenta p_1 and p_2 are the momenta of the initial state partons, the massive partons Q and \bar{Q} have momenta k_Q and $k_{\bar{Q}}$, while the momenta of the remaining $(m - 2)$ massless final state partons are labelled $k_1 \dots k_{m-2}$. S_{m-2} is a symmetry factor for identical massless partons in the final state. $J_m^{(m)}(k_Q, k_{\bar{Q}}, k_1, \dots, k_{m-2})$ is the jet function. It ensures that out of $(m - 2)$ massless partons and a pair of heavy quarks Q and \bar{Q} present in the final state at parton level, an observable with a pair of heavy quark jets in association with $(m - 2)$ jets is built. At this order each massless or massive parton forms a jet on its own. The normalization factor \mathcal{N} includes all QCD-independent factors as well as the dependence on the renormalised QCD coupling constant α_s . \sum_{m-2} denotes the sum over all configurations with $(m - 2)$ massless partons. $d\Phi_m$ is the phase space for an m -parton final state containing $(m - 2)$ massless and two massive partons with total four-momentum $p_1^\mu + p_2^\mu$. In $d = 4 - 2\epsilon$ space-time dimensions, this phase space takes the form:

$$d\Phi_m(k_Q, k_{\bar{Q}}, k_1, \dots, k_{m-2}; p_1, p_2) = \frac{d^{d-1}k_Q}{2E_Q(2\pi)^{d-1}} \frac{d^{d-1}k_{\bar{Q}}}{2E_{\bar{Q}}(2\pi)^{d-1}} \times \frac{d^{d-1}k_1}{2E_1(2\pi)^{d-1}} \dots \frac{d^{d-1}k_{m-2}}{2E_{m-2}(2\pi)^{d-1}} (2\pi)^d \delta^d(p_1 + p_2 - k_Q - k_{\bar{Q}} - k_1 - \dots - k_{m-2}), \quad (2.2)$$

In eq. (2.1) $|\mathcal{M}_m|^2$ denotes a colour-ordered tree-level m -parton matrix element squared for m partons out of which two are massive. These terms only account for the leading colour contributions to the squared matrix elements. On the other hand, colour subleading contributions are, in general, given by the interference between two colour-ordered n -parton amplitudes. However, to keep the notation simpler we denote these interference contributions also as $|\mathcal{M}_m|^2$. Related to these interference terms, it is here worth noting the following: As soon as more than five coloured partons are present in a given partonic process, the subtraction of infrared singularities present in interference terms is more involved than for colour ordered squared matrix elements. This particular issue will be treated in section 7.2.1.

At, NLO the real radiation partonic contribution to the production of the heavy quark pair production in association with $(m - 2)$ jets involves $(m + 1)$ -final state partons with two of them being massive. It may be written as,

$$d\hat{\sigma}_{\text{NLO}}^R(p_1, p_2) = \mathcal{N} \sum_{m+1} d\Phi_{m+1}(k_Q, k_{\bar{Q}}, k_1, \dots, k_{m-1}; p_1, p_2) \times \frac{1}{S_{m+1}} |\mathcal{M}_{m+1}(k_Q, k_{\bar{Q}}, k_1, \dots, k_{m-1}; p_1, p_2)|^2 J_m^{(m+1)}(k_Q, k_{\bar{Q}}, k_1, \dots, k_{m-1}). \quad (2.3)$$

The jet function $J_m^{(m+1)}$ ensures that out of $(m-1)$ -massless partons and a $Q\bar{Q}$ pair, an observable with a pair of heavy quark jets in addition to $(m-2)$ jets, are built. In other words, an m -jet observable is formed.

In this contribution, when the real matrix element squared $|\mathcal{M}_{m+1}|^2$ is integrated over the phase space, it develops singularities when one parton in the final state is unresolved. In the presence of massive partons in the final state, a parton is called unresolved, either when it becomes soft or collinear to another massless parton or when it is *quasi-collinear* to a massive parton. In this latter case, it leads to finite logarithmic terms involving the mass of the massive parton. The notion of *quasi-collinear* limit will be explicitly presented in section 4. To extract the unresolved behaviours of the real matrix element, subtraction terms which take both the massless and massive effects need to be considered.

At the next-to-leading order, the subtraction terms derived in the antenna formalism [62–64, 79] are constructed solely with tree-level three-parton antenna functions. Those functions encapsulate all singular limits due to the emission of one unresolved parton between two colour-connected hard partons, called radiators.

Depending where the two radiators are located, in the initial or in the final state, we distinguish three types of configurations: final-final, initial-final and initial-initial. In any of those configurations, the radiated parton is always in the final state.

The subtraction terms in a given configuration are constructed from products of the corresponding antenna functions with reduced matrix elements. Those can be integrated over a phase space which is factorised into an antenna phase space (involving all unresolved partons and the two radiators) multiplied by a reduced phase space, where the momenta of radiators and unresolved radiation are replaced by two redefined momenta. These redefined momenta can be in the initial or in the final state depending on where the corresponding radiator momenta are and are defined by appropriate mappings. The full subtraction term is then obtained by summing over all antennae required in one configuration and by summing over all configurations needed for the problem under consideration.

The antenna subtraction terms do not provide a strictly local subtraction of collinear singularities in the case of a gluon splitting to two gluons or to a quark-antiquark pair. In these, the antenna subtraction term accounts for the singular behavior only after integration over the azimuthal angle of the two parton system with respect to the collinear direction. As a consequence, the numerical integration of the difference of matrix element and antenna subtraction term is potentially unstable. By an appropriate partitioning of the final state phase space [65, 87], this azimuthal integration variable can be separated off for each limit. Once this variable is separated, the angular terms can be averaged out by a smooth one-dimensional integration, or by combining different phase space points.

The massless and massive three parton final-final antenna functions, besides being fundamental entities of the antenna subtraction formalism developed for colourless initial states and for massless partons in [64] or for massive partons in [86] have another fundamental role. Those can be used as basic ingredients in parton showers. The event generator VINCIA uses these antenna functions as evolution kernels. In its present formulation [88, 89], VINCIA describes the evolution of timelike showers based on the massless [64]

and the massive [86] antenna functions. A study concerning the importance of quark mass effects is currently ongoing. The results presented in this paper concerning the massive initial-final antennae will become relevant for initial-state parton showers for observables with massive final states.

For the NLO corrections to $Q\bar{Q}$ + jets production in hadronic collisions, we will need all three types of subtraction terms and therefore all three types of antenna functions. Since one massive radiator is always in the final state, the subtraction terms will involve final-final and initial-final massive antennae but no initial-initial antennae with massive particles. Antennae involving only massless partons though, will be needed in all three configurations. Those have been derived and integrated in [64, 79, 82]. All required massive antenna functions will be presented in section 3 and their integrated forms will be given in section 5.

In the following, we shall present the general form of the subtraction terms needed in each of the three configurations (final-final, initial-final and initial-initial) to account for single unresolved radiation in processes involving a heavy quark pair in association with jets in the final state. We will in particular focus on the changes introduced in the subtraction terms due to the presence of massive final state particles compared to those when only massless partons are involved [64, 79, 82].

2.2 Subtraction terms for final-final configurations

In the final-final configuration, the subtraction term related to the real contributions to the partonic process yielding a heavy quark pair in association with $(m-2)$ jets given in eq. (2.3) has to take into account the presence of an unresolved parton j of momentum k_j emitted between two hard final-state radiators i and k of momenta k_i and k_k respectively. It reads,

$$\begin{aligned}
 d\hat{\sigma}_{\text{NLO}}^{S.(ff)} = & \mathcal{N} \sum_{m+1} d\Phi_{m+1}(k_1, \dots, k_i, k_j, k_k, k_{m+1}; p_1, p_2) \frac{1}{S_{m+1}} \\
 & \times \sum_j X_{ijk}^0 |\mathcal{M}_m(k_1, \dots, K_I, K_K, \dots, k_{m+1}; p_1, p_2)|^2 \\
 & \times J_m^{(m)}(k_1, \dots, K_I, K_K, \dots, k_{m+1}). \tag{2.4}
 \end{aligned}$$

This subtraction term involves the phase space for the production of $m+1$ partons, $d\Phi_{m+1}$, the massive final-final three-parton antenna function X_{ijk}^0 , the reduced m -parton amplitude squared $|\mathcal{M}_m|^2$ and the jet function $J_m^{(m)}$. The jet function $J_m^{(m)}$ ensures that out of $(m-2)$ massless partons and a pair of massive partons, $(m-2)$ jets and a $Q\bar{Q}$ jet pair is build. The jet function and the reduced m -parton amplitude do not depend on the individual momenta k_i, k_j and k_k , but will only depend on the redefined momenta k_I and k_K which are linear combinations of the original momenta k_i, k_j, k_k .

Two cases are implicitly considered here. Either i and k are massive hard final state radiators in which case the redefined partons I and K are massive or, i is massless and k is massive and the redefined partons I and K are massless and massive respectively. In this case, one of the parton momenta k_a with $a \neq i, k$ is massive in order to obtain a reduced matrix element with two massive final state partons.

Eq. (2.4) holds strictly for the subtraction of singularities of colour ordered matrix elements squared. Furthermore, as mentioned before, also interferences between partial amplitudes with different colour orderings appear in the subleading colour pieces. It will be seen in section 7.2.1 that the subtraction of infrared singularities appearing in these interferences needs a special treatment. However, to keep equations as brief and clear as possible, we still write our subtraction terms in the final-final configuration symbolically as in eq. (2.4).

Most of the massive final-final three-parton antenna functions X_{ijk}^0 needed to evaluate the NLO corrections to $Q\bar{Q} + \text{jets}$ production have been derived in [86]. Those will be listed in section 3. Solely, the new flavour-violating massive antennae, which are related to flavour violating vertices, will be explicitly derived in that section.

The phase space $d\Phi_{m+1}$ can be factorised as follows,

$$\begin{aligned} d\Phi_{m+1}(k_1, \dots, k_i, k_j, k_k, \dots, k_{m+1}; p_1, p_2) &= \\ &= d\Phi_m(k_1, \dots, K_I, K_K, \dots, k_{m+1}; p_1, p_2) \cdot d\Phi_{X_{ijk}}(k_i, k_j, k_k; K_I + K_K). \end{aligned} \quad (2.5)$$

$d\Phi_m$ is the d -dimensional phase space for m outgoing particles with momenta k_1, \dots, k_{m+1} with two of those momenta being massive, $d\Phi_{X_{ijk}}$ is the NLO final-final antenna phase space. It is proportional to a massive three-particle phase space relevant to a $1 \rightarrow 3$ decay [86]. Depending whether the two radiators are of equal masses or whether one of them is massless, different parametrisations of this antenna phase space are obtained. The parametrisations necessary to integrate the final-final massive antenna functions will be given in section 5.

Appropriate final-final phase space mappings are furthermore required to define the final-state momenta k_I and k_K in the relevant kinematical configurations. Those are however not unique. A possible mapping can be found in [83]. As in the massless case, the mapping given in [83] is not symmetric under the exchange of i and k . A possible symmetric version for a massive final-final definition of the mapped momenta will be given elsewhere.

For the analytic integration, we can use the phase space factorisation formula given in eq. (2.5) to rewrite each of the subtraction terms in eq. (2.4) in the form

$$|\mathcal{M}_m(k_1, \dots, K_I, K_K, \dots, k_{m+1}; p_1, p_2)|^2 J_m^{(m)} d\Phi_m \int d\Phi_{X_{ijk}} X_{ijk}^0. \quad (2.6)$$

The integrated massive final-final antennae, normalised appropriately are defined by analogy to the massless case by

$$\mathcal{X}_{ijk}^0(s_{ijk}) = \frac{1}{C(\epsilon)} \int d\Phi_{X_{ijk}} X_{ijk}^0 \quad (2.7)$$

with, the normalisation factor

$$C(\epsilon) = (4\pi)^\epsilon \frac{e^{-\epsilon\gamma_E}}{8\pi^2}. \quad (2.8)$$

The integration is performed analytically in d -dimensions such that the integrated subtraction terms can be combined with the one-loop m -parton contribution.

2.3 Subtraction terms for initial-final configurations

In the initial-final configuration, the subtraction term related to the real contributions given in eq. (2.3) has to take into account the presence of an unresolved parton j of momentum k_j , (which can be massive), emitted between a massless initial state radiator i of momentum p_i and a massive final state radiator k of momentum k_k and mass m_k . It is given by,

$$\begin{aligned}
 d\hat{\sigma}_{\text{NLO}}^{S,(if)} = & \mathcal{N} \sum_{m+1} d\Phi_{m+1}(k_1, \dots, k_j, k_k, \dots, k_{m+1}; p_i, p_2) \frac{1}{S_{m+1}} \\
 & \times \sum_j X_{i,jk}^0 |\mathcal{M}_m(k_1, \dots, K_K, \dots, k_m; p_i, p_2)|^2 J_m^{(m)}(k_1, \dots, K_K, \dots, k_m).
 \end{aligned}
 \tag{2.9}$$

This subtraction term involves the phase space for the production of $(m + 1)$ partons, $d\Phi_{m+1}$, the massive three parton initial-final antenna function denoted by $X_{i,jk}^0$, the reduced m -parton amplitude squared $|\mathcal{M}_m|^2$ and the jet function $J_m^{(m)}$. The reduced m -parton matrix element squared $|\mathcal{M}_m|^2$ does not contain any explicit dependence on the original final state momenta k_j and k_k , but only depends on them through the redefined momentum K_K . The same holds for the jet function $J_m^{(m)}$. The three parton massive antenna functions $X_{i,jk}^0$ depend only on the momenta p_i , k_j and k_k and on the masses of the final state partons m_j and m_k . These will be presented in section 3.

In this configuration, the initial state radiator i is always massless while the final state radiator k is always massive. The unresolved parton j can be either massless or massive such that two situations are covered in eq. (2.9). In case it is massless, one of the partons with k_a where $a \neq j, k$ has to be taken massive in order to build an event with a pair of massive partons in addition to $(m - 2)$ jets. Depending whether the unresolved parton is massless or massive, different phase space parametrisations will be needed. Those will be presented in section 5 when the massive initial-final antennae are integrated over the unresolved phase space.

As it was mentioned for the final-final case, the subtraction terms as presented in eq. (2.9) are strictly valid for the subtraction of infrared singularities of colour ordered matrix elements squared. Interferences of different partial amplitudes which also generate infrared singularities need a special treatment which we shall present in section 7.2.1. To keep our equations as brief and clear as possible, we still write our subtraction terms in the initial-final configuration symbolically as in eq. (2.9).

2.3.1 Phase space factorisation and mappings

For the subtraction term to be integrated and then added to the virtual contributions and mass factorisation counterterms, we need the phase space to factorise adequately. As the presence of massive final states lead to differences in this factorisation with respect to the massless case considered in [79] we derive this phase space factorisation explicitly below.

We start from the $2 \rightarrow (m + 1)$ -particle phase space with a priori two massive final state particles of momenta k_j and k_k and the remaining $(m - 1)$ particles staying massless.

The momenta of the initial state partons are named p_i and p_2 .

$$d\Phi_{m+1}(k_1, \dots, k_{m+1}; p_i, p_2) = (2\pi)^d \delta\left(p_i + p_2 - \sum_l k_l\right) \prod_l [dk_l], \quad (2.10)$$

where the phase space measure for massless and massive partons are respectively given by: $[dk_l] = d^d k \delta^+(k_l^2) / (2\pi)^{d-1}$ for $l \neq j, k$ and $[dk_l] = d^d k_l \delta^+(k_l^2 - m_l^2) / (2\pi)^{d-1}$ for $l = j, k$. We insert

$$1 = \int d^d q \delta(q + p_i - k_j - k_k), \quad (2.11)$$

and, to take into account the masses of the final state particles j and k we insert

$$1 = \frac{Q^2 + m_K^2}{2\pi} \int \frac{dx_i}{x_i} \int [dK_K] (2\pi)^d \delta(q + p_I - K_K) \quad (2.12)$$

with

$$Q^2 = -q^2 \quad x_i = \frac{Q^2 + m_K^2}{2p_i \cdot q} \quad \text{and} \quad p_I = x_i p_i. \quad (2.13)$$

We find that the original phase space for $(m+1)$ partons given in eq. (2.10) can be written as the product of an m -parton ($d\Phi_m$) and a two-to-two particle phase space ($d\Phi_2$) as follows,

$$d\Phi_{m+1}(k_1, \dots, k_{m+1}; p_i, p_2) = d\Phi_m(k_1, \dots, K_K, \dots, k_{m+1}; x_i p_i, p_2) \times \frac{(Q^2 + m_K^2)}{2\pi} d\Phi_2(k_j, k_k; p_i, q) \frac{dx_i}{x_i}. \quad (2.14)$$

Replacing this phase space in the subtraction term given in eq. (2.9) we can explicitly carry out the integration of the antenna functions over the two-to-two particle phase space and get the integrated expression for the subtraction term to be added to the virtual contributions and the mass factorisation counterterms. For this purpose it is convenient to define the integrated massive initial-final antenna as,

$$\mathcal{X}_{i,jk} = \frac{1}{C(\epsilon)} \int d\Phi_2 \frac{(Q^2 + m_K^2)}{2\pi} X_{i,jk}, \quad (2.15)$$

where $C(\epsilon)$ is given in eq. (2.8) and the initial-final massive antenna phase space denoted by $d\Phi_{X_{i,jk}}$ is given by,

$$d\Phi_{X_{i,jk}} = d\Phi_2 \frac{(Q^2 + m_K^2)}{2\pi}. \quad (2.16)$$

Due to the presence of massive particles, the phase space mapping for initial-final configurations derived in the massless case in [79] needs to be generalized as well: In a process of the form $p_i + q \rightarrow k_j + k_k$ with $(q^2 < 0)$, to find a mapping from the original momenta p_i in the initial state, k_j and k_k in the final state, to the redefined momenta p_I and K_K that ensures phase space factorisation, the following conditions need to be fulfilled: The remapped final state momentum K_K must be on-shell, and momentum must be conserved. This implies,

$$p_i - k_j - k_k = x_i p_i - K_K = p_I - K_K \quad (2.17)$$

for the phase space to factorise as in eq. (2.14).

In addition, concerning the single unresolved behaviour of the remapped momenta, the following requirements have to be satisfied:

$$\begin{aligned}
 x_i p_i \rightarrow p_i & & K_K \rightarrow k_k & & \text{when } j \text{ becomes soft} & & (2.18) \\
 x_i p_i \rightarrow p_i & & K_K \rightarrow k_j + k_k & & \text{when } j \text{ becomes collinear or quasi-collinear with } k \\
 x_i p_i \rightarrow p_i - k_j & & K_K \rightarrow k_k & & \text{when } j \text{ becomes collinear or quasi-collinear with } i.
 \end{aligned}$$

It can be seen that when parton j becomes soft, collinear or quasi-collinear to parton k we have $x_i \rightarrow 1$, while for the case in which partons i and j are collinear or quasi-collinear, $x_i \rightarrow 1 - z$, with z being the fraction of the initial state momentum p_i carried by the unresolved momentum j . The notion of quasi-collinear limit can be viewed as an extension of the collinear limit when at least one of the partons becoming collinear to another is massive. Its explicit definition in this initial-final configuration will be given in section 4.

All these conditions given above are interpolated by,

$$\begin{aligned}
 K_K &= k_j + k_k - (1 - x_i)p_i \\
 x_i &= \frac{s_{ij} + s_{ik} - s_{jk}}{s_{ij} + s_{ik}}.
 \end{aligned} \tag{2.19}$$

With this choice, the mass relation for momenta before and after mapping reads,

$$m_K^2 = m_k^2 + m_j^2. \tag{2.20}$$

Note also that in eq. (2.19), s_{ab} stand for $2p_a \cdot p_b$. as everywhere in this paper. In terms of those, massless [79] and massive definitions of x_i are the same. The massless phase space factorisation and mappings given in [79] can be recovered by setting the masses of the final state particles to zero.

2.4 Subtraction terms for initial-initial configurations

Additional divergent contributions may finally also occur in the real matrix element squared given in eq. (2.3) when a massless final state parton becomes unresolved with respect to two initial state partons. In this case, the subtraction terms are constructed exclusively with massless three parton initial-initial antennae. In those, the initial state partons are the hard radiators and this situation was studied in detail in [79].

The subtraction term associated to an unresolved massless parton j and two hard initial state radiators i and k with momenta p_i and p_k in the partonic process $p_i + p_k \rightarrow k_j + k_Q + k_{\bar{Q}} + (m - 2)$ partons takes the form

$$\begin{aligned}
 d\sigma_{\text{NLO}}^{S,(ii)} &= \mathcal{N} \sum_{m+1} d\Phi_{m+1}(k_Q, k_{\bar{Q}}, k_1, \dots, k_j, \dots, k_{m-1}; p_i, p_k) \frac{1}{S_{m+1}} \\
 &\times \sum_j X_{ik,j}^0 |\mathcal{M}_m(\tilde{k}_Q, \tilde{k}_{\bar{Q}}, \tilde{k}_1, \dots, \tilde{k}_{j-1}, \tilde{k}_{j+1}, \dots, k_{m-1}; x_i p_i, x_k p_k)|^2 \\
 &\times J_m^{(m)}(\tilde{k}_Q, \tilde{k}_{\bar{Q}}, \tilde{k}_1, \dots, \tilde{k}_{j-1}, \tilde{k}_{j+1}, \dots, \dots, \tilde{k}_{m-1}).
 \end{aligned} \tag{2.21}$$

This subtraction term involves the phase space for the production of $(m + 1)$ partons, $d\Phi_{m+1}$, the massless three parton initial-initial antenna function denoted by $X_{ik,j}^0$, the reduced m -parton amplitude squared $|\mathcal{M}_m|^2$ and the jet function $J_m^{(m)}$. As mentioned for the final-final and initial-final cases, the subtraction terms as presented in eq. (2.21) are strictly valid for the subtraction of infrared singularities of colour ordered matrix elements squared only. Interferences of different partial amplitudes need a special treatment which we shall present in section 7.2.1. However, to keep our equations as brief and clear as possible, we still write our subtraction terms in the initial-initial configuration symbolically as in eq. (2.21).

All momenta (massless and massive) in the arguments of the reduced matrix elements $|\mathcal{M}_m|^2$ and the jet function $J_m^{(m)}$ have to be redefined. They are denoted with tildes in eq. (2.21). The two hard radiators are simply rescaled by factors x_i and x_k respectively. The other momenta present in the reduced matrix element squared are boosted by a Lorentz transformation onto the new set of momenta $\{\tilde{k}_l, l \neq j\}$ as described for massless partons in [79]. We have checked that the same boost [51] required to redefine the momenta of the massless partons can be used to redefine the momenta of the massive partons. Consequently, the presence of massive partons in the final state does not influence the way the phase space factorises and how the mapping is defined so that the massless factorisation [79] can be used in this context.

The phase space $d\Phi_{m+1}$ factorises into the convolution of a massive m -particle phase space involving only redefined momenta and a massless initial-initial antenna phase space related to the phase space of the unresolved parton j , as in the massless case.

All required initial-initial massless antennae $X_{ik,j}^0$ needed for the construction of the subtraction terms for $t\bar{t}$ production in association with jets have been derived and integrated in [79, 82].

3 Massive antenna functions

In this section, we aim to list all massive antenna functions which are needed to construct the subtraction terms for the hadronic production of a heavy quark pair and for the production of a heavy quark pair in association with one additional jet at NLO.

3.1 General features of antenna functions

In section 2, we saw that the subtraction terms defined in the three configurations (final-final, initial-final and initial-initial) are built with products of the corresponding antenna functions (denoted by X) with reduced matrix element squared. At this order (NLO), only tree level three-particle antenna functions are required. Those describe all unresolved (soft, collinear and quasi-collinear) radiation emitted between a pair of colour-ordered hard partons, the radiators. Originally, in [64], the antenna functions were derived for massless final-final configurations. Those are defined by the pair of hard partons they collapse to in the unresolved limits and in all cases, the antenna functions are derived from physical matrix elements. Generally, the quark-antiquark antenna functions are obtained from

$\gamma \rightarrow q\bar{q} + (\text{partons})$, the quark-gluon antenna functions from $\tilde{\chi} \rightarrow \tilde{g} + (\text{partons})$ [90] and the gluon-gluon antenna functions from $H \rightarrow (\text{partons})$ [91].

The three parton final-final antenna functions were obtained by normalising the colour-ordered three-parton tree level matrix elements squared to the matrix element squared for the basic two-parton process, omitting all couplings and colour factors. As such the tree-level three parton massless or massive final-final antenna functions are *scalars in colour space*, have mass dimension (-2) and are defined by [64, 86] as,

$$X_{ijk}^0 = S_{ijk,IK} \frac{|\mathcal{M}_{ijk}^0|^2}{|\mathcal{M}_{IK}^0|^2}. \tag{3.1}$$

S denotes the symmetry factor associated to the antenna, which accounts both for potential identical particle symmetries and for the presence of more than one antenna in the basic two-parton process. It is chosen such that the antenna function reproduces the unresolved limits of a matrix element with identified particles.

At NLO the existing three parton tree level massless final-final X_{ijk}^0 antennae are:

- Quark-antiquark: the only antenna functions of this kind at NLO are the A-Type antennae, and they are obtained from the ratio $|\mathcal{M}(\gamma^* \rightarrow q\bar{q}g)|^2/|\mathcal{M}(\gamma^* \rightarrow q\bar{q})|^2$. Since the quark and the antiquark are of the same flavour, in the following these antennae will be referred as flavour conserving A-Type antennae.
- Quark-gluon: there are two different antenna functions of this kind: D-Type and E-Type. The D-Type antennae are obtained from the ratio $|\mathcal{M}(\tilde{\chi} \rightarrow \tilde{g}gg)|^2/|\mathcal{M}(\tilde{\chi} \rightarrow \tilde{g}g)|^2$, while the E-Type are computed from $|\mathcal{M}(\tilde{\chi} \rightarrow \tilde{g}q\bar{q})|^2/|\mathcal{M}(\tilde{\chi} \rightarrow \tilde{g}g)|^2$ [90].
- Gluon-gluon: there are also two different antennae of this kind: F-Type and G-Type. The F-Type antenna functions are obtained from $|\mathcal{M}(H \rightarrow ggg)|^2/|\mathcal{M}(H \rightarrow gg)|^2$, while the G-Type are calculated from the ratio $|\mathcal{M}(H \rightarrow gq\bar{q})|^2/|\mathcal{M}(H \rightarrow gg)|^2$ [91].

The initial-final antennae and the initial-initial antennae denoted respectively by $X_{i,jk}^0$ and $X_{ik,j}^0$ are in principle defined by crossing one or two *massless* partons from the final to the initial state in the final-final antennae X_{ijk}^0 . However, this crossing may not be unambiguous for initial-final configurations [79].

All these types of antenna functions are needed for the construction of the subtraction terms for heavy quark pair production except the G-Type antennae. Those antennae are required in all three (final-final, initial-final or initial-initial) configurations and in their massless or massive versions. In the latter case, the final state quark (q) or gluino (\tilde{g}) is taken massive. Note that if both of those partons are present in the final state, the corresponding massive antenna is obtained by taking only the gluino \tilde{g} to be massive. The antenna obtained taking both of these partons massive is not required for the construction of the subtraction terms for heavy quark pair production +jets at NLO, considered here.

In addition to these flavour conserving antennae, we will also need flavour violating massive quark-antiquark A-Type antennae in both final-final and initial-final configurations. Those involve flavour violating vertices and will be defined below.

In all these antennae denoted by X_{ijk} in the final-final configuration and crossings of those, the labels i, j, k can either stand for massless or massive partons. The particle with label j always denotes the unresolved parton. The massive antennae will depend on the invariants s_{ij}, s_{ik} and s_{jk} (with $s_{ab} = 2p_a p_b$ throughout this paper) and on the masses of the final states heavy partons.

In the following, when presenting the expressions of the required massive antennae we will use a clear labeling of those antennae. We shall specify which partons in them are taken massless, those will be indexed with q , or taken massive and then indexed with Q . For conciseness, the $\mathcal{O}(\epsilon)$ in the expression of the unintegrated antennae will be omitted.

3.1.1 Flavour-violating antennae

For the construction of the subtraction terms for $t\bar{t}$ +jet production at NLO, we encounter two types of flavour violating antennae. Both are A-Type quark-antiquark antennae and are labelled: $A_{qg\bar{Q}}^0$ and A_{qgQ}^0 . In those, q represents a massless quark and Q and \bar{Q} represent a massive quark or antiquark of different flavour than the massless quark q . Note that for symmetry reasons of the A-Type antenna, the role of q and \bar{Q} in $A_{qg\bar{Q}}^0$ or of q and Q in A_{qgQ}^0 are interchangeable resulting in the same two flavour-violating antennae. We encounter these two types of flavour violating antennae in final-final and initial-final configurations. In the latter case, a massless quark is always in the initial state and plays the role of the initial state radiator. It is worth noting that the limits covered by flavour violating antennae with a gluon in the initial state can be covered by flavour conserving initial-final antennae, such that these gluon initiated flavour violating antennae are not required here.

The massive final-final flavour violating antennae have their massless counterparts given by $A_{qg\bar{q}'}^0$ and $A_{qgq'}^0$ where q' and \bar{q}' are massless quarks or antiquarks of different flavour than q . These massless counterparts have been used as essential ingredients for the construction of the subtraction terms required for the computation of $e^+e^- \rightarrow 3$ jets at NNLO [65]. Those massless A-Type flavour violating antennae have exactly the same properties and singular structure as a massless flavour-conserving A-Type quark-antiquark antenna given by $A_{qg\bar{q}}^0$ [64]. Those have also the same unintegrated form as this antenna.

In the massive case, the final-final flavour violating A-Type antennae $A_{qg\bar{Q}}^0$ and A_{qgQ}^0 , having one massive particle in the final state, have the same properties and singular structure as a massive flavour-conserving A-Type antenna of the form $A_{Qg\bar{Q}}^0$ where the massive quark Q or the massive antiquark \bar{Q} (by symmetry arguments) present in this flavour conserving antenna is taken massless.

By construction, within the antenna formalism, in the unresolved (collinear, soft or quasi-collinear) limits, the three parton antennae yield massless or massive universal unresolved factors given either by splitting functions or eikonal factors in their massless or massive forms. The spin properties of the final states are determinant to obtain these unresolved factors. Consequently, all flavour-violating antennae (massless or massive) can be generated by processes resulting from the decay of a charged W-boson into massless or massive final state fermions. When the particle in the final state is a (massless or massive) antiquark such that the flavour violating antenna contains a *quark and an antiquark* of different flavours, those final states can be Dirac fermions. When the particles in the an-

tennae are *two quarks* of different flavours, those final states can be produced in extended versions of the Standard Model [92, 93] and will involve Dirac and Majorana fermions.

3.2 Massive NLO final-final antenna functions

For the construction of the subtraction terms relevant for heavy quark pair production at hadron colliders at NLO, we will need three-parton massive final-final flavour conserving and flavour violating A-Type antennae and flavour conserving quark-gluon antenna functions. All flavour-conserving massive final-final antennae required have been derived in [86]. They are given below for completeness. The flavour violating massive final-final antennae are new.

The generic process considered to define the massive three-parton final-final antenna X_{ijk}^0 is given by $q \rightarrow k_i + k_j + k_k$ where q is the virtuality of the colourless initial state, with $q^2 > 0$. In the following, the center of mass energy of the decaying particle E_{cm} will be used instead of q with $q^2 = E_{\text{cm}}^2$. The final state radiator partons i and k are either both massive or only one of them is such. Parton j defines always the unresolved massless final state parton.

The massive three-parton final-final antenna X_{ijk}^0 will depend on on the invariants s_{ij} , s_{jk} and s_{ik} , on the masses m_i and m_k of the final state partons i and k and on E_{cm}^2 .

3.2.1 Quark-Antiquark antennae

The flavour conserving quark-antiquark massive antenna function denoted by $A_{Qg\bar{Q}}^0$ has a massive quark Q and a massive antiquark \bar{Q} as radiators and reads,

$$\begin{aligned}
 A_3^0(1_Q, 3_g, 2_{\bar{Q}}) = & \frac{1}{4(E_{\text{cm}}^2 + 2m_Q^2)} 4 \left(\frac{2s_{12}^2}{s_{13}s_{23}} + \frac{2s_{12}}{s_{13}} + \frac{2s_{12}}{s_{23}} + \frac{s_{23}}{s_{13}} + \frac{s_{13}}{s_{23}} \right. \\
 & + m_Q^2 \left(\frac{8s_{12}}{s_{13}s_{23}} - \frac{2s_{12}}{s_{13}^2} - \frac{2s_{12}}{s_{23}^2} - \frac{2s_{23}}{s_{13}^2} - \frac{2}{s_{13}} - \frac{2}{s_{23}} - \frac{2s_{13}}{s_{23}^2} \right) \\
 & \left. + m_Q^4 \left(-\frac{8}{s_{23}^2} - \frac{8}{s_{13}^2} \right) \right) + O(\epsilon). \tag{3.2}
 \end{aligned}$$

This function is normalised to the two-particle process $\gamma^* \rightarrow Q\bar{Q}$, whose matrix element squared (omitting couplings) is given by

$$A_2^0(1_Q, 2_{\bar{Q}}) = 4 [(1 - \epsilon) E_{\text{cm}}^2 + 2 m_Q^2]. \tag{3.3}$$

3.2.2 Quark-Gluon antennae

For the subtraction terms required for the production of heavy particles in addition to jets, only massive quark-gluon antennae with one massive radiator in the final state are needed.

The quark-gluon massive antennae with either two gluons or a massless quark-antiquark pair in the final state are normalised by the two-particle matrix element squared relevant for the process $\tilde{\chi} \rightarrow \tilde{g}g$, with the gluino \tilde{g} being massive with mass m_Q and the gluon g being massless. This two-particle matrix element squared omitting couplings reads,

$$X_2^0(1_Q, 2_g) = 4(1 - \epsilon) (E_{\text{cm}}^2 - m_Q^2)^2, \tag{3.4}$$

where X can stand for the E or D -Type antennae here.

The quark-gluon massive antenna $E_{Qq'\bar{q}'}$ with a pair of massless quarks $q'\bar{q}'$ and a massive radiator quark Q in the final state reads,

$$E_3^0(1_Q, 3_{q'}, 4_{\bar{q}'}) = \frac{1}{4(E_{\text{cm}}^2 - m_Q^2)^2} 4 \left(s_{13} + s_{14} + \frac{s_{13}^2}{s_{34}} + \frac{s_{14}^2}{s_{34}} - 2E_{\text{cm}}m_Q \right) + O(\epsilon). \quad (3.5)$$

The quark-gluon massive antenna D_{Qgg}^0 with two gluons and a massive radiator Q in the final state reads,

$$\begin{aligned} D_3^0(1_Q, 3_g, 4_g) = & \frac{1}{4(E_{\text{cm}}^2 - m_Q^2)^2} 4 \left(\left(9s_{13} + 9s_{14} + \frac{4s_{13}^2}{s_{14}} + \frac{4s_{14}^2}{s_{13}} + \frac{4s_{13}^2}{s_{34}} + \frac{2s_{13}^3}{s_{14}s_{34}} \right. \right. \\ & \left. \left. + \frac{6s_{13}s_{14}}{s_{34}} + \frac{4s_{14}^2}{s_{34}} + \frac{2s_{13}^3}{s_{13}s_{34}} + 6s_{34} + \frac{3s_{13}s_{34}}{s_{14}} + \frac{3s_{14}s_{34}}{s_{13}} + \frac{s_{34}^2}{s_{13}} + \frac{s_{34}^2}{s_{14}} \right) \\ & - m_Q^2 \left(6 + \frac{2s_{13}^2}{s_{14}^2} + \frac{4s_{13}}{s_{14}} + \frac{4s_{14}}{s_{13}} + \frac{2s_{14}^2}{s_{13}^2} + \frac{6s_{34}}{s_{13}} + \frac{4s_{13}s_{34}}{s_{14}^2} \right. \\ & \left. + \frac{6s_{34}}{s_{14}} + \frac{4s_{14}s_{34}}{s_{13}^2} + \frac{2s_{34}^2}{s_{13}^2} + \frac{2s_{34}^2}{s_{14}^2} + \frac{2s_{34}^2}{s_{13}s_{14}} \right) \\ & \left. + 2E_{\text{cm}}m_Q - E_{\text{cm}}m_Q^3 \frac{2s_{34}}{s_{13}s_{14}} + m_Q^4 \frac{2s_{34}}{s_{13}s_{14}} \right) + O(\epsilon). \end{aligned} \quad (3.6)$$

This tree-level antenna function $D_3^0(1_Q, 3_g, 4_g)$ contains two antennae, corresponding to the following configurations: gluon (3_g) radiated between the massive quark and gluon (4_g) and also the configuration where gluon (4_g) is radiated between the quark and gluon (3_g). The separation between these two configurations is not free from ambiguity, since the collinear limit of the two gluons has to be split between the two configurations. We consider the following decomposition

$$D_3^0(1_Q, 3_g, 4_g) = d_3^0(1_Q, 3_g, 4_g) + d_3^0(1_Q, 4_g, 3_g) \quad (3.7)$$

where the sub-antenna denoted by d_3^0 is given by

$$\begin{aligned} d_3^0(1_Q, 3_g, 4_g) = & \frac{1}{(E_{\text{cm}}^2 - m_Q^2)^2} \\ & \times \left[\frac{9s_{13} + 9s_{14} + 6s_{34}}{2} + \frac{4s_{14}^2}{s_{13}} + \frac{4s_{14}^2}{s_{34}} + \frac{2s_{14}^3}{s_{13}s_{34}} + \frac{3s_{13}s_{14}}{s_{34}} + \frac{3s_{14}s_{34}}{s_{13}} + \frac{s_{34}^2}{s_{13}} \right. \\ & \left. - m_Q^2 \left(3 + \frac{2s_{14}^2}{s_{13}^2} + \frac{4s_{14}}{s_{13}} + \frac{6s_{34}}{s_{13}} + \frac{6s_{14}}{s_{13}} + \frac{4s_{14}s_{34}}{s_{13}^2} + \frac{2s_{34}^2}{s_{13}^2} + \frac{s_{34}^2}{s_{13}s_{14}} \right) \right. \\ & \left. + E_{\text{cm}}m_Q - (E_{\text{cm}} - m_Q)m_Q^3 \frac{s_{34}}{s_{13}s_{14}} \right] + O(\epsilon) \end{aligned} \quad (3.8)$$

Both of these sub-antennae will be needed individually to construct the subtraction term for $t\bar{t}$ +jet at NLO. Those will not need to be integrated separately though.

3.2.3 Flavour violating antennae

In addition to the massive flavour-conserving final-final antennae given above, for the construction of the subtraction terms for $t\bar{t}$ +jet production at NLO, we also need two types of massive flavour violating final-final antennae. We will require an antenna involving as radiators: one massive quark Q and one massless antiquark \bar{q} , denoted by $A_{Qg\bar{q}}^0$ and one antenna with a massive quark Q and a massless quark q denoted by A_{Qgq}^0 . The expressions of these two antennae are related by the exchange of a massless quark versus an antiquark in the final state. Therefore, we here present only one of those namely $A_{Qg\bar{q}}^0$. It is given by,

$$A_3^0(1_Q, 3_g, 2_{\bar{q}}) = \frac{1}{(E_{\text{cm}}^2 + m_Q^2)} \left[\frac{2s_{12}}{s_{13}} + \frac{2s_{12}}{s_{23}} + \frac{2s_{12}^2}{s_{13}s_{23}} + \frac{s_{13}}{s_{23}} + \frac{s_{23}}{s_{13}} - m_Q^2 \left(\frac{2s_{12}}{s_{13}^2} + \frac{2s_{23}}{s_{13}^2} + \frac{2}{s_{13}} \right) \right] + O(\epsilon) \quad (3.9)$$

This antenna function is normalised with the following two-parton matrix element squared:

$$A_2^0(1_Q, 2_{\bar{q}}) = 4(1 - \epsilon) [E_{\text{cm}}^2 + m_Q^2]. \quad (3.10)$$

It is worth noting that the expression of this final-final flavour violating antenna $A_{Qg\bar{q}}^0$ given in eq. (3.9) differs significantly from the flavour conserving antenna $A_{Qg\bar{Q}}^0$ given above in eq. (3.2) as expected.

3.3 Massive NLO initial-final antenna functions

To construct our subtraction terms for heavy quark pair production in association with jets at NLO, we will also need massive initial-final antennae of different types. Flavour-conserving and flavour-violating massive quark-antiquark A-Type antennae and flavour conserving massive quark-gluon antennae are required.

In principle, the massive initial-final massive antennae can be obtained from the corresponding expressions for the massive final-final antennae given above by appropriately crossing one massless parton from the final to the initial state. By this crossing procedure, the presence of an overall uneven number of fermions crossed to define the initial-final antenna leads to an overall minus sign in the definition of the antennae. Furthermore, this crossing procedure is not non ambiguous for the quark-gluon D-Type antennae initiated by a gluon.

In general, the initial-final antennae $X_{i,jk}$ are normalised to the reduced two-parton matrix element squared $|\mathcal{M}_{I,K}^0|^2$ to which the three-parton matrix element squared $|\mathcal{M}_{i,jk}^0|^2$ tends in the limits. For the D-Type antennae initiated by a gluon, depending which limit is considered, the reduced two-parton matrix element to which the three-parton matrix element collapses to, can be different. In particular, the nature of the initial state parton in the three and two-parton matrix elements squared needed as ratio to define the antennae may change.

The generic process necessary to define the massive initial-final three parton antenna $X_{i,jk}^0$ is given by $q + p_i \rightarrow k_j + k_k$ with $q^2 < 0$ and $q^2 = -Q^2$. In this process, p_i is the momentum of the initial state radiator i whereas k_j and k_k are the momenta of the

unresolved parton j and the final state radiator k respectively. Depending on the situation considered, the one massive parton present in the final state can either be the unresolved parton or the final state radiator.

In any case, the initial-final antennae $X_{i,jk}$ will depend on the invariants s_{ij} , s_{jk} and s_{ik} , on the masses m_j and m_k of the final state partons j and k and on Q^2 .

3.3.1 Quark-Antiquark antennae

By crossing a gluon to the initial state in the expression of the final-final massive quark-antiquark antenna $A_{Qg\bar{Q}}^0$ one obtains the initial-final massive quark-antiquark antenna $A_{g;Q\bar{Q}}^0$. In it, the gluon plays the role of the initial state radiator, the final state radiator is a massive quark or an antiquark (by symmetry arguments). The unresolved parton is correspondingly either a massive antiquark or a quark which can become quasi-collinear to the initial state gluon. This antenna is given by,

$$\begin{aligned}
 A_3^0(3_g; 1_Q, 2_{\bar{Q}}) = & -\frac{1}{[Q^2 - 2m_Q^2]} \left(-\frac{2s_{12}^2}{s_{13}s_{23}} + \frac{2s_{12}}{s_{13}} + \frac{2s_{12}}{s_{23}} - \frac{s_{13}}{s_{23}} - \frac{s_{23}}{s_{13}} \right. \\
 & -m_Q^2 \left(\frac{8s_{12}}{s_{13}s_{23}} - \frac{2s_{12}}{s_{13}^2} - \frac{2s_{12}}{s_{23}^2} + \frac{2s_{13}}{s_{23}^2} + \frac{2s_{23}}{s_{13}^2} + \frac{2}{s_{13}} + \frac{2}{s_{23}} \right) \\
 & \left. -m_Q^4 \left(-\frac{8}{s_{13}^2} - \frac{8}{s_{23}^2} \right) \right) + O(\epsilon). \quad (3.11)
 \end{aligned}$$

This function has been normalised to the two-particle matrix element related to the process $\gamma^*Q \rightarrow Q$. It corresponds to the matrix element in which the process $\gamma^*g \rightarrow Q\bar{Q}$ reduces to in all its limits. Omitting couplings, this normalisation factor is given by

$$A_2^0(1_Q; 2_Q) = 4 [(1 - \epsilon)Q^2 - 2m_Q^2]. \quad (3.12)$$

Note that the resulting antenna has an overall minus sign made explicit in eq. (3.11) due to the uneven number of fermions crossed to define it.

3.3.2 Quark-gluon antennae

As in the massless case, the quark-gluon final-final massive antennae are separated into two categories depending if the final state radiator gluon splits into a quark-antiquark pair (E-Type) or into two gluons (D-Type). These initial-final massive antennae depend furthermore on the mass m_χ of the decaying neutralino with momentum q . Following our definition of Q^2 , it is given by $m_\chi = \sqrt{-Q^2}$.

A) E-Type antennae.

Only one case of E-Type initial-final massive antenna functions is required: $E_3^0(4_q; 3_q, 1_Q)$ which has a massless initial state radiator 4_q , a massive final state radiator denoted by 1_Q and an unresolved parton which is the final state quark 3_q of the same flavour as the initial state quark 4_q . This antenna accounts for the massless initial state collinear behaviour and is given by,

$$E_3^0(4_q; 3_q, 1_Q) = -\frac{1}{(Q^2 + m_Q^2)^2} \left(-s_{14} + s_{13} - \frac{s_{13}^2}{s_{34}} - \frac{s_{14}^2}{s_{34}} - 2m_Q m_\chi \right) + O(\epsilon). \quad (3.13)$$

It is normalised to a gluon initiated process, namely $\tilde{\chi}g \rightarrow Q$. The corresponding two-particle matrix element squared omitting couplings reads,

$$E_2^0(2g; 1Q) = 4(1 - \epsilon) (Q^2 + m_Q^2)^2. \quad (3.14)$$

As a consequence of the crossing procedure, the antenna defined in eq. (3.13) has an overall minus sign made explicit in that expression.

B) D-Type antennae.

The massive final-final D-Type antenna denoted by $D_3^0(1Q, 3g, 4g)$ has only one massive final state particle, the quark denoted by $1Q$. The massive initial-final D-Type antenna is in principle obtained by crossing one of the two gluons in this function to the initial state. However, since the initial state gluon can split either into a quark-antiquark pair or into two gluons, two possible reduced matrix elements can serve as normalization for this antenna. A simple crossing of a gluon from the final-final quark-gluon antenna is not sufficient to define it unambiguously. This ambiguity requires the decomposition of the gluon-initiated D-type antenna function into sub-antennae according to the reduced matrix elements it factorises to in the different limits. According to the limit considered, those reduced matrix elements are related to the process $\tilde{\chi}+g \rightarrow Q$ or to the process $\tilde{\chi}+Q \rightarrow g$. These two different reduced matrix elements define the two different limiting behaviours in which the antenna needs to be decomposed. The decomposition can be achieved by separating the terms in the crossed function according to their contributions to a given limiting behaviour. For those terms which give contributions to more than one limiting behaviour, partial fractioning is applied. The antenna corresponding to a reduced matrix element initiated by a quark will be denoted by $D_{g;Qg}^0$ and the antenna corresponding to a reduced matrix element initiated by a gluon by $D_{g;gQ}^0$.

For $D_{g;Qg}^0$, the final state hard radiator is a gluon and the parton becoming unresolved is a massive quark Q which can become quasi-collinear to the initial state gluon. This antenna is given by,

$$\begin{aligned} D_3^0(4g; 1Q, 3g) = & -\frac{1}{(Q^2 + m_Q^2)^2} \\ & \times \left(-\frac{4s_{13}^2}{s_{14}} + \frac{2s_{13}^3}{(Q^2 + s_{13} + m_Q^2)s_{14}} + \frac{3s_{13}s_{34}}{s_{14}} - \frac{s_{34}^2}{s_{14}} \right. \\ & - m_Q^2 \left(\frac{2s_{13}^2}{s_{14}^2} - \frac{4s_{13}}{s_{14}} + \frac{6s_{34}}{s_{14}} - \frac{4s_{13}s_{34}}{s_{14}^2} + \frac{2s_{34}^2}{s_{14}^2} - \frac{2s_{34}^2}{s_{13}s_{14}} \right) \\ & \left. + 2m_Q^4 \frac{s_{34}}{s_{13}s_{14}} - 2m_Q^3 m_\chi \frac{s_{34}}{s_{13}s_{14}} \right) + O(\epsilon). \quad (3.15) \end{aligned}$$

This antenna is normalised to the matrix element associated to the process $\tilde{\chi}Q \rightarrow g$ with a massive quark in the initial state. Omitting couplings it reads,

$$D_2^0(1Q; 2g) = 4(1 - \epsilon) (Q^2 + m_Q^2)^2. \quad (3.16)$$

As this D-Type antenna is initiated by a gluon but normalised by a quark-initiated process, it has an overall minus sign made explicit in eq. (3.15).

For $D_{g,gQ}^0$, the final state hard radiator is a massive quark and the parton becoming unresolved is a gluon which can become soft, collinear to the initial state gluon or quasi-collinear to the final-state massive quark. This three-parton antenna reads,

$$\begin{aligned}
 D_3^0(4_g; 3_g, 1_Q) = & \frac{1}{(Q^2 + m_Q^2)^2} \\
 & \left(9s_{13} - 9s_{14} - 6s_{34} + \frac{4s_{14}^2}{s_{13}} - \frac{4s_{13}^2}{s_{34}} - \frac{4s_{14}^2}{s_{34}} + \frac{s_{34}^2}{s_{13}} \right. \\
 & + \frac{2s_{13}^3}{(Q^2 + s_{13} + m_Q^2)s_{34}} + \frac{2s_{14}^3}{s_{13}s_{34}} + \frac{6s_{13}s_{14}}{s_{34}} + \frac{3s_{14}s_{34}}{s_{13}} \\
 & - m_Q^2 \left(6 + \frac{2s_{14}^2}{s_{13}^2} - \frac{4s_{14}}{s_{13}} - \frac{6s_{34}}{s_{13}} + \frac{4s_{14}s_{34}}{s_{13}^2} + \frac{2s_{34}^2}{s_{13}^2} \right) \\
 & \left. + 2m_Q m_\chi \right) + O(\epsilon). \tag{3.17}
 \end{aligned}$$

This function is normalised to the matrix element associated to the gluon-initiated process $\tilde{\chi}g \rightarrow Q$ with the corresponding two-particle matrix element squared omitting couplings reading,

$$D_2^0(1_g; 2_g) = 4(1 - \epsilon) (Q^2 + m_Q^2)^2. \tag{3.18}$$

Being initiated and normalised by a gluon, this antenna has no overall minus sign.

3.3.3 Flavour violating antennae

Finally, as for the final-final case, in addition to the flavour conserving antennae given above, for the construction of the subtraction terms for $Q\bar{Q}$ +jets production at NLO, we also need two types of massive flavour violating initial-final antennae. Those are such that in both cases a massless quark is playing the role of the initial state radiator; the final state radiators can either be a massive quark or a massive antiquark. The expressions of these two antennae are related by the exchange of a massive quark versus a massive antiquark in the final state such that only one expression is given below. The unintegrated form of the antenna having a massive quark and a gluon in the final state given by $A_{q,gQ}^0$ is,

$$\begin{aligned}
 A_3^0(2_q; 3_g, 1_Q) = & \frac{1}{Q^2 - m_Q^2} \left[\frac{2s_{12}^2}{s_{13}s_{23}} + \frac{2s_{12}}{s_{13}} - \frac{2s_{12}}{s_{23}} + \frac{s_{13}}{s_{23}} + \frac{s_{23}}{s_{13}} \right. \\
 & \left. - m_Q^2 \left(\frac{2s_{12}}{s_{13}^2} + \frac{2s_{23}}{s_{13}^2} - \frac{2}{s_{13}} \right) \right] + \mathcal{O}(\epsilon). \tag{3.19}
 \end{aligned}$$

This antenna function is normalised with the following two-parton matrix element squared:

$$A_2^0(2_q; 1_Q) = 4(1 - \epsilon) [Q^2 - m_Q^2] \tag{3.20}$$

and has therefore no overall minus sign.

4 Singular limits of the massive antennae

The factorisation properties of tree-level QCD squared matrix elements for massless partons are well known [94]. An $(m + 1)$ -parton squared amplitude factorises into a product of a reduced m -parton squared matrix element and a soft eikonal factor in the soft limit, or an Altarelli-Parisi splitting function in the collinear case. If massive partons are involved the factorisation still takes place but the collinear and soft behaviour have to be generalised to take the mass effects into account. Those generalised limits will be described both in final-final and initial-final configurations below.

In the following, we shall here first recall all unresolved factors (massless and massive) in the final-final and initial-final configurations before tabulating the limits of all antenna functions encountered in section 3.

4.1 Single unresolved massless factors

When only massless partons are involved, when a gluon j emitted between two massless hard radiators i and k becomes soft, the squared matrix element factorises and the eikonal factor that factorises off the squared matrix element is

$$S_{ijk} = \frac{2s_{ik}}{s_{ij}s_{jk}}. \quad (4.1)$$

When two massless partons become collinear, the matrix element factorises yielding specific Altarelli-Parisi splitting functions corresponding to a particular parton-parton splitting. Those functions depend on z , the fractional momentum carried by the unresolved parton. Depending whether the unresolved parton is collinear to an initial or to a final state parton, the definition of z will be different. For two final state particles i and j of momenta p_i and p_j becoming collinear, we have, in the limit,

$$p_i \rightarrow zp_{ij}, \quad p_j \rightarrow (1-z)p_{ij}, \quad s_{ik} \rightarrow zs_{ijk}, \quad s_{jk} \rightarrow (1-z)s_{ijk}, \quad (4.2)$$

whereas for a final state particle j of momentum p_j becoming collinear with an initial state parton i of momentum p_i we have

$$p_j \rightarrow zp_i, \quad p_{ij} \rightarrow (1-z)p_i, \quad s_{ik} \rightarrow \frac{s_{ijk}}{1-z}, \quad s_{jk} \rightarrow \frac{zs_{ijk}}{1-z}. \quad (4.3)$$

The splitting functions denoted by $P_{ij \rightarrow (ij)}(z)$ corresponding to the collinear limit of two final state partons i and j are given in [95] by,

$$P_{qg \rightarrow Q}(z) = \frac{1 + (1-z)^2 - \epsilon z^2}{z} \quad (4.4)$$

$$P_{q\bar{q} \rightarrow G}(z) = \frac{z^2 + (1-z)^2 - \epsilon}{1-\epsilon} \quad (4.5)$$

$$P_{gg \rightarrow G}(z) = 2 \left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right]. \quad (4.6)$$

When the collinearity arises between an initial i and a final state parton j , the splitting functions denoted by $P_{ij \leftarrow (ij)}(z)$ are given in [95] by,

$$P_{gq \leftarrow Q}(z) = \frac{1 + z^2 - \epsilon(1-z)^2}{(1-\epsilon)(1-z)^2} = \frac{1}{1-z} \frac{1}{1-\epsilon} P_{qg \rightarrow Q}(1-z) \quad (4.7)$$

$$P_{qg \leftarrow Q}(z) = \frac{1 + (1 - z)^2 - \epsilon z^2}{z(1 - z)} = \frac{1}{1 - z} P_{qg \rightarrow Q}(z) \quad (4.8)$$

$$P_{q\bar{q} \leftarrow G}(z) = \frac{z^2 + (1 - z)^2 - \epsilon}{1 - z} = \frac{1 - \epsilon}{1 - z} P_{q\bar{q} \rightarrow G}(z) \quad (4.9)$$

$$P_{gg \leftarrow G}(z) = \frac{2(1 - z + z^2)^2}{z(1 - z)^2} = \frac{1}{1 - z} P_{gg \rightarrow G}(z). \quad (4.10)$$

The additional factors $(1 - \epsilon)$ and $1/(1 - \epsilon)$ account for the different number of polarizations of quark and gluons in the cases in which the particle entering the hard processes changes its type.

In all splitting functions defined above, the label q can stand for a massless quark or an antiquark since charge conjugation implies that $P_{qg \rightarrow Q} = P_{\bar{q}g \rightarrow \bar{Q}}$ and $P_{qg \leftarrow Q} = P_{\bar{q}g \leftarrow \bar{Q}}$. The labels Q and G in those denote the parent parton of the two collinear partons, which is massless.

4.2 Single unresolved massive factors

When the final state partons are massive, the emission of extra radiation from those can still lead to soft divergences, but not to collinear singularities since the mass of the final state parton regulates those. The relation between matrix element squared and the splitting functions needs to be extended from massless to massive. Similar factorisation formulae for matrix elements as in the massless case hold provided the collinear limit is generalized [83] to the *quasi-collinear* limit.

4.2.1 Quasi-collinear limit in final-final and initial-final configurations

In the final-final configuration, two final state massive partons can become quasi-collinear to each other resulting in a parent parton which is massive. The limit when a massive parton (ij) of momentum $p_{(ij)}$ and mass $m_{(ij)}$ decays quasi-collinearly into two massive partons i and j of masses m_i and m_j is defined by,

$$p_j^\mu \rightarrow z p_{(ij)}^\mu, \quad p_i^\mu \rightarrow (1 - z) p_{(ij)}^\mu \quad (4.11)$$

$$p_{(ij)}^2 = m_{(ij)}^2 \quad (4.12)$$

with the constraints

$$p_i \cdot p_j, m_i, m_j, m_{(ij)} \rightarrow 0, \quad (4.13)$$

at fixed ratios

$$\frac{m_i^2}{p_i \cdot p_j}, \frac{m_j^2}{p_i \cdot p_j}, \frac{m_{(ij)}^2}{p_i \cdot p_j}. \quad (4.14)$$

The difference obtained between taking the *quasi-collinear limit* between two final state particles or taking this limit when one initial and one final state particles are involved is closely related to the difference obtained in these two situations for the *massless collinear limit*. When a massive parton of momentum p_j becomes quasi-collinear to an initial state massless parton p_i we have,

$$p_j \rightarrow z p_i, \quad p_{(ij)} \rightarrow (1 - z) p_i, \quad (4.15)$$

with the constraints,

$$p_i \cdot p_j, m_j, m_{(ij)} \rightarrow 0, \tag{4.16}$$

at fixed ratios

$$\frac{m_j^2}{p_i \cdot p_j}, \frac{m_{(ij)}^2}{p_i \cdot p_j}. \tag{4.17}$$

Only the fractional momentum z carried by the unresolved parton needs to be defined accordingly in both final-final and initial-final situations. For the quasi-collinear limits, it is defined exactly as for the collinear limits. For the final-final case, z is defined as in eq. (4.2) whereas in the initial-final case it is defined by eq. (4.15).

The key difference between the massless collinear limit and the quasi-collinear limit is given by the constraint that the on-shell masses squared of the final state partons have to be kept of the same order as the invariant $s_{ij} = 2p_i \cdot p_j$, with the latter becoming small.

4.2.2 Factorisation in the quasi-collinear limits

In these quasi-collinear limits (final-final or initial-final), the $(m+1)$ -parton matrix element squared factorises into a reduced m -parton matrix element and unresolved massive factors. These single unresolved massive factors are generalizations of the massless unresolved factors defined above.

The generalized soft eikonal factor $S_{ijk}(m_i, m_k)$ for a massless gluon j emitted between two massive partons i and k depends on the invariants $s_{lm} = 2p_l \cdot p_m$ built with the partons i, j and k but also on the masses m_i and m_k of partons i and k . It is given by [83, 86]

$$S_{ijk}(m_i, m_k) = \frac{2s_{ik}}{s_{ij}s_{jk}} - \frac{2m_i^2}{s_{ij}^2} - \frac{2m_k^2}{s_{jk}^2}. \tag{4.18}$$

The massive splitting functions will depend on z , the fractional momentum carried by the unresolved parton j and on the masses m_i and m_j of the partons i and j becoming quasi-collinear. All the mass dependence can be parametrized by $\mu_{(ij)}$ given by,

$$\mu_{(ij)} = \frac{m_i^2 + m_j^2}{(p_i + p_j)^2 - m_{(ij)}^2}. \tag{4.19}$$

The massive splitting functions denoted by $P_{ij \rightarrow (ij)}(z, \mu_{ij}^2)$ for a massive parton (ij) which splits into partons i and j , both being in the final state, have been given in the appendix of [84] and in [86]. Those read,

$$P_{qg \rightarrow Q}(z, \mu_{qg}^2) = \frac{1 + (1-z)^2 - \epsilon z^2}{z} - 2\mu_{qg}^2 \tag{4.20}$$

$$P_{q\bar{q} \rightarrow G}(z, \mu_{q\bar{q}}^2) = \frac{z^2 + (1-z)^2 - \epsilon + \mu_{q\bar{q}}^2}{1 - \epsilon}, \tag{4.21}$$

where

$$\mu_{qg}^2 = \frac{m_Q^2}{s_{qg}} \quad \text{and} \quad \mu_{q\bar{q}}^2 = \frac{2m_Q^2}{s_{q\bar{q}}}. \tag{4.22}$$

Naturally, the gluon-gluon splitting function $P_{gg \rightarrow G}(z)$ is left unchanged. The splitting function $P_{q\bar{q} \rightarrow G}(z, \mu_{q\bar{q}}^2)$ given in eq. (4.21) and related to the quasi-collinear limit of two massive partons does not correspond to a limiting behaviour of the antenna functions required here and given in section 3. It is given here for completeness.

So far we have treated all generalisations of soft and collinear massless factors needed to treat final-final configurations involving massive partons. For the initial-final situations, since all the initial state partons are taken massless the only initial-final splitting function that changes when we allow massive partons in the final state is $P_{q\bar{q} \leftarrow G}(z, \mu_{q\bar{q}}^2)$ given by,

$$P_{q\bar{q} \leftarrow G}(z, \mu_{q\bar{q}}^2) = \frac{z^2 + (1-z)^2 - \epsilon}{1-z} + 2\mu_{q\bar{q}}^2. \quad (4.23)$$

The definition of the momentum fraction z present in this formula will be the same as in the massless case given by eq. (4.3).

4.3 Singular limits of the massive antenna functions

In this subsection we list all the non-vanishing soft, collinear and quasi-collinear limits of the massive final-final and initial-final antenna functions given in section 3. The limits of the massless antennae also needed to construct the subtraction terms for $Q\bar{Q}$ + jet production can be found in [64, 79, 82].

4.3.1 Final-final antenna functions

The limits of the massive flavour conserving quark-antiquark and quark-gluon antenna functions have been derived in [86]. We give them here for completeness. The limits of the flavour violating antennae are new.

The singular limits of the massive quark-antiquark antenna $A_{Qg\bar{Q}}^0$ are

$$\begin{aligned} A_3^0(1_Q, 3_g, 2_{\bar{Q}}) &\xrightarrow{3_g \rightarrow 0} S_{132}(m_Q, m_Q), \\ A_3^0(1_Q, 3_g, 2_{\bar{Q}}) &\xrightarrow{3_g \parallel 1_Q} \frac{1}{s_{13}} P_{qg \rightarrow Q}(z, \mu_{qg}^2), \\ A_3^0(1_Q, 3_g, 2_{\bar{Q}}) &\xrightarrow{3_g \parallel 2_{\bar{Q}}} \frac{1}{s_{23}} P_{qg \rightarrow Q}(z, \mu_{qg}^2). \end{aligned} \quad (4.24)$$

The only non-vanishing singular limit of the quark-gluon E-Type antenna $E_{Qq'\bar{q}'}^0$ is the collinear massless limit of the massless quark-antiquark pair,

$$E_3^0(1_Q, 3_{q'}, 4_{\bar{q}'}) \xrightarrow{3_{q'} \parallel 4_{\bar{q}'}} \frac{1}{s_{34}} P_{q\bar{q} \rightarrow G}(z), \quad (4.25)$$

while for the D-type D_{Qgg}^0 and d-type d_{Qgg}^0 antennae we have,

$$\begin{aligned} D_3^0(1_Q, 3_g, 4_g) &\xrightarrow{3_g \rightarrow 0} S_{134}(m_Q, 0) \\ D_3^0(1_Q, 3_g, 4_g) &\xrightarrow{4_g \rightarrow 0} S_{143}(m_Q, 0) \\ d_3^0(1_Q, i_g, j_g) &\xrightarrow{i_g \rightarrow 0} S_{1ij}(m_Q, 0) \end{aligned}$$

$$\begin{aligned}
d_3^0(1_Q, i_g, j_g) &\xrightarrow{j_g \rightarrow 0} 0 \\
D_3^0(1_Q, 3_g, 4_g) &\xrightarrow{1_Q \parallel 3_g} \frac{1}{s_{13}} P_{qg \rightarrow Q}(z, \mu_{qg}^2) \\
D_3^0(1_Q, 3_g, 4_g) &\xrightarrow{1_Q \parallel 4_g} \frac{1}{s_{14}} P_{qg \rightarrow Q}(z, \mu_{qg}^2) \\
d_3^0(1_Q, i_g, j_g) &\xrightarrow{1_Q \parallel i_g} \frac{1}{s_{1i}} P_{qg \rightarrow Q}(z, \mu_{qg}^2) \\
d_3^0(1_Q, i_g, j_g) &\xrightarrow{1_Q \parallel j_g} 0 \\
D_3^0(1_Q, 3_g, 4_g) &\xrightarrow{3_g \parallel 4_g} \frac{1}{s_{34}} P_{gg \rightarrow G}(z) \\
d_3^0(1_Q, 3_g, 4_g) &\xrightarrow{3_g \parallel 4_g} \frac{1}{s_{34}} \left(P_{gg \rightarrow G}(z) - \frac{2z}{1-z} - z(1-z) \right) \\
d_3^0(1_Q, 4_g, 3_g) &\xrightarrow{3_g \parallel 4_g} \frac{1}{s_{34}} \left(P_{gg \rightarrow G}(z) - \frac{2(1-z)}{z} - z(1-z) \right).
\end{aligned} \tag{4.26}$$

For the massive A-Type flavour violating antenna $A_{Qg\bar{q}}^0$ we have

$$A_3^0(1_Q, 3_g, 2_{\bar{q}}) \xrightarrow{3_g \rightarrow 0} S_{132}(m_Q, 0) \tag{4.27}$$

$$A_3^0(1_Q, 3_g, 2_{\bar{q}}) \xrightarrow{1_Q \parallel 3_g} \frac{1}{s_{13}} P_{qg \rightarrow Q}(z, \mu_{qg}^2) \tag{4.28}$$

$$A_3^0(1_Q, 3_g, 2_{\bar{q}}) \xrightarrow{2_{\bar{q}} \parallel 3_g} \frac{1}{s_{23}} P_{qg \rightarrow Q}(z). \tag{4.29}$$

Again, for symmetry reasons the role of Q and \bar{q} are interchangeable in these formulae.

4.3.2 Initial-final antenna functions

For the quark-antiquark initial-final massive antenna $A_{g,Q\bar{Q}}^0$, only quasi-collinear limits are present. Those are:

$$A_3^0(3_g; 1_Q, 2_{\bar{Q}}) \xrightarrow{1_Q \parallel 3_g} \frac{1}{s_{13}} P_{q\bar{q} \leftarrow G}(z, \mu_{qg}^2) \tag{4.30}$$

$$A_3^0(3_g; 1_Q, 2_{\bar{Q}}) \xrightarrow{2_{\bar{Q}} \parallel 3_g} \frac{1}{s_{23}} P_{q\bar{q} \leftarrow G}(z, \mu_{qg}^2) \tag{4.31}$$

For the E-Type antenna $E_{q;qQ}^0$, the only non-vanishing singular limit is,

$$E_3^0(4_q; 3_q, 1_Q) \xrightarrow{3_q \parallel 4_q} \frac{1}{s_{34}} P_{gq \leftarrow Q}(z). \tag{4.32}$$

The limits of the two quark-gluon D-Type antennae initiated by a gluon $D_{g,Qg}$ and $D_{g,gQ}$ are given by:

$$D_3^0(4_g; 1_Q, 3_g) \xrightarrow{1_Q \parallel 4_g} \frac{1}{s_{14}} P_{q\bar{q} \leftarrow G}(z, \mu_{Qg}^2), \tag{4.33}$$

which has only a quasi-collinear limit when the unresolved quark 1_Q is collinear to the initial-state gluon. The limits of $D_3^0(4_g; 3_g, 1_Q)$ with an unresolved gluon are

$$D_3^0(4_g; 3_g, 1_Q) \xrightarrow{1_Q \parallel 3_g} \frac{1}{s_{13}} P_{qg \rightarrow Q}(z, \mu_{qg}^2) \tag{4.34}$$

$$D_3^0(4_g; 3_g, 1_Q) \xrightarrow{3_g || 4_g} \frac{1}{s_{34}} P_{gg \leftarrow G}(z) \tag{4.35}$$

$$D_3^0(4_g; 3_g, 1_Q) \xrightarrow{3_g \rightarrow 0} S_{134}(m_Q, 0). \tag{4.36}$$

Finally, the initial-final flavour violating antenna $A_{q,gQ}^0$ has its limits given by,

$$A_3^0(2_q; 3_g, 1_Q) \xrightarrow{1_Q || 3_g} \frac{1}{s_{13}} P_{qg \rightarrow Q}(z, \mu_{qg}^2) \tag{4.37}$$

$$A_3^0(2_q; 3_g, 1_Q) \xrightarrow{2_q || 3_g} \frac{1}{s_{23}} P_{qg \leftarrow Q}(z) \tag{4.38}$$

$$A_3^0(2_q; 3_g, 1_Q) \xrightarrow{3_g \rightarrow 0} S_{132}(m_Q, 0). \tag{4.39}$$

5 Integrated massive antenna functions

To combine the antenna subtraction terms with the virtual corrections and the mass factorization counterterms in a given kinematical configuration, the antenna functions have to be integrated over the appropriate factorised antenna phase space. After integration, the implicit soft and collinear singularities present in the antenna functions turn into explicit poles in the dimensional regularization parameter ϵ , and the remaining phase space corresponds to the same n -particle kinematics as the virtual contributions or the mass factorisation counterterms. In this section, we derive the integrated forms of the massive antenna functions defined in section 3. Only full antennae, denoted by capital letters X_{ijk} (or crossing of those), need to be integrated, while partial antennae denoted x_{ijk} sum up to X_{ijk} prior to integration.

5.1 Properties of the integrated massive antennae

The results will be presented in two forms, in expanded and unexpanded forms in the dimensional regularization parameter ϵ . The unexpanded forms of the integrated antennae are functions of a few master integrals obtained after standard reduction techniques [96–100] have been applied. The masters will be given here analytically to all orders in ϵ . Additionally, the integrated antennae will be presented after an ϵ -expansion has been performed on these all-order results up to finite order in ϵ . In this expanded form, the poles in ϵ become explicit and can be related to process independent infrared singularity operators and splitting kernels.

Final-final integrated antennae have their pole part entirely related to colour ordered massive infrared singularity operators $\mathbf{I}_{ij}^{(1)}$, which will be defined below. For initial-final antennae, those infrared operators are not sufficient to capture all singularities present in the integrated antennae. Additional pole terms can arise due to the presence of massless initial-final parton-parton collinear singularities which cancel against the mass factorisation counterterms. These pole terms are proportional to universal and process-independent splitting kernels $p_{ij}^{(0)}(x)$. Those are defined for example in [79] and will be given below.

As a check on our results for the expanded forms of the integrated antennae we consider the following: At NLO, a particular antenna can be regarded as the sum of two

particular dipoles in the dipole formalism of [51, 83]. The two radiators present in a given antenna play then both the role of emitter and spectator in the corresponding two dipoles. Up to terms which do not give rise to singularities when integrated over the phase space, the sum of these two dipoles and the antenna are the same (up to coupling factors). The pole parts of a given integrated antenna and those of the corresponding integrated dipoles can therefore be related. For each of the integrated antenna we will specify how this comparison is performed.

5.1.1 Infrared singularity structure

Owing to the universal factorisation properties of QCD amplitudes in infrared singular limits, it is possible to describe the infrared pole structure of virtual loop corrections and, consequently, also of integrated subtraction terms, by the product of infrared singularity operators with tree level amplitudes. These infrared singularity operators are a priori tensors in colour space. In a colour ordered framework, they decompose into different colour-ordered infrared singularity operators. In massless QCD, there is solid evidence to assume that the infrared singularity operators consist only of combinations of two-particle correlations [101–104] at all orders in perturbation theory. Explicit forms of the massless operators are known to three-loop order [105]. In massive QCD, only the one-loop infrared singularity operator is made up entirely of two-parton correlations [84], while multi-particle correlations can contribute at higher loop order [106, 107]. The explicit form of the massive infrared singularity operators is known to two loops [106, 107] and was used to predict the pole structure of the two-loop matrix elements for $q\bar{q} \rightarrow t\bar{t}$ and $gg \rightarrow t\bar{t}$.

We are concerned only with the infrared singularity structure at NLO in the present study. Consequently, the integrated massive antenna functions will contain a pole structure in terms of the massive one-loop infrared singularity operators. Containing only two-parton correlations irrespective of the particle masses involved, these can be expressed straightforwardly in a colour-ordered form. We introduce the mass-dependent colour-ordered NLO real radiation singularity operator

$$\mathbf{I}_{ij}^{(1)}(\epsilon, s_{ij}, m_i, m_j, \lambda\mu^2), \tag{5.1}$$

which describes the unresolved real radiation between partons i and j . It is a function of the invariant mass of the parton pair, of the masses of the partons and of a product of kinematical parameters $\lambda\mu^2$, which determines the logarithmic pole coefficient. In this form, μ^2 is dependent only on the mass combination of the radiator partons, while λ takes account of the nature of the kinematically allowed endpoint in the different kinematical configurations. $\lambda = 1$ for all final-final antenna functions and for initial-final antenna functions with equal masses, while $\lambda = x_0^2$ with

$$x_0 = \frac{Q^2}{Q^2 + m_Q^2} \tag{5.2}$$

for initial-final antenna functions with one massless and one massive radiator.

In their most general form those infrared operators have their functional dependence given by,

$$\mathbf{I}_{Q\bar{Q}}^{(1)}(\epsilon, s_{Qg}, m_Q, m_Q, \mu^2), \quad \mathbf{I}_{Qg}^{(1)}(\epsilon, s_{Qg}, m_Q, m_Q, \lambda\mu^2) \quad \text{and} \quad \mathbf{I}_{Qg,F}^{(1)}(\epsilon, s_{ij}, m_Q, 0, \mu^2).$$

The third operator describes contributions arising from the splitting of a gluon into a quark-antiquark pair which are proportional to the number of light quark flavours N_F .

The massive infrared operators which have a non-trivial mass dependence, required in both final-final and initial-final configurations are,

$$\begin{aligned} \mathbf{I}_{Q\bar{Q}}^{(1)}\left(\epsilon, s_{Q\bar{Q}}, m_Q, m_Q, \lambda\frac{1-\sqrt{r}}{1+\sqrt{r}}\right) &= \\ &= -\frac{e^{\epsilon\gamma_E}}{2\Gamma(1-\epsilon)} \left[\frac{s_{Q\bar{Q}}+2m_Q^2}{s_{Q\bar{Q}}^2}\right]^\epsilon \left\{\frac{1}{\epsilon}\left(\frac{(1+r)}{2\sqrt{r}}\right)\ln\left(\frac{1-\sqrt{r}}{1+\sqrt{r}}\right)+1\right\} \\ \mathbf{I}_{Qg}^{(1)}\left(\epsilon, s_{Qg}, m_Q, 0, \lambda\frac{m_Q^2}{s_{Qg}+m_Q^2}\right) &= \\ &= -\frac{e^{\epsilon\gamma_E}}{2\Gamma(1-\epsilon)} \left[\frac{s_{Qg}+m_Q^2}{s_{Qg}^2}\right]^\epsilon \left\{\frac{1}{2\epsilon^2}+\frac{1}{2\epsilon}\left(\frac{17}{6}\right)+\frac{1}{2\epsilon}\ln\left(\frac{\lambda m_Q^2}{s_{Qg}+m_Q^2}\right)\right\} \\ \mathbf{I}_{Qg,F}^{(1)}\left(\epsilon, s_{Qg}, m_Q, 0, \frac{m_Q^2}{s_{Qg}+m_Q^2}\right) &= \frac{e^{\epsilon\gamma_E}}{2\Gamma(1-\epsilon)} \left[\frac{s_{Qg}+m_Q^2}{s_{Qg}^2}\right]^\epsilon \left(\frac{1}{6\epsilon}\right), \end{aligned} \tag{5.3}$$

with

$$r = 1 - \frac{4m_Q^2}{s_{Q\bar{Q}}+2m_Q^2}. \tag{5.4}$$

The antiquark-gluon operators are obtained by charge conjugation:

$$\begin{aligned} \mathbf{I}_{g\bar{Q}}^{(1)}(\epsilon, s_{g\bar{Q}}, m_Q, 0, \lambda\mu^2) &= \mathbf{I}_{Qg}^{(1)}(\epsilon, s_{gQ}, m_Q, 0, \lambda\mu^2), \\ \mathbf{I}_{g\bar{Q},F}^{(1)}(\epsilon, s_{g\bar{Q}}, m_Q, 0, \mu^2) &= \mathbf{I}_{gQ,F}^{(1)}(\epsilon, s_{gQ}, m_Q, 0, \mu^2). \end{aligned}$$

In addition to these infrared operators which have their massless counterparts defined in [64], we also have an infrared operator associated with the flavour-violating antennae $A_{Qg\bar{q}}$ and A_{Qgq} defined in section 3. As those antennae have the same unintegrated form both of them are related to one infrared operator denoted by $\mathbf{I}_{q\bar{Q}}^{(1)}$, where \bar{Q} can stand for a massive quark or antiquark. As the quark-gluon operator $\mathbf{I}_{Qg}^{(1)}$, this operator has a mass-dependent logarithmic term proportional to $\lambda\mu^2$ with $\lambda = x_0^2$ in the initial-final configuration. It is given by,

$$\begin{aligned} \mathbf{I}_{q\bar{Q}}^{(1)}\left(\epsilon, s_{q\bar{Q}}, m_{\bar{Q}}, 0, \lambda\frac{m_{\bar{Q}}^2}{s_{q\bar{Q}}+m_{\bar{Q}}^2}\right) &= \\ &= -\frac{e^{\epsilon\gamma_E}}{2\Gamma(1-\epsilon)} \left[\frac{s_{q\bar{Q}}+m_{\bar{Q}}^2}{s_{q\bar{Q}}^2}\right]^\epsilon \left\{\frac{1}{2\epsilon^2}+\frac{1}{2\epsilon}\left(\frac{5}{2}\right)+\frac{1}{2\epsilon}\ln\left(\frac{\lambda m_{\bar{Q}}^2}{s_{q\bar{Q}}+m_{\bar{Q}}^2}\right)\right\}. \end{aligned} \tag{5.5}$$

5.2 Integrated massive final-final antennae

The integrated massive final-final flavour conserving antennae have been derived as function of a few master integrals in [86]. In this section, all final-final integrated antennae will be presented in expanded form such that their pole part can be related to the colour-ordered massive singularity operators $\mathbf{I}_{ij}^{(1)}$ defined above. Before we shall derive the integrated massive flavour-violating antenna and give its expanded and unexpanded forms.

In general, the integrated final-final antennae denoted by \mathcal{X}_{ijk} are given as the integration over the final-final antenna phase space $d\Phi_{X_{ijk}}$ of the unintegrated final-final antennae X_{ijk} as given in eq. (2.7). Those will depend on the masses of the final state particles and on E_{cm}^2 .

We start by giving the NLO massive final-final antenna phase space $d\Phi_{X_{ijk}}$ necessary to evaluate these integrated final-final antennae. In the most general case, where the involved partons i, j, k have three different masses m_i, m_j and m_k , the massive antenna phase space $d\Phi_{X_{ijk}}^{(m_i, m_j, m_k)}$ is given by [85, 86],

$$\begin{aligned} \int d\Phi_{X_{ijk}}^{(m_i, m_j, m_k)}(s_{ij}, s_{jk}, s_{ik}) = & \\ & (2\pi)^{1-d} \frac{2\pi^{d/2-1}}{\Gamma(\frac{d}{2}-1)} \frac{1}{4} \left((E_{\text{cm}}^2 - m_I^2 - m_K^2)^2 - 4m_I^2 m_K^2 \right)^{\frac{3-d}{2}} \\ & \int ds_{ij} ds_{jk} ds_{ik} \delta(E_{\text{cm}}^2 - m_i^2 - m_j^2 - m_k^2 - s_{ij} - s_{jk} - s_{ik}) \\ & [4 \Delta_3(p_i, p_j, p_k)]^{\frac{d-4}{2}} \theta(\Delta_3(p_i, p_j, p_k)). \end{aligned} \quad (5.6)$$

The masses m_I and m_K appearing in this equation are combinations of the masses m_i, m_j and m_k . The function $\Delta_3(p_i, p_j, p_k)$ is the Gram determinant for massive particles of momenta p_i, p_j, p_k given in terms of invariants $s_{ij} = 2p_i \cdot p_j$ and masses m_i, m_j, m_k by,

$$\Delta_3(p_i, p_j, p_k) = \frac{1}{4} (s_{ij}s_{ik}s_{jk} - m_i^2 s_{jk}^2 - m_k^2 s_{ij}^2 - m_j^2 s_{ik}^2 + 4m_i^2 m_j^2 m_k^2). \quad (5.7)$$

To be able to evaluate the integrated flavour-violating massive final-final antennae we need to consider the case where only one particle in the final state is massive while the other two are taken massless. We consider, $m_i = m_j = 0$ and $m_k = m_Q$, in which case the masses of the remapped momenta p_I and p_K have their masses given by $m_I = m_i = 0$, $m_K = m_k$. We use the following parametrisation of the final-final massive antenna phase space with one massive parton $d\Phi_{X_{ijk}}^{(0,0,m)}$ given by,

$$\begin{aligned} d\Phi_{X_{ijk}}^{(0,0,m)} = & \frac{(4\pi)^{\epsilon-2}}{\Gamma(1-\epsilon)} (E_{\text{cm}}^2)^{1-\epsilon} (u_0)^{2-2\epsilon} \\ & \int_0^1 du u^{1-2\epsilon} (1-u)^{1-2\epsilon} (1-u_0 u)^{-1+\epsilon} \int_0^1 dv v^{-\epsilon} (1-v)^{-\epsilon}, \end{aligned} \quad (5.8)$$

with

$$u_0 = 1 - \frac{m_Q^2}{E_{\text{cm}}^2},$$

$$\begin{aligned}
 s_{ij} &= E_{\text{cm}}^2 u_0^2 \frac{u(1-u)v}{(1-u_0u)}, \\
 s_{ik} &= E_{\text{cm}}^2 u_0(1-u).
 \end{aligned}
 \tag{5.9}$$

Integrating the flavour violating antenna $A_3^0(1_Q, 3_g, 2_{\bar{q}})$ given in eq. (3.9) over this phase space $d\Phi_{X_{ijk}}^{(0,0,m)}$ we obtain,

$$\begin{aligned}
 \mathcal{A}_3^0(1_Q, 3_g, 2_{\bar{q}}) &= \frac{8\pi^2(4\pi)^{-\epsilon} e^{\epsilon\gamma_E}}{2E_{\text{cm}}^4 u_0^2(1-u_0)\epsilon^2(1-2\epsilon)} \\
 &\times \left\{ E_{\text{cm}}^2 [12(1-u_0)^2 - 2\epsilon(29-54u_0+24u_0^2) + \epsilon^2(98-176u_0+73u_0^2)] \right. \\
 &\quad \left. + \epsilon^3(-68+120u_0-47u_0^2) + 2\epsilon^4(8-12u_0+3u_0^2) \right] I_1^{(0,0,m)} \\
 &\quad \left. + 3(1-\epsilon) [4(1-u_0) + 2\epsilon(-7+6u_0) + \epsilon^2(14-11u_0) + 2\epsilon^3(-2+u_0)] I_2^{(0,0,m)} \right\}.
 \end{aligned}
 \tag{5.10}$$

In this expression, the master integrals required are $I_1^{(0,0,m)}$ and $I_2^{(0,0,m)}$ were derived in [86]. The master integral $I_1^{(0,0,m)}$ corresponding to the integrated $1 \rightarrow 3$ phase space measure with one massive final state is given by,

$$\begin{aligned}
 I_1^{(0,0,m)} &= \int d\Phi_{X_{ijk}}^{(0,0,m)} \\
 &= (E_{\text{cm}}^2)^{1-\epsilon} u_0^{2-2\epsilon} 2^{-4+2\epsilon} \pi^{-2+\epsilon} \frac{\Gamma(2-2\epsilon)\Gamma(1-\epsilon)}{\Gamma(4-4\epsilon)} {}_2F_1(1-\epsilon, 2-2\epsilon, 4-4\epsilon; u_0),
 \end{aligned}
 \tag{5.11}$$

while $I_2^{(0,0,m)}$ is given by,

$$\begin{aligned}
 I_2^{(0,0,m)} &= \int d\Phi_{X_{ijk}}^{(0,0,m)}(s_{ik}) \\
 &= (E_{\text{cm}}^2)^{2-\epsilon} u_0^{3-2\epsilon} 2^{-5+2\epsilon} \pi^{-2+\epsilon} \frac{\Gamma(2-2\epsilon)\Gamma(1-\epsilon)}{\Gamma(4-4\epsilon)} {}_2F_1(1-\epsilon, 2-2\epsilon, 5-4\epsilon; u_0),
 \end{aligned}
 \tag{5.12}$$

Expanding the integrated flavour-violating A-Type antenna $\mathcal{A}_{Qg\bar{q}}^0$ given in eq. (5.10) in powers of ϵ up to finite order, we obtain

$$\begin{aligned}
 \mathcal{A}_{Qg\bar{q}}^0(1_Q, 3_g, 2_{\bar{q}}) &= -2\mathbf{I}_{\bar{Q}\bar{q}}^{(1)} \left(\epsilon, s_{Qg\bar{q}}, m_Q, 0, \frac{m_Q^2}{m_Q^2 + s_{Qg\bar{q}}} \right) - \left[\frac{1}{1-\mu^2} - \frac{19}{4} + \frac{5\pi^2}{12} \right. \\
 &\quad \left. + \left(\frac{1}{(1-\mu^2)^2} - \frac{5}{4} \right) \ln(\mu^2) + \frac{1}{4} \ln^2(\mu^2) + \text{Li}_2(1-\mu^2) \right] + \mathcal{O}(\epsilon),
 \end{aligned}
 \tag{5.13}$$

where

$$\mu^2 = \frac{m_Q^2}{E_{\text{cm}}^2}.
 \tag{5.14}$$

All other integrated flavour conserving final-final antennae given in terms of masters in [86] can be written in terms of the real infrared singularity operators $\mathbf{I}_{ij}^{(1)}$ defined above up to finite order as follows,

$$\mathcal{A}_3^0(1_Q, 3_g, 2_{\bar{Q}}) = -2\mathbf{I}_{Q\bar{Q}}^{(1)} \left(\epsilon, s_{Qg\bar{Q}}, m_Q, m_Q, \frac{1-\sqrt{r}}{1+\sqrt{r}} \right)$$

$$\begin{aligned}
 & + \frac{1}{8\sqrt{r}} \left\{ \frac{1}{r-3} \left[-40 \ln(2)(r-3)(1+r) \tanh^{-1}(\sqrt{r}) \right. \right. \\
 & \quad + 2\sqrt{r}(-39 + 17r - 16 \ln(2)(r-3)) \\
 & \quad + (-33 + 10r - r^2 + 16 \ln(2)(r-3)(1+r)) \ln \left(\frac{1+\sqrt{r}}{1-\sqrt{r}} \right) \\
 & \quad - 16(r-3) \tanh^{-1}(1-2r) \left(-2\sqrt{r} + (1+r) \ln \left(\frac{1+\sqrt{r}}{1-\sqrt{r}} \right) \right) \\
 & \quad \left. \left. + 10(r-3)(1+r) (\ln^2(1+\sqrt{r}) - \ln^2(1-\sqrt{r})) \right] \right\} \\
 & + 4(1+r) \left[5\text{Li}_2 \left(\frac{1+\sqrt{r}}{2} \right) - 5\text{Li}_2 \left(\frac{1-\sqrt{r}}{2} \right) + 3\text{Li}_2(r) - 12\text{Li}_2(\sqrt{r}) \right. \\
 & \quad \left. + 2 \ln \left(\frac{1+r}{2} \right) \left(1 + \ln \left(\frac{1-\sqrt{r}}{1+\sqrt{r}} \right) \right) \right] \Big\} + \mathcal{O}(\epsilon), \tag{5.15}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{E}_3^0(1_Q, 3_{q'}, 4_{\bar{q}'}) & = -4\mathbf{I}_{Qg,F}^{(1)} \left(\epsilon, s_{Qq'\bar{q}'}, m_Q, 0, \frac{m_Q^2}{m_Q^2 + s_{Qq'\bar{q}'}} \right) \\
 & - \frac{1}{6(1-\mu^2)^3} \left[6 + 3\mu - 14\mu^2 + 14\mu^4 - 3\mu^5 - 14\mu^2 + 14\mu^4 - 3\mu^5 - 6\mu^6 \right. \\
 & \quad \left. - 2\mu^3(-3 - 3\mu + \mu^3) \ln(\mu^2) \right] + \mathcal{O}(\epsilon), \tag{5.16}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{D}_3^0(1_Q, 3_g, 4_g) & = -4\mathbf{I}_{Qg}^{(1)} \left(\epsilon, s_{Qgg}, m_Q, 0, \frac{m_Q^2}{m_Q^2 + s_{Qgg}} \right) + \frac{1}{12(1-\mu^2)^3} \\
 & \times \left[-2(1-\mu^2)(4(-14 + \pi^2) - 3\mu + \mu^2(126 - 8\pi^2) + 9\mu^3 + \mu^4(-74 + 4\pi^2)) \right. \\
 & \quad - 2(-3 + 33\mu^2 + 6\mu^3 - 51\mu^4 + 17\mu^6) \ln(\mu^2) + 3(1-\mu)^2(-2 - 4\mu - \mu^2 + \mu^3) \ln^2(\mu^2) \\
 & \quad \left. + 12(1-\mu)^2(-2 - 4\mu - \mu^2 + \mu^3) \text{Li}_2(1-\mu^2) \right] + \mathcal{O}(\epsilon), \tag{5.17}
 \end{aligned}$$

with r and μ defined as in eq. (5.4) and eq. (5.14) respectively.

Let us notice that the last term present in the expansion of $\mathcal{A}_{Qg\bar{Q}}$ arises through the expansion at finite order of

$$\frac{1}{\epsilon} \left(\frac{(1+r)}{2\sqrt{r}} \right) \ln \left(\frac{1-\sqrt{r}}{1+\sqrt{r}} \right) \left[1 - \frac{2m_Q^2}{E_{\text{cm}}^2} \right]^{+2\epsilon}$$

with $E_{\text{cm}}^2 = s_{Qg\bar{Q}} + 2m_Q^2$. This term arises since we have chosen to factor

$$\left(\frac{s_{Q\bar{Q}} + 2m_Q^2}{s_{Q\bar{Q}}^2} \right)^{+\epsilon}$$

in $\mathbf{I}_{Q\bar{Q}}^{(1)}$ as unexpanded overall factor whereas the integrated antenna $\mathcal{A}_{Qg\bar{Q}}$ is naturally proportional to $\left[s_{Qg\bar{Q}} + 2m_Q^2 \right]^{-\epsilon}$. We have chosen to define the $\mathbf{I}_{Q\bar{Q}}^{(1)}$ operator such, in order to have similar overall factors in all $\mathbf{I}_{ij}^{(1)}$ -type operators.

From these results on the expanded integrated final-final antennae, we see that the pole parts of all of those can be captured solely with poles present in the massive colour-ordered $\mathbf{I}_{ij}^{(1)}$ operators defined above.

Finally, for all integrated final-final antennae, we have compared the pure pole parts of those with the pure pole parts of the corresponding sums of the integrated dipoles presented in [85]. Setting couplings and colour factors to one we found full agreement.

More explicitly, the poles of $\mathcal{A}_{Qg\bar{Q}}$ can be compared with the poles of two dipoles having both two massive final states, a spectator Q and an emitter \bar{Q} . The poles of $\mathcal{E}_{Qq'\bar{q}'}$ can be compared with those of two dipoles with a massless final state emitter q and a massive final-state spectator Q while the poles of \mathcal{D}_{Qgg} have to be compared with the sum of two dipoles where the massive quark and the gluon in the final state can both be emitter or spectator.

5.3 Integrated massive initial-final antennae

In order to obtain the integrated massive initial-final antennae $\mathcal{X}_{i,jk}$, the massive unintegrated initial-final antennae $X_{i,jk}$ defined in section 3 need to be integrated over the initial-final massive antenna phase space $d\Phi_{X_{i,jk}}$ as given in eq. (2.16). As a result, the integrated antennae will all depend on $Q^2 = -q^2$, on x , the momentum fraction carried by the initial state parton p_i , and on the masses of the final states present in a given antenna. As we saw in section 2, the momentum fraction x depends on the number of massive particles present in the final state in a given antenna. For two massive final states j and k of masses m_j and m_k x is given by,

$$x = \frac{Q^2 + m_j^2 + m_k^2}{2p_i \cdot q}.$$

Generally, the pole parts of the integrated massive initial-final antennae are related to the massive $\mathbf{I}_{ij}^{(1)}$ operators defined above and to the x -dependent colour ordered splitting kernels $p_{ij}^{(0)}(x)$. The splitting kernels describe the initial-final massless collinear singularities and are given by [79],

$$\begin{aligned} p_{qq}^{(0)}(x) &= \frac{3}{2} \delta(1-x) + 2\mathcal{D}_0(x) - 1 - x, \\ p_{qg}^{(0)}(x) &= 1 - 2x + 2x^2, \\ p_{gq}^{(0)}(x) &= \frac{2}{x} - 2 + x, \\ p_{gg}^{(0)}(x) &= \frac{11}{6} \delta(1-x) + 2\mathcal{D}_0(x) + \frac{2}{x} - 4 + 2x - 2x^2, \\ p_{gg,F}^{(0)}(x) &= -\frac{1}{3} \delta(1-x), \end{aligned} \tag{5.18}$$

with the distributions

$$\mathcal{D}_n(x) = \left(\frac{\ln^n(1-x)}{1-x} \right)_+.$$

In the following, we shall specify the phase space parametrisations for the initial-final antenna phase space $d\Phi_{X_{i,jk}}$ needed to integrate the different antenna types before giving the results for the integrated initial-final massive antennae in unexpanded and expanded forms.

5.3.1 Phase space parametrisations for initial-final configurations

In the most general case, the massive two-to-two phase space $d\Phi_2$ to which the initial-final antenna phase space $d\Phi_{X_{i,jk}}$ defined in eq. (2.16) is proportional to, is related to the process $q + p_i \rightarrow p_j + p_k$ with p_j and p_k the momenta of the final state partons with masses m_j and m_k . It can be written as

$$d\phi_2(m_j, m_k) = \frac{(4\pi)^\epsilon}{8\pi\Gamma(1-\epsilon)} E_{\text{cm}}^{2\epsilon-2} \left[(E_{\text{cm}}^2 - m_j^2 - m_k^2)^2 - 4m_j^2 m_k^2 \right]^{\frac{1-2\epsilon}{2}} dy y^{-\epsilon} (1-y)^{-\epsilon}, \quad (5.19)$$

where y runs from 0 to 1. In this parametrisation the invariants take the following form

$$2p_i \cdot p_j = \frac{Q^2 + m_j^2 + m_k^2}{2xE_{\text{cm}}^2} \times \left(E_{\text{cm}}^2 + m_j^2 - m_k^2 - (2y-1) \sqrt{(E_{\text{cm}}^2 - m_j^2 - m_k^2)^2 - 4m_j^2 m_k^2} \right) \quad (5.20)$$

$$2p_i \cdot p_k = \frac{Q^2 + m_j^2 + m_k^2}{2xE_{\text{cm}}^2} \times \left(E_{\text{cm}}^2 - m_j^2 + m_k^2 + (2y-1) \sqrt{(E_{\text{cm}}^2 - m_j^2 - m_k^2)^2 - 4m_j^2 m_k^2} \right), \quad (5.21)$$

and the partonic center of mass energy reads

$$E_{\text{cm}} = \sqrt{(p+q)^2} = \sqrt{\frac{Q^2(1-x) + m_j^2 + m_k^2}{x}}. \quad (5.22)$$

In this case also x is defined as,

$$x = \frac{Q^2 + m_j^2 + m_k^2}{2p_i \cdot q}.$$

In our application to $t\bar{t}$ +jet production, we are only interested in the cases where $m_j = m_k = m_Q$ or in the case where $m_j \neq 0$ and $m_k = 0$. For the first case, the parametrisation given above simplifies to

$$d\phi_2(m_Q, m_Q) = \frac{(4\pi)^{\epsilon-1}}{2\Gamma(1-\epsilon)} E_{\text{cm}}^{2\epsilon-2} \left[(E_{\text{cm}}^2 - 2m_Q^2)^2 - 4m_Q^4 \right]^{\frac{1-2\epsilon}{2}} dy y^{-\epsilon} (1-y)^{-\epsilon}, \quad (5.23)$$

while the invariants read

$$2p_i \cdot p_j = \frac{Q^2 + 2m_Q^2}{2xE_{\text{cm}}^2} \left(E_{\text{cm}}^2 - (2y-1) \sqrt{(E_{\text{cm}}^2 - 2m_Q^2)^2 - 4m_Q^4} \right) \quad (5.24)$$

$$2p_i \cdot p_k = \frac{Q^2 + 2m_Q^2}{2xE_{\text{cm}}^2} \left(E_{\text{cm}}^2 + (2y-1) \sqrt{(E_{\text{cm}}^2 - 2m_Q^2)^2 - 4m_Q^4} \right), \quad (5.25)$$

and the center of mass energy is

$$E_{\text{cm}} = \sqrt{\frac{Q^2(1-x) + 2m_Q^2}{x}} \quad \text{with} \quad x = \frac{Q^2 + 2m_Q^2}{2p_i \cdot q}. \quad (5.26)$$

In the case in which $m_k = 0$ and $m_j = m_Q$ the phase space reduces to

$$d\phi_2(m_Q, 0) = \frac{(4\pi)^{\epsilon-1}}{2\Gamma(1-\epsilon)} E_{\text{cm}}^{2\epsilon-2} (E_{\text{cm}}^2 - m_Q^2)^{1-2\epsilon} dy y^{-\epsilon} (1-y)^{-\epsilon}, \quad (5.27)$$

the invariants are

$$2p_i \cdot p_j = \frac{Q^2 + m_Q^2}{xE_{\text{cm}}^2} [E_{\text{cm}}^2 - y(E_{\text{cm}}^2 - m_Q^2)] \quad (5.28)$$

$$2p_i \cdot p_k = \frac{Q^2 + m_Q^2}{xE_{\text{cm}}^2} [y(E_{\text{cm}}^2 - m_Q^2)] \quad (5.29)$$

and the center of mass energy is

$$E_{\text{cm}} = \sqrt{\frac{Q^2(1-x) + m_Q^2}{x}} \quad \text{with} \quad x = \frac{Q^2 + m_Q^2}{2p_i \cdot q}. \quad (5.30)$$

5.3.2 Integrated forms of the initial-final massive antennae

The integrated forms of the antennae can be obtained using reduction techniques, using the extension of the integration by parts method [96, 97] in [98, 99] to reduce the phase space integrals to master integrals. In this task, we express all the invariants in the antenna functions as massive propagators and write the three on-shell conditions $p_a^2 = m_a^2$ ($a = j, k$) as cut propagators. Since the invariants s_{ab} appearing in the antennae are not all independent from each other, the integrated antennae are written as a one-loop diagrams with on-shell conditions and not as two-loop diagrams as in the final-final case. The reduction to master integrals is therefore easier. It was done using the Laporta algorithm [100] with two independent implementations: the Mathematica package FIRE [108], and an in-house implementation in FORM [109]. For all integrated initial-final antennae required and defined in section 3, we find four master integrals. Those can be evaluated analytically in terms of gamma functions and hypergeometric functions. The all order expressions as well as the expanded expressions of all initial-final massive antenna functions will be presented below.

A) Quark-Antiquark antennae.

The integrated form of the massive three-parton initial-final quark antiquark antenna given by $\mathcal{A}_{g,Q\bar{Q}}$ is obtained by integrating its unintegrated form $A_{g,Q,\bar{Q}}$ defined in eq. (3.11) over the initial-final massive antenna phase $d\Phi_{X_{i,jk}}$ given in eq. (2.16) using the parametrisation of the two-to-two parton phase space $d\Phi_2(m_Q, m_Q)$ given in eq. (5.23). It depends on the virtuality of the incoming boson Q^2 , the masses of the final states both being m_Q and on x given by $x = \frac{Q^2 + 2m_Q^2}{2p_i \cdot q}$. It reads,

$$\begin{aligned} \mathcal{A}_3^0(3g; 1Q, 2\bar{Q}) &= \frac{2(4\pi)^{1-\epsilon} e^{\epsilon\gamma_E}}{Q^2 + 2m_Q^2} \\ &\times \left\{ \left[-1 - \frac{2m_Q^2(2-\epsilon)}{Q^2(1-\epsilon) - 2m_Q^2} + 2x(1-2\epsilon) \left(1 - \frac{Q^2 x}{Q^2 + 2m_Q^2} \right) \right] I_2(m_Q, m_Q) \right\} \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{x} \left[\frac{4(2-\epsilon)^2 m_Q^4}{Q^2(1-\epsilon) - 2m_Q^2} + 2m_Q^2(2-\epsilon + 2x^2\epsilon) \right. \\
 & \left. + (1-2x+2x^2)(1-\epsilon)(Q^2 + 2m_Q^2) \right] I_3(m_Q, m_Q, s_{13}) \Big\}. \tag{5.31}
 \end{aligned}$$

$I_2(m_Q, m_Q)$ is the integrated phase space measure with two massive final states. It is given by

$$I_2(m_Q, m_Q) = (4\pi)^{\epsilon-1} \frac{\Gamma(1-\epsilon)}{2\Gamma(2-2\epsilon)} \left(\frac{x}{Q^2(1-x) + 2m_Q^2} \right)^\epsilon \left(1 - \frac{4m_Q^2 x}{Q^2(1-x) + 2m_Q^2} \right)^{\frac{1-2\epsilon}{2}}, \tag{5.32}$$

while the remaining master integral reads

$$\begin{aligned}
 I_3(m_Q, m_Q, s_{ij}) &= \int d\Phi_2(m_Q, m_Q) \frac{1}{s_{ij}} \\
 &= (4\pi)^{\epsilon-1} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} \left(\frac{x}{Q^2 + 2m_Q^2} \right) \left(\frac{x}{Q^2(1-x) + 2m_Q^2} \right)^\epsilon \\
 &\quad \times \frac{v^{1-2\epsilon}}{1+v} {}_2F_1 \left(1, 1-\epsilon, 2-2\epsilon; \frac{2v}{1+v} \right) \tag{5.33}
 \end{aligned}$$

where,

$$v = \left(1 - \frac{4m_Q^2}{E_{\text{cm}}^2} \right)^{\frac{1}{2}} = \left(1 - \frac{4m_Q^2 x}{Q^2(1-x) + 2m_Q^2} \right)^{\frac{1}{2}}. \tag{5.34}$$

The integrated form of the gluon-initiated quark-antiquark massive antenna $A_{g;Q\bar{Q}}^0$ is finite, since it has only quasi-collinear limits. Expanding the all order result given in eq. (5.31) in powers of ϵ up to finite order, we find

$$\begin{aligned}
 \mathcal{A}_3^0(3_g; 1_Q, 2_{\bar{Q}}) &= \frac{1}{(Q^2 + 2m_Q^2)^2(Q^2 - 2m_Q^2)} \\
 &\times \left\{ [Q^4(1-2x+2x^2) + 4m_Q^2 Q^2 + m_Q^4(4+8x-8x^2)] \ln \left(\frac{1-v}{1+v} \right) \right. \\
 &\left. + v [Q^4(1-2x+2x^2) + 4m_Q Q^2(1-x) + m_Q^4(4-8x)] \right\} + \mathcal{O}(\epsilon). \tag{5.35}
 \end{aligned}$$

B) Quark-Gluon antennae.

As only one massive parton is present in the final state, the fractional momentum x carried by the initial state momentum p_i is defined as $x = \frac{Q^2+m_Q^2}{2p_i \cdot q}$. For the initial-final antenna phase space $d\Phi_{X_{i,jk}}$ over which the antennae are integrated, we use the parametrisation of the two-by-two parton phase space $d\Phi_2(m_Q, 0)$ given in eq. (5.27). After reduction, we found that two master integrals are needed.

The master integral $I_2(m_Q, 0)$ corresponds to the integrated phase space in eq. (5.27) and reads,

$$I_2(m_Q, 0) = (4\pi)^{\epsilon-1} \frac{\Gamma(1-\epsilon)}{2\Gamma(2-2\epsilon)} \left(\frac{x}{Q^2(1-x) + m_Q^2} \right)^\epsilon \left(1 - \frac{m_Q^2 x}{Q^2(1-x) + m_Q^2} \right)^{1-2\epsilon}, \quad (5.36)$$

while the remaining integral is given by

$$\begin{aligned} I_3(m_Q, 0, s_{ik}) &= \int d\Phi_2(m_Q, 0) \frac{1}{s_{ik}} \\ &= (4\pi)^{1-\epsilon} \frac{\Gamma(1-\epsilon)}{2\Gamma(2-2\epsilon)} \left(\frac{x}{Q^2 + m_Q^2} \right) \left(\frac{x}{Q^2(1-x) + m_Q^2} \right)^\epsilon \\ &\quad \times u^{1-2\epsilon} {}_2F_1(1, 1-\epsilon, 2-2\epsilon; u) \end{aligned} \quad (5.37)$$

with u being given by

$$u = 1 - \frac{xm_Q^2}{Q^2(1-x) + m_Q^2}.$$

B.1) E-Type antennae.

The integrated form of $E_{q,qQ}^0$ is only proportional to the integrated phase space measure $I_2(m_Q, 0)$ and reads,

$$\begin{aligned} \mathcal{E}_3^0(4_q; 3_q, 1_Q) &= \frac{(4\pi)^{1-\epsilon} e^{\epsilon\gamma_E}}{[Q^2 + m_Q^2]} \\ &\times \left\{ 2m_Q m_\chi - \frac{(Q^2(1-x) + m_Q^2)(1+(1-x)^2)}{\epsilon x(1-x)} + m_Q^2 \left(\frac{1+x}{1-x} \right) - Q^2(1-2x) \right\} I_2(m_Q, 0) \end{aligned} \quad (5.38)$$

Its ϵ expansion is given by,

$$\begin{aligned} \mathcal{E}_3^0(4_q; 3_q, 1_Q) &= \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} [Q^2 + m_Q^2]^{-\epsilon} \times \left\{ -\frac{1}{2\epsilon} p_{qg}^{(0)}(x) \right. \\ &\quad + \frac{1}{2x} (2-2x+x^2) [-2+2\ln(1-x) - \ln(x) - \ln(1-x_0x)] \\ &\quad \left. + \frac{Q^2(-1+3x-2x^2) + m_Q^2(1+x) + 2m_Q m_\chi(1-x)}{2(Q^2(1-x) + m_Q^2)} + \mathcal{O}(\epsilon) \right\}, \end{aligned} \quad (5.39)$$

with x_0 being given in eq. (5.2), and where we have kept the natural phase space factor for initial-final quark-gluon integrated antennae,

$$\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} [Q^2 + m_Q^2]^{-\epsilon}$$

unexpanded. Since this antenna has only one massless initial-final collinear limit, the pole part in its integrated form is only related to the x -dependent splitting kernel $p_{qg}^{(0)}(x)$. This expression can be directly compared with the expression obtained in [85] for the integrated dipole involving a massless initial emitter q and a massive final-state spectator Q and full agreement is found.

B.2) D-Type antennae.

The integrated form of the gluon-initiated initial-final antenna $D_{g,Qg}^0$, which is such that the reduced matrix element is induced by a quark and which has a massive quark Q as unresolved parton, is

$$\begin{aligned}
 \mathcal{D}_3^0(4_g; 1_Q, 3_g) &= \frac{(4\pi)^{1-\epsilon} e^{\epsilon\gamma_E}}{(Q^2 + m_Q^2)} \frac{1}{(1-\epsilon)x(1-x)} \\
 &\times \left\{ \frac{1}{2(Q^2 + m_Q^2)(Q^2(1-x) + m_Q^2)} \right. \\
 &\quad \times \left[Q^6(1-x)^2 [-3 + 6x - 4x^2 + 4x^3 + 2\epsilon(1 - 4x + 6x^2 - 6x^3) \right. \\
 &\quad \quad \left. + \epsilon^2(1 + 2x - 8x^2 + 8x^3)] \right. \\
 &\quad \left. - Q^4 m_Q^2(1-x) [9 - 25x + 32x^2 - 16x^3 + 4x^4 - 2\epsilon(3 - 14x + 30x^2 \right. \\
 &\quad \quad \left. - 24x^3 + 6x^4) - \epsilon^2(3 + 5x - 24x^2 + 32x^3 - 8x^4)] \right. \\
 &\quad \left. + Q^2 m_Q^4 [-9 + 32x - 57x^2 + 38x^3 - 8x^4 + \epsilon(6 - 32x + 82x^2 - 76x^3 + 24x^4) \right. \\
 &\quad \quad \left. + \epsilon^2(3 + 4x - 25x^2 + 38x^3 - 16x^4)] \right. \\
 &\quad \left. + m_Q^6 [-3 + 10x - 19x^2 + 4x^3 + 2\epsilon(1 - 5x + 12x^2 - 6x^3) \right. \\
 &\quad \quad \left. + \epsilon^2(1 + 2x - 7x^2 + 8x^3)] \right. \\
 &\quad \left. + 2\epsilon(1 - \epsilon)Q^4 m_Q m_\chi (-1 - x + 2x^2) \right. \\
 &\quad \left. - 4Q^2 m_Q^3 m_\chi x [-x(1-x) + \epsilon(1 - \epsilon)(1 + x - x^2)] \right. \\
 &\quad \left. + 2m_Q^5 m_\chi x [2x - \epsilon(1 + x) + \epsilon^2(1 + x)] \right] I_2(m_Q, 0) \\
 &\quad - \left[(1 - \epsilon)^2 Q^4 x(1 - 3x + 4x^2 - 2x^3) \right. \\
 &\quad \left. + 2Q^2 m_Q^2 [1 - 4x + 4x^2 - 2x^3 + \epsilon(-2 + 7x - 6x^2 + 2x^3) + \epsilon^2(1 - 2x + 2x^2)] \right. \\
 &\quad \left. + m_Q^4 [1 - 7x + 4x^2 - 2x^3 + \epsilon(-2 + 8x - 4x^2) + \epsilon^2(1 - x + 2x^3)] \right. \\
 &\quad \left. - 2\epsilon(1 - \epsilon)Q^2 m_Q m_\chi x + 2(1 - \epsilon + \epsilon^2)m_Q^3 m_\chi x \right] I_3(m_Q, 0, s_{ik}) \left. \right\}
 \end{aligned} \tag{5.40}$$

Its expansion up to finite order in ϵ reads,

$$\begin{aligned}
 \mathcal{D}_3^0(4_g; 1_Q, 3_g) &= \frac{1}{4(Q^2 + m_Q^2)^2(1-x)} \\
 &\times \left\{ \frac{(Q^2 + m_Q^2)(1-x)}{(Q^2(1-x) + m_Q^2)^2 x} \left[Q^6(1-x)^2(-3 + 6x - 4x^2 + 4x^3) - Q^4 m_Q^2(1-x)(9 - 25x \right. \right. \\
 &\quad \left. \left. + 32x^2 - 16x^3 + 4x^4) + Q^2 m_Q^4(-9 + 32x - 57x^2 + 38x^3 - 8x^4) \right. \right. \\
 &\quad \left. \left. + m_Q^6(-3 + 10x - 19x^2 + 4x^3) + 4Q^2 m_Q^3 m_\chi x^2(1-x) + 4m_Q^5 m_\chi x^2 \right] \right. \\
 &\quad \left. - 2 \ln \left(\frac{m_Q^2 x}{Q^2(1-x) + m_Q^2} \right) \left[Q^4(1-x)(1 - 2x + 2x^2) - 2Q^2 m_Q^2(-1 + 4x - 4x^2 + 2x^3) \right. \right. \\
 &\quad \left. \left. - m_Q^4(-1 + 7x - 4x^2 + x^3) + 2m_Q^3 m_\chi x \right] \right\} + \mathcal{O}(\epsilon)
 \end{aligned} \tag{5.41}$$

This integrated antenna is finite since its unintegrated form has only quasi-collinear limits.

Finally, the integrated form of the initial-final massive D-Type quark-gluon antenna $D_{g,gQ}^0$ which is initiated by a gluon, which has a reduced matrix element induced by a gluon and where the final state gluon is the unresolved parton, is given by,

$$\begin{aligned}
 \mathcal{D}_3^0(4_g; 3_g, 1_Q) &= \frac{(4\pi)^{1-\epsilon} e^{\epsilon\gamma_E}}{2} \\
 &\times \left\{ \frac{Q^2(1-\epsilon)^2 + 2m_Q m_\chi \epsilon(1-\epsilon) + m_Q^2(3-4\epsilon-\epsilon^2)}{(1-\epsilon)[Q^2(1-x) + m_Q^2]} \right. \\
 &\quad - \frac{4(1-2\epsilon)[Q^2(1-x) + m_Q^2](1-x+x^2)^2}{\epsilon x(1-x)^2(Q^2 + m_Q^2)} \\
 &\quad + \frac{2}{(1-\epsilon)(1-x)^2(Q^2 + m_Q^2)} \left[2m_Q m_\chi(1-\epsilon + \epsilon^2)(1-x)^2 \right. \\
 &\quad \left. + Q^2(1-x)[-5 + 3x + \epsilon(8-6x) - \epsilon^2(1-x)] \right. \\
 &\quad \left. + m_Q^2[-9 + 16x - 9x^2 + 2\epsilon(4-7x+4x^2) + \epsilon^2(1-x)^2] \right\} I_2(m_Q, 0)
 \end{aligned} \tag{5.42}$$

The ϵ expansion of this integrated antenna reads

$$\begin{aligned}
 \mathcal{D}_3^0(4_g; 3_g, 1_Q) &= -2\mathbf{I}_{Qg}^{(1)} \left(\epsilon, Q^2, m_Q, 0, x_0^2 \frac{m_Q^2}{Q^2 + m_Q^2} \right) \delta(1-x) - \frac{1}{2\epsilon} p_{gg}^{(0)}(x) \\
 &\quad - \frac{3}{2} + \left(1 - \frac{\pi^2}{12} \right) \delta(1-x) - \mathcal{D}_0(x) + 2\mathcal{D}_1(x) \\
 &\quad + \frac{1}{4(Q^2(1-x) + m^2)^2} \left[Q^4(1-x) - 2Q^2 m_Q^2(1-x)(2-9x) - 5m_Q^4(1-3x) \right. \\
 &\quad \left. + 4Q^2 m_Q m_\chi(1-x)^2 + 4m_Q^3 m_\chi(1-x) \right] \\
 &\quad + 2 \left(-2 + \frac{1}{x} + x - x^2 \right) \ln(1-x) + \left(\frac{1}{2} \delta(1-x) + \frac{1}{1-x} - \mathcal{D}_0(x) \right) \ln(1-x_0) \\
 &\quad + \frac{1}{2} \delta(1-x) \left(\frac{17}{6} \ln(x_0^2) + \ln(x_0^2) \ln(1-x_0) + \frac{1}{2} \ln^2(x_0^2) + \frac{1}{2} \ln^2(1-x_0) \right) \\
 &\quad - \frac{(1-x+x^2)^2}{x(1-x)} [\ln(x) + \ln(1-x_0x)] - \frac{1}{2} p_{gg}^{(0)}(x) \ln \left(\frac{x_0^2(Q^2 + m^2)}{Q^4} \right) + \mathcal{O}(\epsilon),
 \end{aligned} \tag{5.43}$$

where x_0 is given in eq. (5.2).

From this expansion it can be seen that all the pole parts are contained in the infrared singularity operator

$$\mathbf{I}_{Qg}^{(1)} \left(\epsilon, Q^2, m_Q, 0, x_0^2 \frac{m_Q^2}{Q^2 + m_Q^2} \right)$$

as well as in the splitting kernel $p_{gg}^{(0)}(x)$. A factor x_0^2 appears explicitly in the mass-dependent logarithmic term in the infrared operator $\mathbf{I}_{Qg}^{(1)}$. It is necessary in order to capture all poles parts proportional to $\delta(1-x)$ of the integrated antenna, $\mathcal{D}_3^0(4g; 3g, 1Q)$.

Furthermore, we can compare this expanded result for $\mathcal{D}_{g,gQ}^0$ with the corresponding integrated dipoles in [85]. The pole parts of our expression given in eq. (5.43) corresponds to the pole part of the sum of two dipoles which have either a massless initial spectator and a massive final-state emitter or a massless initial state emitter and a massive final state spectator. Full agreement is found providing us with a strong check on our result for $\mathcal{D}_{g,gQ}^0$.

C) Flavour violating antennae.

In addition to the flavour conserving integrated initial-final antennae, we also need to consider the integration over the massive initial-final antenna phase space of the unintegrated initial-final flavour-violating antenna $A_{q;gQ}^0$. The phase space parametrisation of the initial-final antenna phase required here is the same parametrisation as the one used for integrating the initial-final quark-gluon antennae. One uses the parametrisation of the two-to-two particle phase space given by $d\Phi_2(m_Q, 0)$ in eq. (5.27). The integrated flavour violating antenna $\mathcal{A}_{q;gQ}^0$ can be written in term of the phase space measure $I_2(m_Q, 0)$ only. In term of this master integral, its integrated form reads,

$$\begin{aligned} \mathcal{A}_3^0(2q; 3g, 1Q) &= -\frac{(4\pi)^{1-\epsilon} e^{\epsilon\gamma_E}}{2\epsilon(Q^2 + m_Q^2)(Q^2(1-x) + m_Q^2)(1-x)^2} \\ &\times \{ Q^4(1-x)^2 [2(1+x^2) - \epsilon(7-8x+6x^2) + \epsilon^2(1-2x)^2] \\ &\quad + 2m_Q^2 Q^2(1-x) [2(1+x^2) - \epsilon(7-9x+6x^2) + \epsilon^2(1-3x+2x^2)] \\ &\quad + m_Q^4 [2(1+x^2) - \epsilon(7-10x+7x^2) + \epsilon^2(1-x)^2] \} I_2(m_Q, 0) \end{aligned} \quad (5.44)$$

Expanding in powers of ϵ we obtain

$$\begin{aligned} \mathcal{A}_3^0(2q; 1Q, 3g) &= -2\mathbf{I}_{q\bar{Q}}^{(1)}\left(\epsilon, Q^2, m_Q, 0, x_0^2 \frac{m_Q^2}{Q^2 + m_Q^2}\right) \delta(1-x) - \frac{1}{2\epsilon} p_{qq}^{(0)}(x) \\ &\quad + \frac{3}{2} + \left(1 - \frac{\pi^2}{12}\right) \delta(1-x) - \mathcal{D}_0(x) + 2\mathcal{D}_1(x) - \frac{x}{2} - \frac{m_Q^2}{4Q^2(1-x_0x)^2} \\ &\quad + \frac{1}{4(1-x_0)(1-x_0x)} + \left(\frac{1}{2}\delta(1-x) + \frac{1}{1-x} - \mathcal{D}_0(x)\right) \ln(1-x_0) \\ &\quad + \frac{1}{4}\delta(1-x) (5\ln(x_0^2) + 2\ln(x_0^2)\ln(1-x_0) + \ln^2(x_0^2) + \ln^2(1-x_0)) \\ &\quad - (1+x)\ln(1-x) - \frac{6(1+x^2)}{1-x} [\ln(x) - \ln(1-x_0x)] \\ &\quad - \frac{1}{2} p_{qq}^{(0)}(x) \ln\left(\frac{x_0^2(Q^2 + m^2)}{Q^4}\right) + \mathcal{O}(\epsilon), \end{aligned} \quad (5.45)$$

All the pole pieces are contained in the following infrared singularity operator

$$\mathbf{I}_{q\bar{Q}}^{(1)} \left(\epsilon, Q^2, m_Q, 0, x_0^2 \frac{m_Q^2}{Q^2 + m_Q^2} \right)$$

and the splitting kernel $p_{q\bar{q}}^{(0)}(x)$. As for the integrated gluon-initiated quark-gluon antenna $\mathcal{D}_{g,gQ}^0$, a factor x_0^2 appears in the mass-dependent logarithmic term in the infrared operator $\mathbf{I}_{q\bar{Q}}^{(1)}$. It is necessary in order to capture all poles parts proportional to $\delta(1-x)$ of the integrated flavour violating antenna, $\mathcal{A}_3^0(2_q; 3_g, 1_Q)$.

In summary, in this subsection 5.3 we have shown that all integrated massive initial-final antennae have their pole parts related either to massive infrared operators or to splitting kernels or both.

6 Check of $\mathcal{A}_{g;Q\bar{Q}}^0$

A strong check can be performed on the integrated quark-antiquark antenna $\mathcal{A}_{g;Q\bar{Q}}^0$ by comparing its expression given in eq. (5.31) and known results from the literature on the leading order heavy-quark coefficient functions.

To compare our results with γ induced deep inelastic scattering we consider the contraction of the hadronic tensor $W^{\mu\nu}$ with the metric tensor $-g_{\mu\nu}$. This corresponds to the trace of the hadronic tensor, which in terms of the structure functions \mathcal{F}_2 and \mathcal{F}_L is given by,

$$-W_\mu^\mu = -\frac{d-1}{2}\mathcal{F}_L(z, Q^2) + \frac{d-2}{2}\mathcal{F}_2(z, Q^2), \quad (6.1)$$

where the structure functions can be expanded in powers of the strong coupling constant.

To zeroth order in α_s these structure functions are given by the simple parton model result

$$\mathcal{F}_{L,q}^{(0)} = \mathcal{F}_{L,g}^{(0)} = 0, \quad \mathcal{F}_{2,q}^{(0)} = \delta(1-z), \quad \mathcal{F}_{2,g}^{(0)} = 0. \quad (6.2)$$

We find that the correct normalisation of W_μ^μ to be checked against the antenna $\mathcal{A}_{g;Q\bar{Q}}^0$ is as in the massless case [80] given by

$$-\frac{2}{d-2}W_\mu^\mu = \mathcal{F}_2 - \frac{d-1}{d-2}\mathcal{F}_L. \quad (6.3)$$

such that the following relation should hold

$$\mathcal{A}_{g;Q\bar{Q}}^0 \times [Q^2(1-\epsilon) - 2m_Q^2] = -\frac{1-\epsilon}{2} \left[\mathcal{F}_{2,g}^{(1)}(x, Q^2) - \frac{3-2\epsilon}{2-2\epsilon} \mathcal{F}_{L,g}^{(1)}(x, Q^2) \right] \quad (6.4)$$

with $\mathcal{F}_{2,g}^{(1)}$ and $\mathcal{F}_{L,g}^{(1)}$ being the leading order heavy quark coefficient functions.

The factor on the left-hand side of this equation which multiplies the integrated antenna corresponds to the massive two-parton antenna $A_{Q;Q}$ serving to normalise the three parton antenna and given in eq. (3.12).

The coefficient functions are given up to finite order in ϵ in [110] with a different choice of variable x . $x \equiv x_B = \frac{Q^2}{2p \cdot q}$ is used instead of our expression for x given in eq. (5.26) by $x = \frac{Q^2 + 2m_Q^2}{2p \cdot q}$, which includes the masses m_Q of the final states Q and \bar{Q} .

We used this second definition of x in our expression of the integrated antenna $\mathcal{A}_{g;Q\bar{Q}}$ since this definition of x is required in order to guarantee phase space factorisation in our subtraction formalism. Adapting the results given in [110] to this mass-dependent definition of x , the heavy quark coefficient functions read,

$$\mathcal{F}_{2,g}^{(1)}(x, Q^2) = -\frac{2}{(Q^2 + 2m_Q^2)^2} \left\{ v [Q^4(1 - 8x + 8x^2) - 4Q^2m_Q^2(-1 + 3x + x^2) + m_Q^4(4 + 8x)] + [Q^4(1 - 2x + 2x^2) + 4Q^2m_Q^2(1 - 3x^2) + m^4(4 + 8x - 8x^2)] \log\left(\frac{1-v}{1+v}\right) \right\} + \mathcal{O}(\epsilon) \quad (6.5)$$

$$\mathcal{F}_{L,g}^{(1)}(x, Q^2) = \frac{8Q^2x}{(Q^2 + 2m_Q^2)^2} \left\{ v [Q^2(1 - x) + 2m_Q^2] + 2m_Q^2x \log\left(\frac{1-v}{1+v}\right) \right\} + \mathcal{O}(\epsilon). \quad (6.6)$$

To the finite order in ϵ , one can show that

$$\begin{aligned} & -\frac{1}{2}\mathcal{F}_{2,g}^{(1)} + \frac{3}{4}\mathcal{F}_{L,g}^{(1)} \\ &= \frac{1}{(Q^2 + 2m_Q^2)^2} \left\{ v [Q^4(1 - 2x + 2x^2) + 4Q^2m_Q^2(1 - x^2) + m_Q^4(4 + 8x)] \right. \\ & \quad \left. + [Q^4(1 - 2x + 2x^2) + 4Q^2m_Q^2 + m_Q^4(4 + 8x - 8x^2)] \log\left(\frac{1-v}{1+v}\right) \right\}, \end{aligned} \quad (6.7)$$

with the variable v defined as in eq. (5.34) since the redefinition of x in [110] does not affect the definition of v . The expression given in eq. (6.7) coincides with the $\mathcal{O}(\epsilon^0)$ term of our expanded form for $\mathcal{A}_{g;Q\bar{Q}}$ after we undo the normalisation of this antenna function by multiplying it by $[Q^2 - 2m_Q^2]$. The relation given by eq. (6.4) is therefore herewith proven at $\mathcal{O}(\epsilon^0)$ giving us a strong check on the integrated antenna $\mathcal{A}_{g;Q\bar{Q}}$ itself.

7 Application to top quark pair production at LHC

In this section we shall give the colour-ordered real emission contributions for all partonic processes contributing to $t\bar{t}$ and $t\bar{t} + jet$ production at the LHC present at NLO. Together with these, we shall present their corresponding antenna subtraction terms which capture all single unresolved (soft, collinear and quasi-collinear) radiation of the real matrix-element squared for each partonic process involved.

The results presented here for the real contributions and subtraction terms for the process $pp \rightarrow t\bar{t} + 1 \text{ jet}$ at NLO are essential ingredients for the computation of the double real contributions and their subtraction terms to the production of $t\bar{t}$ at NNLO. Our subtraction terms enable to capture all single unresolved radiation present in those double real contributions. Furthermore, concerning the colour decomposition of the real matrix

element squared for $t\bar{t} + 1$ jet production at NLO, in the limit where the heavy quarks present in the final state are taken massless, the colour decomposition provided here corresponds to the colour decomposition of the real matrix elements to the processes involving a massless quark-antiquark pair for two jet production at NNLO.

For all partonic processes involved, starting from well-known amplitudes given in [28–30, 33, 34], the colour decomposition of the real matrix elements squared is presented and where possible, checked against results in the literature. Wherever possible, decoupling identities are used to reduce the size of the colour ordered decomposition of the real matrix-element squared and in some cases to eliminate the interference terms in those.

The colour ordered subtraction terms are explicitly constructed as sums of terms involving the product of antenna functions with reduced matrix element squared and jet functions, as explained in section 2.

Concerning the notation of the matrix elements appearing in the real contributions, those matrix elements denoted by \mathcal{M} represent colour-ordered amplitudes in which the coupling constants and colour factors have been omitted. Furthermore, to explicitly visualize the colour connection between particles in these colour ordered amplitudes, a double semicolon is used in the labeling of the partons contributing to a given matrix elements. This double semi colon serves to separate strings of colour connected partons. Partons within a pair of double semicolons are colour-connected.

Since in the antenna framework a parton can only be unresolved with respect to its colour-connected neighbours, this notation helps to identify the unresolved limits present in a given colour ordered amplitude and therefore helps to construct the corresponding subtraction terms. Notationwise, we also denote gluons which are photon-like and only couple to quark lines, with the index γ instead of g , to manifestly separate leading from subleading contributions. In amplitudes where all gluons are photon-like no semicolons are used, since the concept of colour connection is not meaningful in those configurations.

Concerning the notation in the subtraction terms themselves, the reduced matrix-element squared present in those are also to be taken without coupling constants and coupling factors. In those matrix elements, the crossed momenta are denoted with a hat, the remapped final-state momenta are denoted with tildes and the remapped momenta of initial state hard radiators are denoted by a bar and a hat.¹

7.1 $t\bar{t}$ production at LHC

Following the general factorisation formula given in eq. (1.1) for hadronic collision processes, the real NLO contributions to the production of a massive quark-antiquark pair in a hadronic collision can be written as

$$d\sigma^R = \int \frac{d\xi_1}{\xi_1} \frac{d\xi_2}{\xi_2} \left\{ \sum_q \left[f_q(\xi_1) f_{\bar{q}}(\xi_2) d\hat{\sigma}_{q\bar{q} \rightarrow Q\bar{Q}g} + f_q(\xi_1) f_g(\xi_2) d\hat{\sigma}_{qg \rightarrow Q\bar{Q}q} \right. \right. \tag{7.1}$$

$$\left. \left. + f_{\bar{q}}(\xi_1) f_g(\xi_2) d\hat{\sigma}_{\bar{q}g \rightarrow Q\bar{Q}\bar{q}} \right] + f_g(\xi_1) f_g(\xi_2) d\hat{\sigma}_{gg \rightarrow Q\bar{Q}g} \right\}.$$

¹This notation for the remapped initial state momenta was already used in [82].

$f_i(\xi_j)$ denotes the parton distribution function of parton i , which carries a fraction ξ_j of one of the incoming hadron momenta and $d\hat{\sigma}$ are the partonic cross sections. These process-dependent partonic cross sections $d\hat{\sigma}$ are, in turn given by

$$d\hat{\sigma}_{q\bar{q}\rightarrow Q\bar{Q}g} = d\Phi_3(k_Q, k_{\bar{Q}}, k_g; p_q, p_{\bar{q}}) |M_{q\bar{q}\rightarrow Q\bar{Q}g}^0|^2 J_2^{(3)}(k_Q, k_{\bar{Q}}, k_g), \quad (7.2)$$

$$d\hat{\sigma}_{qg\rightarrow Q\bar{Q}q} = d\Phi_3(k_Q, k_{\bar{Q}}, k_q; p_q, p_g) |M_{qg\rightarrow Q\bar{Q}q}^0|^2 J_2^{(3)}(k_Q, k_{\bar{Q}}, k_q), \quad (7.3)$$

$$d\hat{\sigma}_{gg\rightarrow Q\bar{Q}g} = d\Phi_3(k_Q, k_{\bar{Q}}, k_g; p_{g1}, p_{g2}) |M_{gg\rightarrow Q\bar{Q}g}^0|^2 J_2^{(3)}(k_Q, k_{\bar{Q}}, k_g), \quad (7.4)$$

where the momenta labels in the matrix elements are omitted for the sake of conciseness. Each of these process-dependent partonic cross sections involves a massive two-to-three parton phase space, $d\Phi_3$, the corresponding matrix element squared $|M^0|^2$ and the appropriate jet function $J_2^{(3)}$ related to the selection criteria of 2-jet events. Out of three partons, from which two are a Q and \bar{Q} , an event with two jets having each a heavy quark Q or a heavy anti-quark \bar{Q} in them is formed. In the corresponding subtraction terms, the jet functions will all be of the type $J_2^{(2)}$ and will only depend on the hard final-state momenta appearing in the reduced matrix-element squared.

In order to obtain the subtraction terms, the colour decomposition of the real matrix elements present in the partonic cross sections given above in eqs. (7.2), (7.3), (7.4) has to be performed. However, since colour ordering does not distinguish between initial and final state partons, it is sufficient to consider the colour decomposition for the two unphysical processes $0 \rightarrow Q\bar{Q}q\bar{q}g$ and $0 \rightarrow Q\bar{Q}ggg$, and obtain the matrix elements needed for $t\bar{t}$ production from two initial state partons by appropriate crossings. We shall follow this strategy in the following.

7.1.1 Processes derived from $0 \rightarrow Q\bar{Q}q\bar{q}g$

At amplitude level, the colour decomposition for this unphysical process $0 \rightarrow Q\bar{Q}q\bar{q}g$ is

$$\begin{aligned} M_5^0(1_Q, 2_{\bar{Q}}, 3_q, 4_{\bar{q}}, 5_g) = g^3 \sqrt{2} & \left[(T^{a_5})_{i_1, i_4} \delta_{i_2, i_3} \mathcal{M}_5^0(1_Q, 5_g, 4_{\bar{q}}; ; 2_{\bar{Q}}, 3_q) \right. \\ & + (T^{a_5})_{i_3, i_2} \delta_{i_1, i_4} \mathcal{M}_5^0(1_Q, 4_{\bar{q}}; ; 2_{\bar{Q}}, 5_g, 3_q) \\ & - \frac{1}{N_c} (T^{a_5})_{i_1, i_2} \delta_{i_3, i_4} \mathcal{M}_5^0(1_Q, 5_g, 2_{\bar{Q}}; ; 3_q, 4_{\bar{q}}) \\ & \left. - \frac{1}{N_c} (T^{a_5})_{i_3, i_4} \delta_{i_1, i_2} \mathcal{M}_5^0(1_Q, 2_{\bar{Q}}; ; 3_q, 5_g, 4_{\bar{q}}) \right], \end{aligned} \quad (7.5)$$

Squaring, and using the photon decoupling identities stating that

$$\begin{aligned} \mathcal{M}_5^0(1_Q, 2_{\bar{Q}}, 3_q, 4_{\bar{q}}, 5_g) & = \mathcal{M}_5^0(1_Q, 5_g, 4_{\bar{q}}; ; 2_{\bar{Q}}, 3_q) + \mathcal{M}_5^0(1_Q, 4_{\bar{q}}; ; 2_{\bar{Q}}, 5_g, 3_q) \\ & = \mathcal{M}_5^0(1_Q, 5_g, 2_{\bar{Q}}; ; 3_q, 4_{\bar{q}}) + \mathcal{M}_5^0(1_Q, 2_{\bar{Q}}; ; 3_q, 5_g, 4_{\bar{q}}) \end{aligned} \quad (7.6)$$

gives

$$\begin{aligned} |M_5^0(1_Q, 2_{\bar{Q}}, 3_q, 4_{\bar{q}}, 5_g)|^2 & = g^6 (N_c^2 - 1) \\ & \times \left[N_c (|\mathcal{M}_5^0(1_Q, 5_g, 4_{\bar{q}}; ; 2_{\bar{Q}}, 3_q)|^2 + |\mathcal{M}_5^0(1_Q, 4_{\bar{q}}; ; 2_{\bar{Q}}, 5_g, 3_q)|^2) \right. \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} E_3^0(4_q; 3_q, 2_{\bar{Q}}) \left(|\mathcal{M}_4^0(1_Q, \hat{5}_g, \hat{4}_g, (\tilde{2}\bar{3})_{\bar{Q}})|^2 + |\mathcal{M}_4^0(1_Q, \hat{4}_g, \hat{5}_g, (\tilde{2}\bar{3})_{\bar{Q}})|^2 \right) J_2^{(2)}(k_1, k_{\tilde{2}\bar{3}}) \Big] \\
 & - \frac{1}{N_c} \left[\frac{1}{2} A_3^0(5_g; 1_Q, 2_{\bar{Q}}) \left(|\mathcal{M}_4^0((\tilde{1}\bar{2})_Q, \hat{5}_{\bar{Q}}, 3_q, \hat{4}_q)|^2 + |\mathcal{M}_4^0(\hat{5}_{\bar{Q}}, (\tilde{1}\bar{2})_{\bar{Q}}, 3_q, \hat{4}_q)|^2 \right) J_2^{(2)}(k_{\tilde{1}\bar{2}}, k_3) \right. \\
 & + A_3^0(4_q, 5_g; 3_q) |\mathcal{M}_4^0(\tilde{1}_Q, \tilde{2}_{\bar{Q}}, \hat{4}_q, \hat{5}_{\bar{q}})|^2 J_2^{(2)}(\tilde{k}_1, \tilde{k}_2) \\
 & + \frac{1}{2} E_3^0(4_q; 3_q, 1_Q) |\mathcal{M}_4^0((\tilde{1}\bar{3})_Q, \hat{4}_\gamma, \hat{5}_\gamma, 2_{\bar{Q}})|^2 J_2^{(2)}(k_{\tilde{1}\bar{3}}, k_2) \\
 & \left. + \frac{1}{2} E_3^0(4_q; 3_q, 2_{\bar{Q}}) |\mathcal{M}_4^0(1_Q, \hat{4}_\gamma, \hat{5}_\gamma, (\tilde{2}\bar{3})_{\bar{Q}})|^2 J_2^{(2)}(k_1, k_{\tilde{2}\bar{3}}) \right] \Big\}. \tag{7.9}
 \end{aligned}$$

7.1.2 Partonic process $gg \rightarrow Q\bar{Q}g$

The colour decomposition for the unphysical process $0 \rightarrow Q\bar{Q}ggg$ is

$$M_5^0(1_Q, 2_{\bar{Q}}, 3_g, 4_g, 5_g) = (g\sqrt{2})^3 \sum_{(i,j,k) \in P(3,4,5)} (T^{a_i} T^{a_j} T^{a_k})_{i_2}^{i_1} \mathcal{M}_5^0(1_Q, i_g, j_g, k_g, 2_{\bar{Q}}). \tag{7.10}$$

Squaring and crossing gluons 4_g and 5_g to the initial state gives

$$\begin{aligned}
 & |M_5^0(1_Q, 2_{\bar{Q}}, 3_g, \hat{4}_g, \hat{5}_g)|^2 = g^6 (N_c^2 - 1) \\
 & \times \left\{ \sum_{(i,j) \in P(4,5)} \left[N_c^2 \left(|\mathcal{M}_5^0(1_Q, 3_g, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}})|^2 + |\mathcal{M}_5^0(1_Q, \hat{i}_g, 3_g, \hat{j}_g, 2_{\bar{Q}})|^2 \right. \right. \right. \\
 & \quad \left. \left. + |\mathcal{M}_5^0(1_Q, \hat{i}_g, \hat{j}_g, 3_g, 2_{\bar{Q}})|^2 \right) \right. \\
 & \quad \left. - \left(|\mathcal{M}_5^0(1_Q, 3_g, \hat{i}_g, \hat{j}_\gamma, 2_{\bar{Q}})|^2 + |\mathcal{M}_5^0(1_Q, \hat{i}_g, 3_g, \hat{j}_\gamma, 2_{\bar{Q}})|^2 \right) \right. \\
 & \quad \left. + |\mathcal{M}_5^0(1_Q, \hat{i}_g, \hat{j}_g, 3_\gamma, 2_{\bar{Q}})|^2 \right] \\
 & \quad \left. + \left(\frac{N_c^2 + 1}{N_c^2} \right) |\overline{\mathcal{M}}_5^0(1_Q, 3_g, \hat{4}_g, \hat{5}_g, 2_{\bar{Q}})|^2 \right\}. \tag{7.11}
 \end{aligned}$$

We have used the following photon decoupling identity,

$$\begin{aligned}
 & \mathcal{M}_5^0(1_Q, i_g, j_g, k_\gamma, 2_{\bar{Q}}) = \\
 & \mathcal{M}_5^0(1_Q, i_g, j_g, k_g, 2_{\bar{Q}}) + \mathcal{M}_5^0(1_Q, i_g, k_g, j_g, 2_{\bar{Q}}) + \mathcal{M}_5^0(1_Q, k_g, i_g, j_g, 2_{\bar{Q}})
 \end{aligned} \tag{7.12}$$

where gluon k is a U(1) boson decoupled from the other gluons, and

$$\overline{\mathcal{M}}_5^0(1_Q, 3_g, 4_g, 5_g, 2_{\bar{Q}}) = \sum_{(i,j,k) \in P(3,4,5)} \mathcal{M}_5^0(1_Q, i_g, j_g, k_g, 2_{\bar{Q}}), \tag{7.13}$$

where all gluons are photon-like.

The colour decomposition of the matrix element squared with two gluons in the initial state given by eq. (7.11) is obtained as follows: After squaring the amplitude for the process $0 \rightarrow Q\bar{Q}ggg$ and expanding the sum of permutations concerning the three final state gluons, the crossing of two gluons 4_g and 5_g is performed. As the final state gluon labelled 3_g is fixed, the terms are then regrouped and a sum over the permutations for the two initial state gluons 4_g and 5_g only is performed.

Only colour ordered squared matrix elements are involved in eq. (7.11) for which unresolved radiation can be captured by a single antenna function. Massive flavour-conserving quark-antiquark A-Type antennae, massive quark-gluon D-type antennae together with massless initial-initial gluon-gluon F-Type antennae are required.

The subtraction term reads,

$$\begin{aligned}
 d\hat{\sigma}_{gg \rightarrow Q\bar{Q}g}^S &= g^6 (N_c^2 - 1) d\Phi_3(k_{1Q}, k_{2\bar{Q}}, k_{3g}; p_{4g}, p_{5g}) \\
 &\times \left\{ \sum_{(i,j) \in P(4,5)} \left[N_c^2 \left[\frac{1}{2} A_3^0(i_g; 1_Q, 2_{\bar{Q}}) (|\mathcal{M}_4^0((\tilde{12})_Q, 3_g, \hat{j}_g, \hat{i}_Q)|^2 + |\mathcal{M}_4^0(\hat{i}_{\bar{Q}}, 3_g, \hat{j}_g, (\tilde{12})_{\bar{Q}})|^2 \right. \right. \right. \\
 &\quad + |\mathcal{M}_4^0((\tilde{12})_Q, \hat{j}_g, 3_g, \hat{i}_Q)|^2 + |\mathcal{M}_4^0(\hat{i}_{\bar{Q}}, \hat{j}_g, 3_g, (\tilde{12})_{\bar{Q}})|^2) J_2^{(2)}(k_{\tilde{12}}, k_3) \\
 &\quad + D_3^0(i_g; 3_g, 1_Q) |\mathcal{M}_4^0((\tilde{13})_Q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}})|^2 J_2^{(2)}(k_{\tilde{13}}, k_2) \\
 &\quad + D_3^0(i_g; 3_g, 2_{\bar{Q}}) |\mathcal{M}_4^0(1_Q, \hat{j}_g, \hat{i}_g, (\tilde{23})_{\bar{Q}})|^2 J_2^{(2)}(k_1, k_{\tilde{23}}) \\
 &\quad \left. \left. \left. + F_3^0(i_g, j_g; 3_g) |\mathcal{M}_4^0(\tilde{1}_Q, \hat{i}_g, \hat{j}_g, \tilde{2}_{\bar{Q}})|^2 J_2^{(2)}(\tilde{k}_1, \tilde{k}_2) \right] \right. \right. \\
 &\quad - A_3^0(1_Q, 3_g, 2_{\bar{Q}}) |\mathcal{M}_4^0((\tilde{13})_Q, i_g, j_g, (\tilde{23})_{\bar{Q}})|^2 J_2^{(2)}(k_{\tilde{13}}, k_{\tilde{23}}) \tag{7.14} \\
 &\quad - \frac{1}{2} A_3^0(i_g; 1_Q, 2_{\bar{Q}}) (|\mathcal{M}_4^0((\tilde{12})_Q, 3_g, \hat{j}_g, \hat{i}_Q)|^2 + |\mathcal{M}_4^0(\hat{i}_{\bar{Q}}, 3_g, \hat{j}_g, (\tilde{12})_{\bar{Q}})|^2 \\
 &\quad + |\mathcal{M}_4^0((\tilde{12})_Q, \hat{j}_g, 3_g, \hat{i}_Q)|^2 + |\mathcal{M}_4^0(\hat{i}_{\bar{Q}}, \hat{j}_g, 3_g, (\tilde{12})_{\bar{Q}})|^2 \\
 &\quad + |\mathcal{M}_4^0((\tilde{12})_Q, 3_\gamma, \hat{j}_\gamma, \hat{i}_Q)|^2 + |\mathcal{M}_4^0(\hat{i}_{\bar{Q}}, 3_\gamma, \hat{j}_\gamma, (\tilde{12})_{\bar{Q}})|^2) J_2^{(2)}(k_{\tilde{12}}, k_3) \\
 &\quad - D_3^0(i_g; 3_g, 1_Q) |\mathcal{M}_4^0((\tilde{13})_Q, \hat{i}_\gamma, \hat{j}_\gamma, 2_{\bar{Q}})|^2 J_2^{(2)}(k_{\tilde{13}}, k_2) \\
 &\quad - D_3^0(i_g; 3_g, 2_{\bar{Q}}) |\mathcal{M}_4^0(1_Q, \hat{j}_\gamma, \hat{i}_\gamma, (\tilde{23})_{\bar{Q}})|^2 J_2^{(2)}(k_1, k_{\tilde{23}}) \\
 &\quad + \frac{1}{2N_c^2} A_3^0(i_g; 1_Q, 2_{\bar{Q}}) (|\mathcal{M}_4^0((\tilde{12})_Q, 3_\gamma, \hat{j}_\gamma, \hat{i}_Q)|^2 \\
 &\quad \left. \left. \left. + |\mathcal{M}_4^0(\hat{i}_{\bar{Q}}, 3_\gamma, \hat{j}_\gamma, (\tilde{12})_{\bar{Q}})|^2) J_2^{(2)}(k_{\tilde{12}}, k_3) \right] \right. \right. \\
 &\quad \left. \left. + \left(\frac{N_c^2 + 1}{N_c^2} \right) A_3^0(1_Q, 3_g, 2_{\bar{Q}}) |\mathcal{M}_4^0((\tilde{13})_Q, \hat{4}_\gamma, \hat{5}_\gamma, (\tilde{23})_{\bar{Q}})|^2 J_2^{(2)}(k_{\tilde{13}}, k_{\tilde{23}}) \right\}.
 \end{aligned}$$

7.1.3 Consistency check

We have performed a powerful check on all subtraction terms required for $t\bar{t}$ production given above. For each of those subtraction terms, which is a sum of terms multiplied by colour factors proportional to N_c , we have verified that it gives the correct non-colour ordered collinear or/and quasi-collinear behaviour. We have checked this feature for all collinear and quasi-collinear limits present in all subtraction terms presented above. We have verified that each subtraction term obey

$$d\hat{\sigma}^S \xrightarrow{a||b} g^2 C \frac{P_{ab}(z)}{s_{ab}} \times |M_m^0|^2 \times d\Phi_m J_m^{(m)}, \tag{7.15}$$

where $C = C_A, C_F, T_R$ is the corresponding Casimir, M_m^0 is the *non-colour ordered* reduced matrix element and $P_{ab}(z)$ stands for a massless or massive spitting function governing the

particular collinear or quasi-collinear limit as defined in section 4.

Let us now check all collinear and quasi-collinear limits of the subtraction term $d\hat{\sigma}_{gg \rightarrow Q\bar{Q}g}^S$ given in eq. (7.14) in this way. This subtraction term has the following collinear and quasi-collinear limits: $1_Q || 3_g, 3_g || \hat{4}_g$ and $1_Q || \hat{4}_g$.

The following relation between the non-colour ordered and colour ordered amplitudes squared for the process $gg \rightarrow Q\bar{Q}$ is needed. It reads,

$$\begin{aligned}
 |M_4^0(1_Q, 2_{\bar{Q}}, \hat{i}_g, \hat{j}_g)|^2 &= g^4 \left(\frac{N_c^2 - 1}{N_c} \right) \left[N_c^2 \left(|\mathcal{M}_4^0(1_Q, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}})|^2 \right. \right. \\
 &\quad \left. \left. + |\mathcal{M}_4^0(1_Q, \hat{j}_g, \hat{i}_g, 2_{\bar{Q}})|^2 \right) - |\mathcal{M}_4^0(1_Q, \hat{i}_\gamma, \hat{j}_\gamma, 2_{\bar{Q}})|^2 \right]. \quad (7.16)
 \end{aligned}$$

When the final-state quasi-collinear limit $1_Q || 3_g$ is taken in the subtraction term given in eq. (7.14), we obtain

$$\begin{aligned}
 d\hat{\sigma}_{gg \rightarrow Q\bar{Q}g}^S &\xrightarrow{1_Q || 3_g} g^6 \left(\frac{N_c^2 - 1}{N_c} \right)^2 \frac{P_{qg \rightarrow Q}(z, \mu_{qg}^2)}{s_{13}} \\
 &\times \left[N_c^2 \left(|\mathcal{M}_4^0((1+3)_Q, \hat{4}_g, \hat{5}_g, 2_{\bar{Q}})|^2 + |\mathcal{M}_4^0((1+3)_Q, \hat{5}_g, \hat{4}_g, 2_{\bar{Q}})|^2 \right) \right. \\
 &\quad \left. - |\mathcal{M}_4^0((1+3)_Q, \hat{4}_\gamma, \hat{5}_\gamma, 2_{\bar{Q}})|^2 \right] d\Phi_2(k_{(1+3)Q}, k_{2\bar{Q}}; p_{4g}, p_{5g}) J_2^{(2)}(k_{1+3}, k_2) \quad (7.17) \\
 &= g^2 C_F \frac{P_{qg \rightarrow Q}(z, \mu_{qg}^2)}{s_{13}} |M_4^0((1+3)_Q, 2_{\bar{Q}}, \hat{4}_g, \hat{5}_g)|^2 \\
 &\quad \times d\Phi_2(k_{(1+3)Q}, k_{2\bar{Q}}; p_{4g}, p_{5g}) J_2^{(2)}(k_{1+3}, k_2),
 \end{aligned}$$

where we have used eq. (7.16) with the momentum of the massive final state quark 1_Q being given by $(k_1 + k_3)$. In the limit $1_Q || 3_g$, the contributing terms in eq. (7.14) are proportional to the antenna $A_3^0(1_Q, 3_g, 2_{\bar{Q}})$ multiplied by a reduced colour-ordered matrix elements squared involving the remapped momenta $k_{(\bar{1}3)}$ and $k_{(\bar{2}3)}$. In the $1_Q || 3_g$ limit, those remapped momenta $k_{(\bar{1}3)}$ and $k_{(\bar{2}3)}$ are respectively given by $(k_1 + k_3)$ and k_2 .

Similarly, in the same subtraction term given in eq. (7.14) we can see that in the $3_g || \hat{4}_g$ limit we obtain

$$\begin{aligned}
 d\hat{\sigma}_{gg \rightarrow Q\bar{Q}g}^S &\xrightarrow{3_g || \hat{4}_g} g^6 \frac{N_c^2 - 1}{N_c} \frac{P_{g\bar{g} \rightarrow G}(z)}{s_{34}} \\
 &\times \left[N_c^2 \left(|\mathcal{M}_4^0(1_Q, (\widehat{4-3})_g, \hat{5}_g, 2_{\bar{Q}})|^2 + |\mathcal{M}_4^0(1_Q, \hat{5}_g, (\widehat{4-3})_g, 2_{\bar{Q}})|^2 \right) \right. \\
 &\quad \left. - |\mathcal{M}_4^0(1_Q, (\widehat{4-3})_\gamma, \hat{5}_\gamma, 2_{\bar{Q}})|^2 \right] d\Phi_2(k_{1Q}, k_{2\bar{Q}}; p_{(4-3)g}, p_{5g}) J_2^{(2)}(k_1, k_2) \quad (7.18) \\
 &= g^2 C_A \frac{P_{g\bar{g} \rightarrow G}(z, \mu_{qg}^2)}{s_{34}} |M_4^0(1_Q, 2_{\bar{Q}}, (\widehat{4-3})_g, \hat{5}_g)|^2 \\
 &\quad \times d\Phi_2(k_{1Q}, k_{2\bar{Q}}; p_{(4-3)g}, p_{5g}) J_2^{(2)}(k_1, k_2).
 \end{aligned}$$

We have used eq. (7.16) with incoming momenta $(p_4 - k_3)$ and p_5 . In the $3_g || \hat{4}_g$ limit, the contributing terms in eq. (7.14) are proportional either to $D_3^0(4_g; 3_g, 1_Q)$ or to

$F_3^0(4_g, 5_g; 3_g)$. In the first case, the remapped momenta (13) and $\hat{4}$ present in the reduced matrix elements multiplying by $D_3^0(4_g; 3_g, 1_Q)$ are given by k_1 and $(p_4 - k_3)$ respectively. In the second case, the remapped momenta $\hat{4}$ and $\hat{5}$ present in the reduced matrix-element squared multiplying $F_3^0(4_g, 5_g; 3_g)$, are given by $(p_4 - k_3)$ and p_5 respectively.

Finally, for the $1_Q || \hat{4}_g$ limit, we see that

$$\begin{aligned}
 & d\hat{\sigma}_{gg \rightarrow Q\bar{Q}g} \xrightarrow{1_Q || \hat{4}_g} g^6 \left(\frac{N_c^2 - 1}{N_c} \right)^2 \frac{P_{q\bar{q} \leftarrow G}(z, \mu_{qg}^2)}{s_{14}} \\
 & \times \left[N_c^2 \left(|\mathcal{M}_4^0((\widehat{4-1})_{\bar{Q}}, \hat{5}_g, 3_g, 2_{\bar{Q}})|^2 + |\mathcal{M}_4^0((\widehat{4-1})_{\bar{Q}}, 3_g, \hat{5}_g, 2_{\bar{Q}})|^2 \right) \right. \\
 & \quad \left. - |\mathcal{M}_4^0((\widehat{4-1})_{\bar{Q}}, 3_\gamma, \hat{5}_\gamma, 2_{\bar{Q}})|^2 \right] d\Phi_2(k_{2\bar{Q}}, k_{3g}; p_{(4-1)\bar{Q}}, p_{5g}) J_2^{(2)}(k_2, k_3) \quad (7.19) \\
 & = g^2 C_F \frac{P_{q\bar{q} \leftarrow G}(z, \mu_{qg}^2)}{s_{14}} |\mathcal{M}_4^0((\widehat{4-1})_{\bar{Q}}, 2_{\bar{Q}}, 3_g, \hat{5}_g)|^2 \\
 & \quad \times d\Phi_2(k_{2\bar{Q}}, k_{3g}; p_{(4-1)\bar{Q}}, p_{5g}) J_2^{(2)}(k_2, k_3).
 \end{aligned}$$

Again, the relation between the full matrix element squared and the partial amplitudes given by eq. (7.16) has been used. In this case the final state quark denoted by 1_Q and the gluon denoted by 4_g have been crossed to the initial state.

Since colour decomposition does not distinguish between initial and final state coloured particles, this crossings can be safely done with the relation still being true. The terms in eq. (7.14) that contribute to this limit are those involving $A_3^0(4_g; 1_Q, 2_{\bar{Q}})$ and the corresponding remapped momenta appearing in the reduced squared amplitude multiplying this antenna become $k_{(\widehat{12})} \rightarrow k_2$ and $p_{\hat{4}} \rightarrow (p_4 - k_1)$.

These same verifications have been performed on all collinear and quasi-collinear limits of all subtraction terms listed in this section providing us with strong consistency check on our results for the subtraction terms required to compute the cross section for $pp \rightarrow t\bar{t}$ at NLO.

7.2 $t\bar{t} + jet$ production at LHC

The real emission correction to the production of a massive quark-antiquark pair in association with a jet in a hadronic collision is given by,

$$\begin{aligned}
 d\sigma^R = \int \frac{d\xi_1}{\xi_1} \frac{d\xi_2}{\xi_2} \left\{ \sum_q \left[f_q(\xi_1) f_{\bar{q}}(\xi_2) d\hat{\sigma}_{q\bar{q} \rightarrow Q\bar{Q}gg} + f_q(\xi_1) f_g(\xi_2) d\hat{\sigma}_{qg \rightarrow Q\bar{Q}qg} \right. \right. \\
 \quad + f_{\bar{q}}(\xi_1) f_g(\xi_2) d\hat{\sigma}_{\bar{q}g \rightarrow Q\bar{Q}\bar{q}g} + f_q(\xi_1) f_{\bar{q}}(\xi_2) d\hat{\sigma}_{q\bar{q} \rightarrow Q\bar{Q}q\bar{q}} \\
 \quad \left. \left. + f_q(\xi_1) f_q(\xi_2) d\hat{\sigma}_{qq \rightarrow Q\bar{Q}qq} + f_{\bar{q}}(\xi_1) f_{\bar{q}}(\xi_2) d\hat{\sigma}_{\bar{q}\bar{q} \rightarrow Q\bar{Q}\bar{q}\bar{q}} \right] \right. \quad (7.20) \\
 \quad + \sum_{q \neq q'} \left[f_q(\xi_1) f_q(\xi_2) d\hat{\sigma}_{q\bar{q} \rightarrow Q\bar{Q}q'q'} + f_q(\xi_1) f_{q'}(\xi_2) d\hat{\sigma}_{qq' \rightarrow Q\bar{Q}qq'} \right. \\
 \quad \left. + f_{\bar{q}}(\xi_1) f_{\bar{q}'}(\xi_2) d\hat{\sigma}_{\bar{q}\bar{q}' \rightarrow Q\bar{Q}\bar{q}\bar{q}'} + f_q(\xi_1) f_{q'}(\xi_2) d\hat{\sigma}_{q\bar{q}' \rightarrow Q\bar{Q}q\bar{q}'} \right] \\
 \quad \left. + f_g(\xi_1) f_g(\xi_2) (d\hat{\sigma}_{gg \rightarrow Q\bar{Q}gg} + d\hat{\sigma}_{gg \rightarrow Q\bar{Q}q\bar{q}}) \right\},
 \end{aligned}$$

with the partonic cross sections given by

$$d\hat{\sigma}_{q\bar{q}\rightarrow Q\bar{Q}gg} = d\Phi_4(k_Q, k_{\bar{Q}}, k_{g_1}, k_{g_2}; p_q, p_{\bar{q}}) |M_{q\bar{q}\rightarrow Q\bar{Q}gg}^0|^2 J_3^{(4)}(k_Q, k_{\bar{Q}}, k_{g_1}, k_{g_2}) \quad (7.21)$$

$$d\hat{\sigma}_{qg\rightarrow Q\bar{Q}qg} = d\Phi_4(k_Q, k_{\bar{Q}}, k_q, k_{q'}; p_q, p_g) |M_{qg\rightarrow Q\bar{Q}qg}^0|^2 J_3^{(4)}(k_Q, k_{\bar{Q}}, k_q, k_{q'}) \quad (7.22)$$

$$d\hat{\sigma}_{q\bar{q}\rightarrow Q\bar{Q}q'q'} = d\Phi_4(k_Q, k_{\bar{Q}}, k_{q'}, k_{q''}; p_q, p_{\bar{q}}) |M_{q\bar{q}\rightarrow Q\bar{Q}q'q'}^0|^2 J_3^{(4)}(k_Q, k_{\bar{Q}}, k_{q'}, k_{q''}) \quad (7.23)$$

$$d\hat{\sigma}_{qg\rightarrow Q\bar{Q}qg} = d\Phi_4(k_Q, k_{\bar{Q}}, k_q, k_{q'}; p_q, p_{g'}) |M_{qg\rightarrow Q\bar{Q}qg}^0|^2 J_3^{(4)}(k_Q, k_{\bar{Q}}, k_q, k_{q'}) \quad (7.24)$$

$$d\hat{\sigma}_{gg\rightarrow Q\bar{Q}gg} = d\Phi_4(k_Q, k_{\bar{Q}}, k_{g_1}, k_{g_2}; p_{g_3}, p_{g_4}) |M_{gg\rightarrow Q\bar{Q}gg}^0|^2 J_3^{(4)}(k_Q, k_{\bar{Q}}, k_{g_1}, k_{g_2}) \quad (7.25)$$

$$d\hat{\sigma}_{gq\rightarrow Q\bar{Q}q\bar{q}} = d\Phi_4(k_Q, k_{\bar{Q}}, k_q, k_{q'}; p_{g_1}, p_{g_2}) |M_{gq\rightarrow Q\bar{Q}q\bar{q}}^0|^2 J_3^{(4)}(k_Q, k_{\bar{Q}}, k_q, k_{q'}) \quad (7.26)$$

The jet functions appearing in the partonic cross section are all of the type $J_3^{(4)}$ and correspond to the selection criteria of 3-jet events. Out of four partons, from which two are a $Q\bar{Q}$ pair, an event with three jets is build. From these 3-jets, one jet is made of massless partons and two other jets have each a heavy quark Q or a heavy antiquark \bar{Q} in them. In the corresponding subtraction terms, the jet functions are of the type $J_3^{(3)}$ and those depend only on the hard final-state momenta appearing in the reduced matrix-element squared.

In order to obtain the colour decomposition of the matrix element squared for the partonic processes defined above and given in eq. (7.26), we will follow the same strategy as for $t\bar{t}$ production. We will use the colour decomposition of unphysical processes and then consider appropriate crossings. For $t\bar{t} + 1$ jet production the unphysical processes to be considered are $0 \rightarrow Q\bar{Q}q\bar{q}'q'$, $0 \rightarrow Q\bar{Q}q\bar{q}gg$ and $0 \rightarrow Q\bar{Q}gggg$.

However, the presence of one additional parton in the final state introduces a few difficulties. In the first place, the number of partial amplitudes as well as the number of unresolved limits to subtract increases. Also, identical quark flavour contributions must be considered. But most importantly, in the contributions related to the partonic processes derived from $0 \rightarrow Q\bar{Q}q\bar{q}gg$ and $0 \rightarrow Q\bar{Q}gggg$, the colour decomposition of the partonic amplitudes squared leads to interferences between partial amplitudes with different colour orderings that cannot be removed using any decoupling identities. Those require subtraction in an uncommon way in the antenna formalism, which we shall explain below.

7.2.1 Interference terms

The subtraction of infrared singularities in a colour ordered squared amplitude is easy within the antenna formalism: a suitable antenna function multiplied by a reduced squared matrix element with its momenta properly remapped accomplishes the task. However, this is not the case for interferences between partial amplitudes with different colour orderings. Those interference terms which are most generally of the form

$$\mathcal{M}_{n+1}^0(\dots, a, s, b, \dots) \mathcal{M}_{n+1}^0(\dots, c, s, d, \dots)^\dagger$$

with gluon s colour connected to partons a and b in one amplitude and colour-connected to partons c and d in the other amplitude, lead to soft singularities when integrated over the phase space. Those singular behaviours cannot be straightforwardly subtracted with

just one suitable antenna function multiplied by a reduced squared matrix element with remapped momenta.

In order to understand how to subtract the soft singularities in these interferences terms, we appeal to the factorisation properties of colour ordered matrix elements at the amplitude level. Quite generally, when a gluon s with helicity λ becomes soft between partons a and b , the colour ordered amplitude factorises [112, 113] as

$$\mathcal{M}_{n+1}^0(\dots, a, s_\lambda, b, \dots) \xrightarrow{k_s \rightarrow 0} \epsilon_\lambda(p_s) \cdot [J_a(p_s) - J_b(p_s)] \mathcal{M}_n^0(\dots, a, b, \dots), \quad (7.27)$$

where $\epsilon_\lambda(p_s)$ is the polarisation vector of the soft gluon, and $J_a^\mu(p_s)$ is the soft gluon current, given by

$$J_a^\mu(p_s) = \frac{p_a^\mu}{\sqrt{2} p_a \cdot p_s}. \quad (7.28)$$

Eq. (7.27) holds in d dimensions and is absolutely general: it does not depend on the identity of partons a and b (they can be either gluons, massive quarks or massless quarks) nor on their helicities. Therefore, summing over the helicities of the soft gluon, we find that the interference of two partial amplitudes with different colour orderings factorises in the soft limit, $k_s \rightarrow 0$, as

$$\begin{aligned} & \sum_{\lambda=\pm} \mathcal{M}_{n+1}^0(\dots, a, s_\lambda, b, \dots) \mathcal{M}_{n+1}^0(\dots, c, s_\lambda, d, \dots)^\dagger \\ & \xrightarrow{k_s \rightarrow 0} \left(\sum_{\lambda=\pm} \epsilon_\lambda^\mu(p_s) \epsilon_{-\lambda}^\nu(p_s) \right) [J_a(p_s) - J_b(p_s)]_\mu [J_c(p_s) - J_d(p_s)]_\nu \\ & \quad \times \mathcal{M}_n(\dots, a, b, \dots) \mathcal{M}_n(\dots, c, d, \dots)^\dagger. \end{aligned} \quad (7.29)$$

Since colour ordered amplitudes are gauge invariant, we can replace

$$\sum_{\lambda=\pm} \epsilon_\lambda^\mu(p_s) \epsilon_{-\lambda}^\nu(p_s) \rightarrow -g^{\mu\nu} \quad (7.30)$$

and thus obtain the following limit for the interference term:

$$\begin{aligned} & \mathcal{M}_{n+1}^0(\dots, a, s, b, \dots) \mathcal{M}_{n+1}^0(\dots, c, s, d, \dots)^\dagger \\ & \xrightarrow{k_s \rightarrow 0} \left(\frac{s_{ad}}{s_{as}s_{ds}} + \frac{s_{bc}}{s_{bs}s_{cs}} - \frac{s_{ac}}{s_{as}s_{cs}} - \frac{s_{bd}}{s_{bs}s_{ds}} \right) \mathcal{M}_n^0(\dots, a, b, \dots) \mathcal{M}_n^0(\dots, c, d, \dots)^\dagger. \end{aligned} \quad (7.31)$$

In this soft limit, we see that the interference term given by

$$\mathcal{M}_{n+1}^0(\dots, a, s, b, \dots) \mathcal{M}_{n+1}^0(\dots, c, s, d, \dots)^\dagger$$

factorises into a difference of four (half) eikonal factors times the interference of the reduced (i.e where the soft gluon is absent) interference term given by

$$\mathcal{M}_n^0(\dots, a, b, \dots) \mathcal{M}_n^0(\dots, c, d, \dots)^\dagger.$$

Each of these half eikonal factors can be reproduced by a single antenna function. The soft singularities of an interference term like $\mathcal{M}_{n+1}^0(\dots, a, s, b, \dots)\mathcal{M}_{n+1}^0(\dots, c, s, d, \dots)^\dagger$ for the case where a, b, c and d are all different can therefore be subtracted with

$$\begin{aligned} & \frac{1}{2}X_3^0(a, s, d)\mathcal{M}_{n,1}^0(\dots, \tilde{a}s, \tilde{b}s, \dots)\mathcal{M}_{n,1}^0(\dots, \tilde{c}s, \tilde{d}s, \dots)^\dagger \\ & + \frac{1}{2}X_3^0(b, s, c)\mathcal{M}_{n,2}^0(\dots, \tilde{a}s, \tilde{b}s, \dots)\mathcal{M}_{n,2}^0(\dots, \tilde{c}s, \tilde{d}s, \dots)^\dagger \\ & - \frac{1}{2}X_3^0(a, s, c)\mathcal{M}_{n,3}^0(\dots, \tilde{a}s, \tilde{b}s, \dots)\mathcal{M}_{n,3}^0(\dots, \tilde{c}s, \tilde{d}s, \dots)^\dagger \\ & - \frac{1}{2}X_3^0(b, s, d)\mathcal{M}_{n,4}^0(\dots, \tilde{a}s, \tilde{b}s, \dots)\mathcal{M}_{n,4}^0(\dots, \tilde{c}s, \tilde{d}s, \dots)^\dagger. \end{aligned} \tag{7.32}$$

In this equation, each of the antenna functions denoted as X_3^0 can be either final-final, initial-final or initial-initial, depending on whether the partons a, b, c and d are in the initial or final state, and they can be either massive or massless depending on whether the hard radiators are massive or massless. The absence of mass terms in eq. (7.31) is respected by eq. (7.32) even when one of the hard radiators (and therefore all the antenna functions involving that parton) is massive: In the soft limit, although a massive antenna yields a massive soft eikonal factor including explicitly mass terms, as seen in section 4, in the combination of massive antennae required in eq. (7.32) those mass terms cancel amongst each other.

In eq. (7.32), the different labels on the reduced matrix elements denoted by $\mathcal{M}_{n,i}^{(0)}$ do not mean that the matrix elements themselves are different, but that the momentum mapping is, in principle, different for each term. The remapping in each case is done in accordance with the type of antenna function involved.

A very important feature about this way of treating the interference terms is that not only soft singularities are subtracted with eq. (7.32), as it was originally intended, but also no “extra” collinear singularities are introduced by this subtraction term itself. The collinear limits of gluon s collinear with partons a, b, c and d are correctly dealt with. Indeed, if a and b are different from c and d , the interference term does not contain any collinear singularities. It has square-root singularities that do not give rise to ϵ poles upon integration. In this case, the subtraction terms in eq. (7.32) do not contain any collinear singularities either: all the collinear limits introduced by the first two terms, are subtracted by the last two. And, in the case where two partons are equal in eq. (7.29), for example, when a and c are the same, the interference term develops a collinear singularity when $a||s$. In the limit where s becomes soft, setting $c = a$ in eq. (7.29) gives

$$\begin{aligned} & \mathcal{M}_{n+1}^0(\dots, a, s, b, \dots)\mathcal{M}_{n+1}^0(\dots, a, s, d, \dots)^\dagger \\ & \xrightarrow{k_s \rightarrow 0} \left(\frac{s_{ad}}{s_{as}s_{ds}} + \frac{s_{ab}}{s_{as}s_{bs}} - \frac{s_{bd}}{s_{bs}s_{ds}} - \frac{m_a^2}{s_{as}^2} \right) \mathcal{M}_n^0(\dots, a, b, \dots)\mathcal{M}_n^0(\dots, c, d, \dots)^\dagger. \end{aligned} \tag{7.33}$$

The corresponding subtraction terms are

$$\begin{aligned} & \frac{1}{2}X_3^0(a, s, d)\mathcal{M}_{n,1}^0(\dots, \tilde{a}s, \tilde{b}s, \dots)\mathcal{M}_{n,1}^0(\dots, \tilde{a}s, \tilde{d}s, \dots)^\dagger \\ & + \frac{1}{2}X_3^0(a, s, b)\mathcal{M}_{n,2}^0(\dots, \tilde{a}s, \tilde{b}s, \dots)\mathcal{M}_{n,2}^0(\dots, \tilde{a}s, \tilde{d}s, \dots)^\dagger \end{aligned} \tag{7.34}$$

$$-\frac{1}{2}X_3^0(b, s, d)\mathcal{M}_{n,3}^0(\dots, \tilde{a}s, \tilde{b}s, \dots)\mathcal{M}_{n,3}^0(\dots, \tilde{a}s, \tilde{d}s, \dots)^\dagger,$$

from where it can be seen that these combination of subtraction terms also subtracts the $a||s$ (quasi)-collinear limit, in addition to the soft s limit.

Taking this construction of the subtraction terms required for the interference terms of massive amplitudes into account, let us now present our results for the real contributions and their corresponding subtraction terms for all partonic processes involved in $t\bar{t}$ +jet production at NLO. We start with processes involving only quarks and derived from $0 \rightarrow Q\bar{Q}q\bar{q}q'\bar{q}'$, which is the easiest case since only colour-ordered matrix-element squared are involved.

7.2.2 Processes derived from $0 \rightarrow Q\bar{Q}q\bar{q}q'\bar{q}'$

We choose to separate the colour decompositions of the real contributions related to processes derived from $0 \rightarrow Q\bar{Q}q\bar{q}q'\bar{q}'$ with and without identical quarks explicitly.

The colour decomposition for the unphysical partonic process $0 \rightarrow Q\bar{Q}q\bar{q}q'\bar{q}'$ in the non-identical flavour case ($q \neq q'$) is given by,

$$\begin{aligned} M_6^0(1_Q, 2_{\bar{Q}}, 3_q, 4_{\bar{q}}, 5_{q'}, 6_{\bar{q}'}) &= g^4 \left[\delta_{i_1, i_4} \delta_{i_3, i_6} \delta_{i_5, i_2} \mathcal{M}_6^0(1_Q, 4_{\bar{q}}; ; 3_q, 6_{\bar{q}'}; 5_{q'}, 2_{\bar{Q}}) \right. \\ &\quad + \delta_{i_1, i_6} \delta_{i_3, i_2} \delta_{i_5, i_4} \mathcal{M}_6^0(1_Q, 6_{\bar{q}'}; ; 3_q, 2_{\bar{Q}}; ; 5_{q'}, 4_{\bar{q}}) \\ &\quad - \frac{1}{N_c} \delta_{i_1, i_4} \delta_{i_3, i_2} \delta_{i_5, i_6} \mathcal{M}_6^0(1_Q, 4_{\bar{q}}; ; 3_q, 2_{\bar{Q}}; ; 5_{q'}, 6_{\bar{q}'}) \\ &\quad - \frac{1}{N_c} \delta_{i_1, i_6} \delta_{i_3, i_4} \delta_{i_5, i_2} \mathcal{M}_6^0(1_Q, 6_{\bar{q}'}; ; 3_q, 4_{\bar{q}}; ; 5_{q'}, 2_{\bar{Q}}) \quad (7.35) \\ &\quad - \frac{1}{N_c} \delta_{i_1, i_2} \delta_{i_3, i_6} \delta_{i_5, i_4} \mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, 6_{\bar{q}'}; ; 5_{q'}, 4_{\bar{q}}) \\ &\quad \left. + \frac{1}{N_c^2} \delta_{i_1, i_2} \delta_{i_3, i_4} \delta_{i_5, i_6} \mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, 4_{\bar{q}}; ; 5_{q'}, 6_{\bar{q}'}) \right]. \end{aligned}$$

Squaring, and using the fact that the partial amplitudes satisfy

$$\begin{aligned} &M_6^0(1_Q, 2_{\bar{Q}}; 3_q, 4_{\bar{q}}; 5_{q'}, 6_{\bar{q}'}) \\ &= \mathcal{M}_6^0(1_Q, 4_{\bar{q}}; ; 3_q, 6_{\bar{q}'}; ; 5_{q'}, 2_{\bar{Q}}) + \mathcal{M}_6^0(1_Q, 6_{\bar{q}'}; ; 3_q, 2_{\bar{Q}}; ; 5_{q'}, 4_{\bar{q}}) \quad (7.36) \\ &= \frac{1}{2} \left(\mathcal{M}_6^0(1_Q, 4_{\bar{q}}; ; 3_q, 2_{\bar{Q}}; 5_{q'}, 6_{\bar{q}'}) + \mathcal{M}_6^0(1_Q, 6_{\bar{q}'}; ; 3_q, 4_{\bar{q}}; 5_{q'}, 2_{\bar{Q}}) + \mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, 6_{\bar{q}'}; 5_{q'}, 4_{\bar{q}}) \right) \end{aligned}$$

gives

$$\begin{aligned} &|M_6^0(1_Q, 2_{\bar{Q}}, 3_q, 4_{\bar{q}}, 5_{q'}, 6_{\bar{q}'})|^2 = g^8(N_c^2 - 1) \\ &\quad \left\{ N_c \left(|\mathcal{M}_6^0(1_Q, 4_{\bar{q}}; ; 3_q, 6_{\bar{q}'}; ; 5_{q'}, 2_{\bar{Q}})|^2 + |\mathcal{M}_6^0(1_Q, 6_{\bar{q}'}; ; 3_q, 2_{\bar{Q}}; ; 5_{q'}, 4_{\bar{q}})|^2 \right) \right. \\ &\quad + \frac{1}{N_c} \left(|\mathcal{M}_6^0(1_Q, 4_{\bar{q}}; ; 3_q, 2_{\bar{Q}}; ; 5_{q'}, 6_{\bar{q}'})|^2 + |\mathcal{M}_6^0(1_Q, 6_{\bar{q}'}; ; 3_q, 4_{\bar{q}}; ; 5_{q'}, 2_{\bar{Q}})|^2 \right. \\ &\quad \left. \left. + |\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, 6_{\bar{q}'}; ; 5_{q'}, 4_{\bar{q}})|^2 - 3|\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, 4_{\bar{q}}; ; 5_{q'}, 6_{\bar{q}'})|^2 \right) \right\} \quad (7.37) \end{aligned}$$

which represents the colour decomposition for squared amplitude of the unphysical process $0 \rightarrow Q\bar{Q}q\bar{q}q'\bar{q}'$ (with q and q' of different flavour). It has no interference terms. Depending

on which partons are crossed in eq. (7.37) a sum over final state quark flavours might need to be performed. When a quark-antiquark pair of the same flavour is crossed to the initial state resulting in the the colour ordered squared amplitude for the process $q\bar{q} \rightarrow Q\bar{Q}q'\bar{q}'$, the result must be multiplied² by $(N_F - 1)$.

The identical flavour case is obtained taking the difference of two colour-ordered non-identical cases. At the amplitude level, we use the following relation

$$M_6^0(1_Q, 2_{\bar{Q}}, 3_q, 4_{\bar{q}}, 5_q, 6_{\bar{q}}) = M_6^0(1_Q, 2_{\bar{Q}}, 3_q, 4_{\bar{q}}, 5_{q'}, 6_{\bar{q}'}) - M_6^0(1_Q, 2_{\bar{Q}}, 3_q, 6_{\bar{q}}, 5_{q'}, 4_{\bar{q}'}), \quad (7.38)$$

which, upon squaring, gives the colour decomposition of the matrix-element squared for the identical flavour case,

$$\begin{aligned} & |M_6^0(1_Q, 2_{\bar{Q}}, 3_q, 4_{\bar{q}}, 5_q, 6_{\bar{q}})|^2 = g^8(N_c^2 - 1) \\ & \times \left\{ N_c \left(|\mathcal{M}_6^0(1_Q, 4_{\bar{q}}; ; 3_q, 6_{\bar{q}'}; ; 5_{q'}, 2_{\bar{Q}})|^2 + |\mathcal{M}_6^0(1_Q, 6_{\bar{q}}; ; 3_q, 4_{\bar{q}'}; ; 5_{q'}, 2_{\bar{Q}})|^2 \right. \right. \\ & \quad \left. \left. + |\mathcal{M}_6^0(1_Q, 6_{\bar{q}'}; ; 3_q, 2_{\bar{Q}}; ; 5_{q'}, 4_{\bar{q}})|^2 + |\mathcal{M}_6^0(1_Q, 4_{\bar{q}'}; ; 3_q, 2_{\bar{Q}}; ; 5_{q'}, 6_{\bar{q}})|^2 \right) \right. \\ & \quad + 2\text{Re}(\mathcal{M}_6^0(1_Q, 4_{\bar{q}'}; ; 3_q, 2_{\bar{Q}}; ; 5_{q'}, 6_{\bar{q}})\mathcal{M}_6^0(1_Q, 4_{\bar{q}}; ; 3_q, 2_{\bar{Q}}; ; 5_{q'}, 6_{\bar{q}'}))^\dagger \\ & \quad + 2\text{Re}(\mathcal{M}_6^0(1_Q, 4_{\bar{q}'}; ; 3_q, 6_{\bar{q}}; ; 5_{q'}, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, 4_{\bar{q}}; ; 3_q, 6_{\bar{q}'}; ; 5_{q'}, 2_{\bar{Q}}))^\dagger \\ & \quad + 2\text{Re}(\mathcal{M}_6^0(1_Q, 6_{\bar{q}}; ; 3_q, 2_{\bar{Q}}; ; 5_{q'}, 4_{\bar{q}'})\mathcal{M}_6^0(1_Q, 6_{\bar{q}'}; ; 3_q, 2_{\bar{Q}}; ; 5_{q'}, 4_{\bar{q}}))^\dagger \\ & \quad + 2\text{Re}(\mathcal{M}_6^0(1_Q, 6_{\bar{q}}; ; 3_q, 4_{\bar{q}'}; ; 5_{q'}, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, 6_{\bar{q}'}; ; 3_q, 4_{\bar{q}}; ; 5_{q'}, 2_{\bar{Q}}))^\dagger \\ & \quad \left. - 2\text{Re}(\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, 6_{\bar{q}}; ; 5_{q'}, 4_{\bar{q}'})\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, 4_{\bar{q}}; ; 5_{q'}, 6_{\bar{q}'}))^\dagger \right) \quad (7.39) \\ & + \frac{1}{N_c} \left(|\mathcal{M}_6^0(1_Q, 4_{\bar{q}}; ; 3_q, 2_{\bar{Q}}; ; 5_{q'}, 6_{\bar{q}'})|^2 + |\mathcal{M}_6^0(1_Q, 6_{\bar{q}}; ; 3_q, 2_{\bar{Q}}; ; 5_{q'}, 4_{\bar{q}'})|^2 \right. \\ & \quad + |\mathcal{M}_6^0(1_Q, 6_{\bar{q}'}; ; 3_q, 4_{\bar{q}}; ; 5_{q'}, 2_{\bar{Q}})|^2 + |\mathcal{M}_6^0(1_Q, 4_{\bar{q}'}; ; 3_q, 6_{\bar{q}}; ; 5_{q'}, 2_{\bar{Q}})|^2 \\ & \quad + |\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, 6_{\bar{q}'}; ; 5_{q'}, 4_{\bar{q}})|^2 + |\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, 4_{\bar{q}'}; ; 5_{q'}, 6_{\bar{q}})|^2 \\ & \quad \left. - 3|\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, 4_{\bar{q}}; ; 5_{q'}, 6_{\bar{q}'})|^2 - 3|\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, 6_{\bar{q}}; ; 5_{q'}, 4_{\bar{q}'})|^2 \right) \\ & - \frac{1}{N_c^2} \left(6\text{Re}(\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, 6_{\bar{q}}; ; 5_{q'}, 4_{\bar{q}'})\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, 4_{\bar{q}}; ; 5_{q'}, 6_{\bar{q}'})^\dagger) \right. \\ & \quad - 2\text{Re}(\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, 6_{\bar{q}}; ; 5_{q'}, 4_{\bar{q}'})\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, 6_{\bar{q}'}; ; 5_{q'}, 4_{\bar{q}})^\dagger) \\ & \quad \left. - 2\text{Re}(\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, 4_{\bar{q}'}; ; 5_{q'}, 6_{\bar{q}})\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, 4_{\bar{q}}; ; 5_{q'}, 6_{\bar{q}'})^\dagger) \right) \}. \end{aligned}$$

The interference terms present in eq. (7.39) lead only to finite contributions when integrated over the phase space. Indeed, they lead to square root singularities which do not need subtraction.

Furthermore, apart from those interference terms, the identical-quark flavour contributions involve all six colour-ordered matrix element squared appearing in the non-identical flavour case plus additional ones. In all cases, identical or non-identical quark and in all

²Note that the result needs to be multiplied by $(N_F - 1)$ and not N_F since we are explicitly separating the identical flavour contributions from the non-identical ones.

crossings required, the only singular behaviours which have to be captured by the antennae in the subtraction terms are final-final and initial-final collinear singularities between identical-flavour massless quark-(anti)quarks. Due to the presence of a massive radiator in the final state, only one antenna, the quark-gluon massive antenna E_3^0 is required in final-final and initial-final configurations.

After the crossing of 3_q and $4_{\bar{q}}$ to the initial state, in the non-identical matrix element squared for the process $0 \rightarrow Q\bar{Q}q\bar{q}'\bar{q}'$ given in eq. (7.37) we obtain the colour ordered matrix element squared for $q\bar{q} \rightarrow Q\bar{Q}q'\bar{q}'$. Its subtraction term is,

$$\begin{aligned}
 d\hat{\sigma}_{q\bar{q} \rightarrow Q\bar{Q}q'\bar{q}'}^S &= \frac{g^8(N_c^2 - 1)(N_F - 1)}{2} d\Phi_4(k_{1Q}, k_{2\bar{Q}}, k_{5q'}, k_{6\bar{q}'}; p_{3\bar{q}}, p_{4q}) \\
 &\times \left\{ N_c \left[E_3^0(1_Q, 5_{q'}, 6_{\bar{q}'}') \left(|\mathcal{M}_5^0((\tilde{15})_Q, \hat{4}_q; ; \hat{3}_{\bar{q}}, (\tilde{56})_g, 2_{\bar{Q}})|^2 \right. \right. \right. \\
 &\quad \left. \left. \left. + |\mathcal{M}_5^0((\tilde{15})_Q, (\tilde{56})_g, \hat{4}_q; ; \hat{3}_{\bar{q}}, 2_{\bar{Q}})|^2 \right) J_3^{(3)}(k_{\tilde{15}}, k_2, k_{\tilde{56}}) \right. \right. \\
 &\quad \left. \left. + E_3^0(2_{\bar{Q}}, 5_{q'}, 6_{\bar{q}'}') \left(|\mathcal{M}_5^0(1_Q, \hat{4}_q; ; \hat{3}_{\bar{q}}, (\tilde{56})_g, (25)_{\bar{Q}})|^2 \right. \right. \right. \\
 &\quad \left. \left. \left. + |\mathcal{M}_5^0(1_Q, (\tilde{56})_g, \hat{4}_q; ; \hat{3}_{\bar{q}}, (25)_{\bar{Q}})|^2 \right) J_3^{(3)}(k_1, k_{\tilde{25}}, k_{\tilde{56}}) \right] \right. \\
 &\quad \left. + \frac{1}{N_c} \left[E_3^0(1_Q, 5_{q'}, 6_{\bar{q}'}') \left(|\mathcal{M}_5^0((\tilde{15})_Q, 2_{\bar{Q}}; ; \hat{3}_{\bar{q}}, (\tilde{56})_g, \hat{4}_q)|^2 \right. \right. \right. \\
 &\quad \left. \left. \left. + |\mathcal{M}_5^0((\tilde{15})_Q, (\tilde{56})_g, 2_{\bar{Q}}; ; \hat{3}_{\bar{q}}, \hat{4}_q)|^2 \right) J_3^{(3)}(k_{\tilde{15}}, k_2, k_{\tilde{56}}) \right. \right. \\
 &\quad \left. \left. - 2|\mathcal{M}_5^0((\tilde{15})_Q, 2_{\bar{Q}}, \hat{3}_{\bar{q}}, \hat{4}_q, (\tilde{56})_\gamma)|^2 \right) J_3^{(3)}(k_{\tilde{15}}, k_2, k_{\tilde{56}}) \right. \right. \\
 &\quad \left. \left. + E_3^0(2_{\bar{Q}}, 5_{q'}, 6_{\bar{q}'}') \left(|\mathcal{M}_5^0(1_Q, (2\tilde{5})_{\bar{Q}}; ; \hat{3}_{\bar{q}}, (\tilde{56})_g, \hat{4}_q)|^2 \right. \right. \right. \\
 &\quad \left. \left. \left. + |\mathcal{M}_5^0(1_Q, (\tilde{56})_g, (2\tilde{5})_{\bar{Q}}; ; \hat{3}_{\bar{q}}, \hat{4}_q)|^2 \right. \right. \right. \\
 &\quad \left. \left. \left. - 2|\mathcal{M}_5^0(1_Q, (2\tilde{5})_{\bar{Q}}, \hat{3}_{\bar{q}}, \hat{4}_q, (\tilde{56})_\gamma)|^2 \right) J_3^{(3)}(k_1, k_{\tilde{25}}, k_{\tilde{56}}) \right] \right\}. \tag{7.40}
 \end{aligned}$$

If, instead, $4_{\bar{q}}$ and $6_{\bar{q}'}$ are crossed in eq. (7.37), we obtain the squared matrix element for the process $q\bar{q}' \rightarrow Q\bar{Q}q\bar{q}'$. The corresponding subtraction term reads

$$\begin{aligned}
 d\hat{\sigma}_{q\bar{q}' \rightarrow Q\bar{Q}q\bar{q}'}^S &= \frac{g^8(N_c^2 - 1)}{2} d\Phi_4(k_{1Q}, k_{2\bar{Q}}, k_{3q}, k_{5q'}; p_{4q}, p_{6q'}) \\
 &\times \left\{ N_c \left[E_3^0(4_q; 3_q, 1_Q) \left(|\mathcal{M}_5^0((\tilde{13})_Q, \hat{6}_{q'}; ; 5_{q'}, \hat{4}_g, 2_{\bar{Q}})|^2 \right. \right. \right. \\
 &\quad \left. \left. \left. + |\mathcal{M}_5^0((\tilde{13})_Q, \hat{4}_g, \hat{6}_{q'}; ; 5_{q'}, 2_{\bar{Q}})|^2 \right) J_3^{(3)}(k_{\tilde{13}}, k_2, k_5) \right. \right. \\
 &\quad \left. \left. + E_3^0(4_q; 3_q, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0(1_Q, \hat{6}_{q'}; ; 5_{q'}, \hat{4}_g, (\tilde{23})_{\bar{Q}})|^2 \right. \right. \right. \\
 &\quad \left. \left. \left. + |\mathcal{M}_5^0(1_Q, \hat{4}_g, \hat{6}_{q'}; ; 5_{q'}, (\tilde{23})_{\bar{Q}})|^2 \right) J_3^{(3)}(k_1, k_{\tilde{23}}, k_5) \right. \right. \\
 &\quad \left. \left. + E_3^0(6_{q'}; 5_{q'}, 1_Q) \left(|\mathcal{M}_5^0((\tilde{15})_Q, \hat{4}_q; ; 3_q, \hat{6}_g, 2_{\bar{Q}})|^2 \right. \right. \right. \\
 &\quad \left. \left. \left. + |\mathcal{M}_5^0((\tilde{15})_Q, \hat{6}_g, \hat{4}_q; ; 3_q, 2_{\bar{Q}})|^2 \right) J_3^{(3)}(k_{\tilde{15}}, k_2, k_3) \right. \right. \\
 &\quad \left. \left. + E_3^0(6_{q'}; 5_{q'}, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0(1_Q, \hat{4}_q; ; 3_q, \hat{6}_g, (\tilde{25})_{\bar{Q}})|^2 \right. \right. \right.
 \end{aligned}$$

$$\begin{aligned}
 & + |\mathcal{M}_5^0(1_Q, \hat{6}_g, \hat{4}_q; ; 3_q, (\tilde{25})_{\bar{Q}})|^2 \Big) J_3^{(3)}(k_1, k_{\tilde{25}}, k_3) \Big] \\
 + \frac{1}{N_c} & \left[E_3^0(4_q; 3_q, 1_Q) \left(|\mathcal{M}_5^0((\tilde{13})_Q, 2_{\bar{Q}}; ; 5_{q'}, \hat{4}_g, \hat{6}_{q'})|^2 \right. \right. \\
 & + |\mathcal{M}_5^0((\tilde{13})_Q, \hat{4}_g, 2_{\bar{Q}}; ; 5_{q'}, \hat{6}_{q'})|^2 \\
 & - 2 |\mathcal{M}_5^0((\tilde{13})_Q, 2_{\bar{Q}}, 5_{q'}, \hat{6}_{q'}, \hat{4}_\gamma)|^2 \Big) J_3^{(3)}(k_{\tilde{13}}, k_2, k_5) \\
 + E_3^0(4_q; 3_q, 2_{\bar{Q}}) & \left(|\mathcal{M}_5^0(1_Q, (\tilde{23})_{\bar{Q}}; ; 5_{q'}, \hat{4}_g, \hat{6}_{q'})|^2 \right. \\
 & + |\mathcal{M}_5^0(1_Q, \hat{4}_g, (\tilde{23})_{\bar{Q}}; ; 5_{q'}, \hat{6}_{q'})|^2 \\
 & - 2 |\mathcal{M}_5^0(1_Q, (\tilde{23})_{\bar{Q}}, 5_{q'}, \hat{6}_{q'}, \hat{4}_\gamma)|^2 \Big) J_3^{(3)}(k_1, k_{\tilde{23}}, k_5) \\
 + E_3^0(6_{q'}; 5_{q'}, 1_Q) & \left(|\mathcal{M}_5^0((\tilde{15})_Q, 2_{\bar{Q}}; ; 3_q, \hat{6}_g, \hat{4}_q)|^2 \right. \\
 & + |\mathcal{M}_5^0((\tilde{15})_Q, \hat{6}_g, 2_{\bar{Q}}; ; 3_q, \hat{4}_q)|^2 \\
 & - 2 |\mathcal{M}_5^0((\tilde{15})_Q, 2_{\bar{Q}}, 3_q, \hat{4}_q, \hat{6}_\gamma)|^2 \Big) J_3^{(3)}(k_{\tilde{15}}, k_2, k_3) \\
 + E_3^0(6_{q'}; 5_{q'}, 2_{\bar{Q}}) & \left(|\mathcal{M}_5^0(1_Q, (\tilde{25})_{\bar{Q}}; ; 3_q, \hat{6}_g, \hat{4}_q)|^2 \right. \\
 & + |\mathcal{M}_5^0(1_Q, \hat{6}_g, (\tilde{25})_{\bar{Q}}; ; 3_q, \hat{4}_q)|^2 \\
 & \left. \left. + |\mathcal{M}_5^0(1_Q, (\tilde{25})_{\bar{Q}}, 3_q, \hat{4}_q, \hat{6}_\gamma)|^2 \right) J_3^{(3)}(k_1, k_{\tilde{25}}, k_3) \right] \Big\}.
 \end{aligned} \tag{7.41}$$

For the identical flavour-case, the same two crossings, 3_q and $4_{\bar{q}}$ on one hand and the crossing of $4_{\bar{q}}$ and $6_{\bar{q}}$ on the other hand need to be considered. As the real matrix-element in the identical quark case are given by those in the non-identical quark case plus additional terms, similarly the subtraction terms for the identical-quark case are obtained by adding to the subtraction terms valid for the non-identical case additional subtraction terms related to the new colour ordered matrix-element squared appearing only in the identical-flavour case only and given in eq. (7.39).

If 3_q and $4_{\bar{q}}$ are crossed in the identical flavour squared matrix element given in eq. (7.39), the squared matrix element for $q\bar{q} \rightarrow Q\bar{Q}q\bar{q}$ is obtained in a colour ordered way. The subtraction term for this process is,

$$\begin{aligned}
 d\hat{\sigma}_{q\bar{q} \rightarrow Q\bar{Q}q\bar{q}}^S &= \frac{g^8(N_c^2 - 1)}{2} d\Phi_4(k_{1Q}, k_{2\bar{Q}}, k_{5q}, k_{6\bar{q}}; p_{3\bar{q}}, p_{4q}) \\
 & \times \left\{ N_c \left[E_3^0(1_Q, 5_q, 6_{\bar{q}}) \left(|\mathcal{M}_5^0((\tilde{15})_Q, \hat{4}_q; ; \hat{3}_{\bar{q}}, (\tilde{56})_g, 2_{\bar{Q}})|^2 \right. \right. \right. \\
 & \quad + |\mathcal{M}_5^0((\tilde{15})_Q, (\tilde{56})_g, \hat{4}_q; ; \hat{3}_{\bar{q}}, 2_{\bar{Q}})|^2 \Big) J_3^{(3)}(k_{\tilde{15}}, k_2, k_{\tilde{56}}) \\
 + E_3^0(2_{\bar{Q}}, 5_q, 6_{\bar{q}}) & \left(|\mathcal{M}_5^0(1_Q, \hat{4}_q; ; \hat{3}_{\bar{q}}, (\tilde{56})_g, (\tilde{25})_{\bar{Q}})|^2 \right. \\
 & \quad + |\mathcal{M}_5^0(1_Q, (\tilde{56})_g, \hat{4}_q; ; \hat{3}_{\bar{q}}, (\tilde{25})_{\bar{Q}})|^2 \Big) J_3^{(3)}(k_1, k_{\tilde{25}}, k_{\tilde{56}}) \\
 + E_3^0(4_q; 5_q, 1_Q) & \left(|\mathcal{M}_5^0((\tilde{15})_Q, 6_{\bar{q}}; ; \hat{3}_{\bar{q}}, \hat{4}_g, 2_{\bar{Q}})|^2 \right. \\
 & \quad \left. \left. + |\mathcal{M}_5^0((\tilde{15})_Q, \hat{4}_g, 6_{\bar{q}}; ; \hat{3}_{\bar{q}}, 2_{\bar{Q}})|^2 \right) J_3^{(3)}(k_{\tilde{15}}, k_2, k_6) \right\}
 \end{aligned}$$

$$\begin{aligned}
 & +E_3^0(4_q; 5_q, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0(1_Q, 6_{\bar{q}}; ; \hat{3}_{\bar{q}}, \hat{4}_g, (\tilde{25})_{\bar{Q}})|^2 \right. \\
 & \quad \left. + |\mathcal{M}_5^0(1_Q, \hat{4}_g, 6_{\bar{q}}; ; \hat{3}_{\bar{q}}, (\tilde{25})_{\bar{Q}})|^2 \right) J_3^{(3)}(k_1, k_{\tilde{25}}, k_6) \\
 & +E_3^0(3_{\bar{q}}; 6_{\bar{q}}, 1_Q) \left(|\mathcal{M}_5^0((\tilde{16})_Q, \hat{4}_q; ; 5_q, \hat{3}_g, 2_{\bar{Q}})|^2 \right. \\
 & \quad \left. + |\mathcal{M}_5^0((\tilde{16})_Q, \hat{3}_g, \hat{4}_q; ; 5_q, 2_{\bar{Q}})|^2 \right) J_3^{(3)}(k_{\tilde{16}}, k_2, k_5) \\
 & +E_3^0(3_{\bar{q}}; 6_{\bar{q}}, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0(1_Q, \hat{4}_q; ; 5_q, \hat{3}_g, (\tilde{26})_{\bar{Q}})|^2 \right. \\
 & \quad \left. + |\mathcal{M}_5^0(1_Q, \hat{3}_g, \hat{4}_q; ; 5_q, (\tilde{26})_{\bar{Q}})|^2 \right) J_3^{(3)}(k_1, k_{\tilde{26}}, k_5) \Big] \\
 & + \frac{1}{N_c} \left[E_3^0(1_Q, 5_q, 6_{\bar{q}}) \left(|\mathcal{M}_5^0((\tilde{15})_Q, 2_{\bar{Q}}; ; \hat{3}_{\bar{q}}, (\tilde{56})_g, \hat{4}_q)|^2 \right. \right. \\
 & \quad \left. \left. + |\mathcal{M}_5^0((\tilde{15})_Q, (\tilde{56})_g, 2_{\bar{Q}}; ; \hat{3}_{\bar{q}}, \hat{4}_q)|^2 \right. \right. \\
 & \quad \left. \left. - 2|\mathcal{M}_5^0((\tilde{15})_Q, 2_{\bar{Q}}, \hat{3}_{\bar{q}}, \hat{4}_q, (\tilde{56})_\gamma)|^2 \right) J_3^{(3)}(k_{\tilde{15}}, k_2, k_{\tilde{56}}) \right. \\
 & +E_3^0(2_{\bar{Q}}, 5_q, 6_{\bar{q}}) \left(|\mathcal{M}_5^0(1_Q, (\tilde{25})_{\bar{Q}}; ; \hat{3}_{\bar{q}}, (\tilde{56})_g, \hat{4}_q)|^2 \right. \\
 & \quad \left. + |\mathcal{M}_5^0(1_Q, (\tilde{56})_g, (\tilde{25})_{\bar{Q}}; ; \hat{3}_{\bar{q}}, \hat{4}_q)|^2 \right. \\
 & \quad \left. - 2|\mathcal{M}_5^0(1_Q, (\tilde{25})_{\bar{Q}}, \hat{3}_{\bar{q}}, \hat{4}_q, (\tilde{56})_\gamma)|^2 \right) J_3^{(3)}(k_1, k_{\tilde{25}}, k_{\tilde{56}}) \\
 & +E_3^0(4_q; 5_q, 1_Q) \left(|\mathcal{M}_5^0((\tilde{15})_Q, 2_{\bar{Q}}; ; \hat{3}_{\bar{q}}, \hat{4}_g, 6_{\bar{q}})|^2 \right. \\
 & \quad \left. + |\mathcal{M}_5^0((\tilde{15})_Q, \hat{4}_g, 2_{\bar{Q}}; ; \hat{3}_{\bar{q}}, 6_{\bar{q}})|^2 \right. \\
 & \quad \left. - 2|\mathcal{M}_5^0((\tilde{15})_Q, 2_{\bar{Q}}, \hat{3}_{\bar{q}}, 6_{\bar{q}}, \hat{4}_\gamma)|^2 \right) J_3^{(3)}(k_{\tilde{15}}, k_2, k_6) \\
 & +E_3^0(4_q; 5_q, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0(1_Q, (\tilde{25})_{\bar{Q}}; ; \hat{3}_{\bar{q}}, \hat{4}_g, 6_{\bar{q}})|^2 \right. \\
 & \quad \left. + |\mathcal{M}_5^0(1_Q, \hat{4}_g, (\tilde{25})_{\bar{Q}}; ; \hat{3}_{\bar{q}}, 6_{\bar{q}})|^2 \right. \\
 & \quad \left. - 2|\mathcal{M}_5^0(1_Q, (\tilde{25})_{\bar{Q}}, \hat{3}_{\bar{q}}, 6_{\bar{q}}, \hat{4}_\gamma)|^2 \right) J_3^{(3)}(k_1, k_{\tilde{25}}, k_6) \\
 & +E_3^0(3_{\bar{q}}; 6_{\bar{q}}, 1_Q) \left(|\mathcal{M}_5^0((\tilde{16})_Q, 2_{\bar{Q}}; ; 5_q, \hat{3}_g, \hat{4}_q)|^2 \right. \\
 & \quad \left. + |\mathcal{M}_5^0((\tilde{16})_Q, \hat{3}_g, 2_{\bar{Q}}; ; 5_q, \hat{4}_{\bar{q}})|^2 \right. \\
 & \quad \left. - 2|\mathcal{M}_5^0((\tilde{16})_Q, 2_{\bar{Q}}, 5_q, \hat{4}_q, \hat{3}_\gamma)|^2 \right) J_3^{(3)}(k_{\tilde{16}}, k_2, k_5) \\
 & +E_3^0(3_{\bar{q}}; 6_{\bar{q}}, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0(1_Q, (\tilde{26})_{\bar{Q}}; ; 5_q, \hat{3}_g, \hat{4}_q)|^2 \right. \\
 & \quad \left. + |\mathcal{M}_5^0(1_Q, \hat{3}_g, (\tilde{26})_{\bar{Q}}; ; 5_q, \hat{4}_q)|^2 \right. \\
 & \quad \left. - 2|\mathcal{M}_5^0(1_Q, (\tilde{26})_{\bar{Q}}, 5_q, \hat{4}_q, \hat{3}_\gamma)|^2 \right) J_3^{(3)}(k_1, k_{\tilde{26}}, k_5) \Big] \Big\}.
 \end{aligned} \tag{7.42}$$

The squared matrix element for $qq \rightarrow Q\bar{Q}qq$ in terms of colour ordered partial amplitudes is obtained by crossing $4_{\bar{q}}$ and $6_{\bar{q}}$ in eq. (7.39). In this case, the subtraction term reads

$$\begin{aligned}
 d\hat{\sigma}_{qq \rightarrow Q\bar{Q}qq}^S &= \frac{g^8(N_c^2 - 1)}{2} d\Phi_4(k_{1Q}, k_{2\bar{Q}}, k_{3q}, k_{5q}; p_{4q}, p_{6q}) \\
 &\times \left\{ N_c \left[E_3^0(4_q; 3_q, 1_Q) \left(|\mathcal{M}_5^0((\tilde{13})_Q, \hat{6}_q; ; 5_q, \hat{4}_g, 2_{\bar{Q}})|^2 \right. \right. \right.
 \end{aligned}$$

$$\begin{aligned}
 & + |\mathcal{M}_5^0((\tilde{13})_Q, \hat{4}_g, \hat{6}_q; ; 5_q, 2_{\bar{Q}})|^2 \Big) J_3^{(3)}(k_{\tilde{13}}, k_2, k_5) \\
 & + E_3^0(4_q; 3_q, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0(1_Q, \hat{6}_q; ; 5_q, \hat{4}_g, (\tilde{23})_{\bar{Q}})|^2 \right. \\
 & \quad \left. + |\mathcal{M}_5^0(1_Q, \hat{4}_g, \hat{6}_q; ; 5_q, (\tilde{23})_{\bar{Q}})|^2 \right) J_3^{(3)}(k_1, k_{\tilde{23}}, k_5) \\
 & + E_3^0(4_q; 5_q, 1_Q) \left(|\mathcal{M}_5^0((\tilde{15})_Q, \hat{6}_q; ; 3_q, \hat{4}_g, 2_{\bar{Q}})|^2 \right. \\
 & \quad \left. + |\mathcal{M}_5^0((\tilde{15})_Q, \hat{4}_g, \hat{6}_q; ; 3_q, 2_{\bar{Q}})|^2 \right) J_3^{(3)}(k_{\tilde{15}}, k_2, k_3) \\
 & + E_3^0(4_q; 5_q, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0(1_Q, \hat{6}_q; ; 3_q, \hat{4}_g, (\tilde{25})_{\bar{Q}})|^2 \right. \\
 & \quad \left. + |\mathcal{M}_5^0(1_Q, \hat{4}_g, \hat{6}_q; ; 3_q, (\tilde{25})_{\bar{Q}})|^2 \right) J_3^{(3)}(k_1, k_{\tilde{25}}, k_3) \\
 & + E_3^0(6_q; 5_q, 1_Q) \left(|\mathcal{M}_5^0((\tilde{15})_Q, \hat{4}_q; ; 3_q, \hat{6}_g, 2_{\bar{Q}})|^2 \right. \\
 & \quad \left. + |\mathcal{M}_5^0((\tilde{15})_Q, \hat{6}_g, \hat{4}_q; ; 3_q, 2_{\bar{Q}})|^2 \right) J_3^{(3)}(k_{\tilde{15}}, k_2, k_3) \\
 & + E_3^0(6_q; 3_q, 1_Q) \left(|\mathcal{M}_5^0((\tilde{13})_Q, \hat{4}_q; ; 5_q, \hat{6}_g, 2_{\bar{Q}})|^2 \right. \\
 & \quad \left. + |\mathcal{M}_5^0((\tilde{13})_Q, \hat{6}_g, \hat{4}_q; ; 5_q, 2_{\bar{Q}})|^2 \right) J_3^{(3)}(k_{\tilde{13}}, k_2, k_5) \\
 & + E_3^0(6_q; 5_q, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0(1_Q, \hat{4}_q; ; 3_q, \hat{6}_g, (\tilde{25})_{\bar{Q}})|^2 \right. \\
 & \quad \left. + |\mathcal{M}_5^0(1_Q, \hat{6}_g, \hat{4}_q; ; 3_q, (\tilde{25})_{\bar{Q}})|^2 \right) J_3^{(3)}(k_1, k_{\tilde{25}}, k_3) \\
 & + E_3^0(6_q; 3_q, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0(1_Q, \hat{4}_q; ; 5_q, \hat{6}_g, (\tilde{23})_{\bar{Q}})|^2 \right. \\
 & \quad \left. + |\mathcal{M}_5^0(1_Q, \hat{6}_g, \hat{4}_q; ; 5_q, (\tilde{23})_{\bar{Q}})|^2 \right) J_3^{(3)}(k_1, k_{\tilde{23}}, k_5) \\
 & + \frac{1}{N_c} \left[E_3^0(4_q; 3_q, 1_Q) \left(|\mathcal{M}_5^0((\tilde{13})_Q, 2_{\bar{Q}}; ; 5_q, \hat{4}_g, \hat{6}_q)|^2 \right. \right. \\
 & \quad \left. + |\mathcal{M}_5^0((\tilde{13})_Q, \hat{4}_g, 2_{\bar{Q}}; ; 5_q, \hat{6}_q)|^2 \right. \\
 & \quad \left. - 2|\mathcal{M}_5^0((\tilde{13})_Q, 2_{\bar{Q}}, 5_q, \hat{6}_q, \hat{4}_\gamma)|^2 \right) J_3^{(3)}(k_{\tilde{13}}, k_2, k_5) \\
 & + E_3^0(4_q; 3_q, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0(1_Q, (\tilde{23})_{\bar{Q}}; ; 5_q, \hat{4}_g, \hat{6}_q)|^2 \right. \\
 & \quad \left. + |\mathcal{M}_5^0(1_Q, \hat{4}_g, (\tilde{23})_{\bar{Q}}; ; 5_q, \hat{6}_q)|^2 \right. \\
 & \quad \left. - 2|\mathcal{M}_5^0(1_Q, (\tilde{23})_{\bar{Q}}, 5_q, \hat{6}_q, \hat{4}_\gamma)|^2 \right) J_3^{(3)}(k_1, k_{\tilde{23}}, k_5) \\
 & + E_3^0(4_q; 5_q, 1_Q) \left(|\mathcal{M}_5^0((\tilde{15})_Q, 2_{\bar{Q}}; ; 3_q, \hat{4}_g, \hat{6}_q)|^2 \right. \\
 & \quad \left. + |\mathcal{M}_5^0((\tilde{15})_Q, \hat{4}_g, 2_{\bar{Q}}; ; 3_q, \hat{6}_q)|^2 \right. \\
 & \quad \left. - 2|\mathcal{M}_5^0((\tilde{15})_Q, 2_{\bar{Q}}, 3_q, \hat{6}_q, \hat{4}_\gamma)|^2 \right) J_3^{(3)}(k_{\tilde{15}}, k_2, k_3) \\
 & + E_3^0(4_q; 5_q, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0(1_Q, (\tilde{25})_{\bar{Q}}; ; 3_q, \hat{4}_g, \hat{6}_q)|^2 \right. \\
 & \quad \left. + |\mathcal{M}_5^0(1_Q, \hat{4}_g, (\tilde{25})_{\bar{Q}}; ; 3_q, \hat{6}_q)|^2 \right. \\
 & \quad \left. - 2|\mathcal{M}_5^0(1_Q, (\tilde{25})_{\bar{Q}}, 3_q, \hat{6}_q, \hat{4}_\gamma)|^2 \right) J_3^{(3)}(k_1, k_{\tilde{25}}, k_3)
 \end{aligned} \tag{7.43}$$

$$\begin{aligned}
 & +E_3^0(6_q; 5_q, 1_Q) \left(|\mathcal{M}_5^0((\widetilde{15})_Q, 2_{\bar{Q}}; ; 3_q, \hat{6}_g, \hat{4}_q)|^2 \right. \\
 & \quad + |\mathcal{M}_5^0((\widetilde{15})_Q, \hat{6}_g, 2_{\bar{Q}}; ; 3_q, \hat{4}_q)|^2 \\
 & \quad \left. - 2|\mathcal{M}_5^0((\widetilde{15})_Q, 2_{\bar{Q}}, 3_q, \hat{4}_q, \hat{6}_\gamma)|^2 \right) J_3^{(3)}(k_{\widetilde{15}}, k_2, k_3) \\
 & +E_3^0(6_q; 5_q, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0(1_Q, (\widetilde{25})_{\bar{Q}}; ; 3_q, \hat{6}_g, \hat{4}_q)|^2 \right. \\
 & \quad + |\mathcal{M}_5^0(1_Q, \hat{6}_g, (\widetilde{25})_{\bar{Q}}; ; 3_q, \hat{4}_q)|^2 \\
 & \quad \left. - 2|\mathcal{M}_5^0(1_Q, (\widetilde{25})_{\bar{Q}}, 3_q, \hat{4}_q, \hat{6}_\gamma)|^2 \right) J_3^{(3)}(k_1, k_{\widetilde{25}}, k_3) \\
 & +E_3^0(6_q; 3_q, 1_Q) \left(|\mathcal{M}_5^0((\widetilde{13})_Q, 2_{\bar{Q}}; ; 5_q, \hat{6}_g, \hat{4}_q)|^2 \right. \\
 & \quad + |\mathcal{M}_5^0((\widetilde{13})_Q, \hat{6}_g, 2_{\bar{Q}}; ; 5_q, \hat{4}_q)|^2 \\
 & \quad \left. - 2|\mathcal{M}_5^0((\widetilde{13})_Q, 2_{\bar{Q}}, 5_q, \hat{4}_q, \hat{6}_\gamma)|^2 \right) J_3^{(3)}(k_{\widetilde{13}}, k_2, k_5) \\
 & +E_3^0(6_q; 3_q, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0(1_Q, (\widetilde{23})_{\bar{Q}}; ; 5_q, \hat{6}_g, \hat{4}_q)|^2 \right. \\
 & \quad + |\mathcal{M}_5^0(1_Q, \hat{6}_g, (\widetilde{23})_{\bar{Q}}; ; 5_q, \hat{4}_q)|^2 \\
 & \quad \left. + |\mathcal{M}_5^0(1_Q, (\widetilde{23})_{\bar{Q}}, 5_q, \hat{4}_q, \hat{6}_\gamma)|^2 \right) J_3^{(3)}(k_1, k_{\widetilde{23}}, k_5) \Big] \Big\}.
 \end{aligned}$$

7.2.3 Processes derived from $0 \rightarrow Q\bar{Q}q\bar{q}gg$

The colour decomposition for the unphysical process $0 \rightarrow Q\bar{Q}q\bar{q}gg$ is

$$\begin{aligned}
 & M_6^0(1_Q, 2_{\bar{Q}}, 3_q, 4_{\bar{q}}, 5_g, 6_g) \\
 & = 2g^4 \sum_{(i,j) \in P(5,6)} \left\{ \delta_{i_3, i_2} (T^{a_i} T^{a_j})_{i_1, i_4} \mathcal{M}_6^0(1_Q, i_g, j_g, 4_{\bar{q}}; ; 3_q, 2_{\bar{Q}}) \right. \\
 & \quad + (T^{a_i})_{i_1, i_4} (T^{a_j})_{i_3, i_2} \mathcal{M}_6^0(1_Q, i_g, 4_{\bar{q}}; ; 3_q, j_g, 2_{\bar{Q}}) \\
 & \quad + \delta_{i_1, i_4} (T^{a_i} T^{a_j})_{i_3, i_2} \mathcal{M}_6^0(1_Q, 4_{\bar{q}}; ; 3_q, i_g, j_g, 2_{\bar{Q}}) \\
 & \quad - \frac{1}{N_c} \delta_{i_3, i_4} (T^{a_i} T^{a_j})_{i_1, i_2} \mathcal{M}_6^0(1_Q, i_g, j_g, 2_{\bar{Q}}; ; 3_q, 4_{\bar{q}}) \\
 & \quad - \frac{1}{N_c} (T^{a_i})_{i_1, i_2} (T^{a_j})_{i_3, i_4} \mathcal{M}_6^0(1_Q, i_g, 2_{\bar{Q}}; ; 3_q, j_g, 4_{\bar{q}}) \\
 & \quad \left. - \frac{1}{N_c} \delta_{i_1, i_2} (T^{a_i} T^{a_j})_{i_3, i_4} \mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, i_g, j_g, 4_{\bar{q}}) \right\}, \tag{7.44}
 \end{aligned}$$

and the partial amplitudes satisfy

$$\begin{aligned}
 & \mathcal{M}_6^0(1_Q, 2_{\bar{Q}}, 3_q, 4_{\bar{q}}, 5_\gamma, 6_\gamma) \\
 & = \sum_{(i,j) \in P(5,6)} \left(\mathcal{M}_6^0(1_Q, i_g, j_g, 4_{\bar{q}}; ; 3_q, 2_{\bar{Q}}) + \mathcal{M}_6^0(1_Q, i_g, 4_{\bar{q}}; ; 3_q, j_g, 2_{\bar{Q}}) \right. \\
 & \quad \left. + \mathcal{M}_6^0(1_Q, 4_{\bar{q}}; ; 3_q, i_g, j_g, 2_{\bar{Q}}) \right) \\
 & = \sum_{(k,l) \in P(5,6)} \left(\mathcal{M}_6^0(1_Q, k_g, l_g, 2_{\bar{Q}}; ; 3_q, 4_{\bar{q}}) + \mathcal{M}_6^0(1_Q, k_g, 2_{\bar{Q}}; ; 3_q, l_g, 4_{\bar{q}}) \right. \\
 & \quad \left. + \mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; ; 3_q, k_g, l_g, 4_{\bar{q}}) \right). \tag{7.45}
 \end{aligned}$$

Squaring, and using the identity eq. (7.45) to rearrange the result, we obtain

$$\begin{aligned}
 |M_6^0(1_Q, 2_{\bar{Q}}, 3_q, 4_{\bar{q}}, 5_g, 6_g)|^2 &= \frac{g^8(N_c^2 - 1)}{N_c^2} \\
 &\times \sum_{(i,j) \in P(5,6)} \left\{ \right. \\
 &\quad (N_c^2 - 1) \left[N_c^2 \left(|\mathcal{M}_6^0(1_Q, i_g, j_g, 4_{\bar{q}}; 3_q, 2_{\bar{Q}})|^2 + |\mathcal{M}_6^0(1_Q, i_g, 4_{\bar{q}}; 3_q, j_g, 2_{\bar{Q}})|^2 \right. \right. \\
 &\quad \left. \left. + |\mathcal{M}_6^0(1_Q, 4_{\bar{q}}; 3_q, i_g, j_g, 2_{\bar{Q}})|^2 \right) \right. \\
 &\quad \left. + |\mathcal{M}_6^0(1_Q, i_g, j_g, 2_{\bar{Q}}; 3_q, 4_{\bar{q}})|^2 + |\mathcal{M}_6^0(1_Q, i_g, 2_{\bar{Q}}; 3_q, j_g, 4_{\bar{q}})|^2 \right. \\
 &\quad \left. + |\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; 3_q, i_g, j_g, 4_{\bar{q}})|^2 \right] + |\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}, 3_q, 4_{\bar{q}}, i_\gamma, j_\gamma)|^2 \\
 &\quad + N_c^2 \left[2\text{Re}(\mathcal{M}_6^0(1_Q, i_g, j_g, 4_{\bar{q}}; 3_q, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, 4_{\bar{q}}; 3_q, i_g, j_g, 2_{\bar{Q}})^\dagger) \right. \\
 &\quad + 2\text{Re}(\mathcal{M}_6^0(1_Q, i_g, j_g, 4_{\bar{q}}; 3_q, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, 4_{\bar{q}}; 3_q, j_g, i_g, 2_{\bar{Q}})^\dagger) \\
 &\quad + \text{Re}(\mathcal{M}_6^0(1_Q, i_g, 4_{\bar{q}}; 3_q, j_g, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, j_g, 4_{\bar{q}}; 3_q, i_g, 2_{\bar{Q}})^\dagger) \\
 &\quad - \text{Re}(\mathcal{M}_6^0(1_Q, i_g, j_g, 4_{\bar{q}}; 3_q, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, j_g, i_g, 4_{\bar{q}}; 3_q, 2_{\bar{Q}})^\dagger) \\
 &\quad - \text{Re}(\mathcal{M}_6^0(1_Q, 4_{\bar{q}}; 3_q, i_g, j_g, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, 4_{\bar{q}}; 3_q, j_g, i_g, 2_{\bar{Q}})^\dagger) \\
 &\quad - 2\text{Re}(\mathcal{M}_6^0(1_Q, i_g, j_g, 4_{\bar{q}}; 3_q, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, i_g, j_g, 2_{\bar{Q}}; 3_q, 4_{\bar{q}})^\dagger) \\
 &\quad - 2\text{Re}(\mathcal{M}_6^0(1_Q, i_g, 4_{\bar{q}}; 3_q, j_g, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, i_g, j_g, 2_{\bar{Q}}; 3_q, 4_{\bar{q}})^\dagger) \\
 &\quad - 2\text{Re}(\mathcal{M}_6^0(1_Q, 4_{\bar{q}}; 3_q, i_g, j_g, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, i_g, j_g, 2_{\bar{Q}}; 3_q, 4_{\bar{q}})^\dagger) \\
 &\quad - 2\text{Re}(\mathcal{M}_6^0(1_Q, i_g, j_g, 4_{\bar{q}}; 3_q, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, i_g, 2_{\bar{Q}}; 3_q, j_g, 4_{\bar{q}})^\dagger) \\
 &\quad - 2\text{Re}(\mathcal{M}_6^0(1_Q, i_g, 4_{\bar{q}}; 3_q, j_g, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, i_g, 2_{\bar{Q}}; 3_q, j_g, 4_{\bar{q}})^\dagger) \\
 &\quad - 2\text{Re}(\mathcal{M}_6^0(1_Q, i_g, 4_{\bar{q}}; 3_q, j_g, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, i_g, 2_{\bar{Q}}; 3_q, i_g, 4_{\bar{q}})^\dagger) \\
 &\quad - 2\text{Re}(\mathcal{M}_6^0(1_Q, i_g, 4_{\bar{q}}; 3_q, i_g, j_g, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, j_g, 2_{\bar{Q}}; 3_q, i_g, 4_{\bar{q}})^\dagger) \\
 &\quad - 2\text{Re}(\mathcal{M}_6^0(1_Q, i_g, j_g, 4_{\bar{q}}; 3_q, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; 3_q, i_g, j_g, 4_{\bar{q}})^\dagger) \\
 &\quad - 2\text{Re}(\mathcal{M}_6^0(1_Q, 4_{\bar{q}}; 3_q, i_g, j_g, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; 3_q, i_g, j_g, 4_{\bar{q}})^\dagger) \\
 &\quad - 2\text{Re}(\mathcal{M}_6^0(1_Q, i_g, 4_{\bar{q}}; 3_q, j_g, 2_{\bar{Q}})\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; 3_q, j_g, i_g, 4_{\bar{q}})^\dagger) \\
 &\quad + \text{Re}(\mathcal{M}_6^0(1_Q, i_g, 2_{\bar{Q}}; 3_q, j_g, 4_{\bar{q}})\mathcal{M}_6^0(1_Q, j_g, 2_{\bar{Q}}; 3_q, i_g, 4_{\bar{q}})^\dagger) \\
 &\quad - \text{Re}(\mathcal{M}_6^0(1_Q, i_g, j_g, 2_{\bar{Q}}; 3_q, 4_{\bar{q}})\mathcal{M}_6^0(1_Q, j_g, i_g, 2_{\bar{Q}}; 3_q, 4_{\bar{q}})^\dagger) \\
 &\quad - \text{Re}(\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; 3_q, i_g, j_g, 4_{\bar{q}})\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; 3_q, j_g, i_g, 4_{\bar{q}})^\dagger) \\
 &\quad + 2\text{Re}(\mathcal{M}_6^0(1_Q, i_g, j_g, 2_{\bar{Q}}; 3_q, 4_{\bar{q}})\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; 3_q, i_g, j_g, 4_{\bar{q}})^\dagger) \\
 &\quad \left. + 2\text{Re}(\mathcal{M}_6^0(1_Q, i_g, j_g, 2_{\bar{Q}}; 3_q, 4_{\bar{q}})\mathcal{M}_6^0(1_Q, 2_{\bar{Q}}; 3_q, j_g, i_g, 4_{\bar{q}})^\dagger) \right\}. \tag{7.46}
 \end{aligned}$$

Besides colour-ordered amplitudes squared this expression includes interference terms between amplitudes with different colour orderings which require subtraction.

After appropriate crossings, for each of the colour-ordered matrix-element squared appearing on the right hand side of this equation, an antenna capturing the single unresolved radiation is determined. For the soft radiation behaviour present in the interference terms, the subtraction terms are constructed with the difference of four antennae as explained in section 7.2.1. For the subtraction terms, we will need all types of massive final-final and initial final antennae defined in section 3 in addition to massless antennae in all three (final-final, initial-final and initial-initial) configurations which were defined in [64, 79, 82].

By crossing 3_q and $4_{\bar{q}}$ in eq. (7.46) to the initial state, the matrix element for the process $q\bar{q} \rightarrow Q\bar{Q}gg$ is obtained. From the unresolved limits of this squared matrix element we obtain the following subtraction term,

$$\begin{aligned}
 d\hat{\sigma}_{q\bar{q} \rightarrow Q\bar{Q}gg}^S &= g^8(N_c^2 - 1)d\Phi_4(k_{1Q}, k_{2\bar{Q}}, k_{5g}, k_{6g}; p_{3\bar{q}}, p_{4q}) \sum_{(i,j) \in (5,6)} \left\{ \right. \\
 &N_c^2 \left[A_3^0(4_q; 1_Q, i_g) |\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{4}_q; ; \hat{3}_{\bar{q}}, j_g, 2_{\bar{Q}})|^2 J_3^{(3)}(k_{\tilde{1}i}, k_2, k_j) \right. \\
 &\quad + A_3^0(3_{\bar{q}}; 2_{\bar{Q}}, i_g) |\mathcal{M}_5^0(1_Q, j_g, \hat{4}_q; ; \hat{3}_{\bar{q}}, (\tilde{2}i)_{\bar{Q}})|^2 J_3^{(3)}(k_1, k_{\tilde{2}i}, k_j) \\
 &\quad + d_3^0(1_Q, i_g, j_g) |\mathcal{M}_5^0((\tilde{1}i)_Q, (\tilde{i}j)_g, \hat{4}_q; ; \hat{3}_{\bar{q}}, 2_{\bar{Q}})|^2 J_3^{(3)}(k_{\tilde{1}i}, k_{\tilde{i}j}, k_2) \\
 &\quad + d_3^0(2_{\bar{Q}}, i_g, j_g) |\mathcal{M}_5^0(1_Q, \hat{4}_q; ; \hat{3}_{\bar{q}}, (\tilde{i}j)_g, (\tilde{2}i)_{\bar{Q}})|^2 J_3^{(3)}(k_1, k_{\tilde{2}i}, k_{\tilde{i}j}) \\
 &\quad + d_3^0(4_q; j_g, i_g) |\mathcal{M}_5^0(1_Q, (\tilde{i}j)_g, \hat{4}_q; ; \hat{3}_{\bar{q}}, 2_{\bar{Q}})|^2 J_3^{(3)}(k_1, k_2, k_{\tilde{i}j}) \\
 &\quad \left. + d_3^0(3_{\bar{q}}; i_g, j_g) |\mathcal{M}_5^0(1_Q, \hat{4}_q; ; \hat{3}_{\bar{q}}, (\tilde{i}j)_g, 2_{\bar{Q}})|^2 J_3^{(3)}(k_1, k_2, k_{\tilde{i}j}) \right] \\
 &+ A_3^0(1_Q, i_g, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0((\tilde{1}i)_Q, (\tilde{2}i)_{\bar{Q}}; ; \hat{3}_{\bar{q}}, j_g, \hat{4}_q)|^2 \right. \\
 &\quad + 2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{4}_q; ; \hat{3}_{\bar{q}}, j_g, (\tilde{2}i)_{\bar{Q}}) \mathcal{M}_5^0((\tilde{1}i)_Q, j_g, \hat{4}_q; ; \hat{3}_{\bar{q}}, (\tilde{2}i)_{\bar{Q}})^\dagger) \\
 &\quad - 2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, (\tilde{2}i)_{\bar{Q}}; ; \hat{3}_{\bar{q}}, j_g, \hat{4}_q) \mathcal{M}_5^0((\tilde{1}i)_Q, \hat{4}_q; ; \hat{3}_{\bar{q}}, j_g, (\tilde{2}i)_{\bar{Q}})^\dagger) \\
 &\quad \left. - 2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, (\tilde{2}i)_{\bar{Q}}; ; \hat{3}_{\bar{q}}, j_g, \hat{4}_q) \mathcal{M}_5^0((\tilde{1}i)_Q, j_g, \hat{4}_q; ; \hat{3}_{\bar{q}}, (\tilde{2}i)_{\bar{Q}})^\dagger) \right) J_3^{(3)}(k_{\tilde{1}i}, k_{\tilde{2}i}, k_j) \\
 &- A_3^0(4_q; 1_Q, i_g) \left(|\mathcal{M}_5^0((\tilde{1}i)_Q, j_g, \hat{4}_q; ; \hat{3}_{\bar{q}}, 2_{\bar{Q}})|^2 + |\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{4}_q; ; \hat{3}_{\bar{q}}, j_g, 2_{\bar{Q}})|^2 \right. \\
 &\quad + 2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{4}_q; ; \hat{3}_{\bar{q}}, j_g, 2_{\bar{Q}}) \mathcal{M}_5^0((\tilde{1}i)_Q, 2_{\bar{Q}}; ; \hat{3}_{\bar{q}}, j_g, \hat{4}_q)^\dagger) \\
 &\quad \left. + 2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{4}_q; ; \hat{3}_{\bar{q}}, j_g, 2_{\bar{Q}}) \mathcal{M}_5^0((\tilde{1}i)_Q, j_g, 2_{\bar{Q}}; ; \hat{3}_{\bar{q}}, \hat{4}_q)^\dagger) \right) J_3^{(3)}(k_{\tilde{1}i}, k_2, k_j) \\
 &+ A_3^0(3_{\bar{q}}; 1_Q, i_g) \left(2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{4}_q; ; \hat{3}_{\bar{q}}, j_g, 2_{\bar{Q}}) \mathcal{M}_5^0((\tilde{1}i)_Q, j_g, 2_{\bar{Q}}; ; \hat{3}_{\bar{q}}, \hat{4}_q)^\dagger) \right. \\
 &\quad + 2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, j_g, \hat{4}_q; ; \hat{3}_{\bar{q}}, 2_{\bar{Q}}) \mathcal{M}_5^0((\tilde{1}i)_Q, 2_{\bar{Q}}; ; \hat{3}_{\bar{q}}, j_g, \hat{4}_q)^\dagger) \\
 &\quad \left. - 2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, j_g, \hat{4}_q; ; \hat{3}_{\bar{q}}, 2_{\bar{Q}}) \mathcal{M}_5^0((\tilde{1}i)_Q, \hat{4}_q; ; \hat{3}_{\bar{q}}, j_g, 2_{\bar{Q}})^\dagger) \right) J_3^{(3)}(k_{\tilde{1}i}, k_2, k_j) \\
 &+ A_3^0(4_q; 2_{\bar{Q}}, i_g) \left(2\text{Re}(\mathcal{M}_5^0(1_Q, \hat{4}_q; ; \hat{3}_{\bar{q}}, j_g, (\tilde{2}i)_{\bar{Q}}) \mathcal{M}_5^0(1_Q, (\tilde{2}i)_{\bar{Q}}; ; \hat{3}_{\bar{q}}, j_g, \hat{4}_q)^\dagger) \right. \\
 &\quad + 2\text{Re}(\mathcal{M}_5^0(1_Q, j_g, \hat{4}_q; ; \hat{3}_{\bar{q}}, (\tilde{2}i)_{\bar{Q}}) \mathcal{M}_5^0(1_Q, j_g, (\tilde{2}i)_{\bar{Q}}; ; \hat{3}_{\bar{q}}, \hat{4}_q)^\dagger) \\
 &\quad \left. - 2\text{Re}(\mathcal{M}_5^0(1_Q, \hat{4}_q; ; \hat{3}_{\bar{q}}, j_g, (\tilde{2}i)_{\bar{Q}}) \mathcal{M}_5^0(1_Q, j_g, \hat{4}_q; ; \hat{3}_{\bar{q}}, (\tilde{2}i)_{\bar{Q}})^\dagger) \right) J_3^{(3)}(k_1, k_{\tilde{2}i}, k_j) \\
 &- A_3^0(3_{\bar{q}}; 2_{\bar{Q}}, i_g) \left(|\mathcal{M}_5^0(1_Q, \hat{4}_q; ; \hat{3}_{\bar{q}}, j_g, (\tilde{2}i)_{\bar{Q}})|^2 + |\mathcal{M}_5^0(1_Q, j_g, \hat{4}_q; ; \hat{3}_{\bar{q}}, (\tilde{2}i)_{\bar{Q}})|^2 \right. \\
 &\quad + 2\text{Re}(\mathcal{M}_5^0(1_Q, j_g, \hat{4}_q; ; \hat{3}_{\bar{q}}, (\tilde{2}i)_{\bar{Q}}) \mathcal{M}_5^0(1_Q, (\tilde{2}i)_{\bar{Q}}; ; \hat{3}_{\bar{q}}, j_g, \hat{4}_q)^\dagger) \\
 &\quad \left. + 2\text{Re}(\mathcal{M}_5^0(1_Q, j_g, \hat{4}_q; ; \hat{3}_{\bar{q}}, (\tilde{2}i)_{\bar{Q}}) \mathcal{M}_5^0(1_Q, j_g, (\tilde{2}i)_{\bar{Q}}; ; \hat{3}_{\bar{q}}, \hat{4}_q)^\dagger) \right) J_3^{(3)}(k_1, k_{\tilde{2}i}, k_j)
 \end{aligned}$$

$$\begin{aligned}
 & +A_3^0(3_{\bar{q}}, 4_q; i_g) \left(|\mathcal{M}_5^0(\tilde{1}_Q, \tilde{j}_g, \tilde{2}_{\bar{Q}}; \hat{3}_{\bar{q}}, \hat{4}_q)|^2 \right. \\
 & \quad +2\text{Re}(\mathcal{M}_5^0(\tilde{1}_Q, \hat{4}_q; \hat{3}_{\bar{q}}, \tilde{j}_g, \tilde{2}_{\bar{Q}})\mathcal{M}_5^0(\tilde{1}_Q, j_g, \hat{4}_q; \hat{3}_{\bar{q}}, \tilde{2}_{\bar{Q}})^\dagger) \\
 & \quad -2\text{Re}(\mathcal{M}_5^0(\tilde{1}_Q, \hat{4}_q; \hat{3}_{\bar{q}}, \tilde{j}_g, \tilde{2}_{\bar{Q}})\mathcal{M}_5^0(\tilde{1}_Q, \tilde{j}_g, \tilde{2}_{\bar{Q}}; \hat{3}_{\bar{q}}, \hat{4}_q)^\dagger) \\
 & \quad \left. -2\text{Re}(\mathcal{M}_5^0(\tilde{1}_Q, \tilde{j}_g, \hat{4}_q; \hat{3}_{\bar{q}}, \tilde{2}_{\bar{Q}})\mathcal{M}_5^0(\tilde{1}_Q, \tilde{j}_g, \tilde{2}_{\bar{Q}}; \hat{3}_{\bar{q}}, \hat{4}_q)^\dagger) \right) J_3^{(3)}(\tilde{k}_1, \tilde{k}_2, \tilde{k}_j) \\
 & +d_3^0(1_Q, i_g, j_g) \left(|\mathcal{M}_5^0((\tilde{1}i)_Q, (\tilde{i}j)_g, 2_{\bar{Q}}; \hat{3}_{\bar{q}}, \hat{4}_q)|^2 \right. \\
 & \quad +2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{4}_q; \hat{3}_{\bar{q}}, (\tilde{i}j)_g, 2_{\bar{Q}})\mathcal{M}_5^0((\tilde{1}i)_Q, 2_{\bar{Q}}; \hat{3}_{\bar{q}}, (\tilde{i}j)_g, \hat{4}_q)^\dagger) \\
 & \quad -2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{4}_q; \hat{3}_{\bar{q}}, (\tilde{i}j)_g, 2_{\bar{Q}})\mathcal{M}_5^0((\tilde{1}i)_Q, (\tilde{i}j)_g, 2_{\bar{Q}}; \hat{3}_{\bar{q}}, \hat{4}_q)^\dagger) \\
 & \quad -2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, (\tilde{i}j)_g, \hat{4}_q; \hat{3}_{\bar{q}}, 2_{\bar{Q}})\mathcal{M}_5^0((\tilde{1}i)_Q, 2_{\bar{Q}}; \hat{3}_{\bar{q}}, (\tilde{i}j)_g, \hat{4}_q)^\dagger) \\
 & \quad \left. -2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, (\tilde{i}j)_g, \hat{4}_q; \hat{3}_{\bar{q}}, 2_{\bar{Q}})\mathcal{M}_5^0((\tilde{1}i)_Q, (\tilde{i}j)_g, 2_{\bar{Q}}; \hat{3}_{\bar{q}}, \hat{4}_q)^\dagger) \right) J_3^{(3)}(k_{\tilde{1}i}, k_2, k_{\tilde{i}j}) \\
 & +d_3^0(2_{\bar{Q}}, i_g, j_g) \left(|\mathcal{M}_5^0(1_Q, (\tilde{i}j)_g, (\tilde{2}i)_{\bar{Q}}; \hat{3}_{\bar{q}}, \hat{4}_q)|^2 \right. \\
 & \quad +2\text{Re}(\mathcal{M}_5^0(1_Q, (\tilde{i}j)_g, \hat{4}_q; \hat{3}_{\bar{q}}, (\tilde{2}i)_{\bar{Q}})\mathcal{M}_5^0(1_Q, (\tilde{2}i)_{\bar{Q}}; \hat{3}_{\bar{q}}, (\tilde{i}j)_g, \hat{4}_q)^\dagger) \\
 & \quad -2\text{Re}(\mathcal{M}_5^0(1_Q, \hat{4}_q; \hat{3}_{\bar{q}}, (\tilde{i}j)_g, (\tilde{2}i)_{\bar{Q}})\mathcal{M}_5^0(1_Q, (\tilde{2}i)_{\bar{Q}}; \hat{3}_{\bar{q}}, (\tilde{i}j)_g, \hat{4}_q)^\dagger) \\
 & \quad -2\text{Re}(\mathcal{M}_5^0(1_Q, \hat{4}_q; \hat{3}_{\bar{q}}, (\tilde{i}j)_g, (\tilde{2}i)_{\bar{Q}})\mathcal{M}_5^0(1_Q, (\tilde{i}j)_g, (\tilde{2}i)_{\bar{Q}}; \hat{3}_{\bar{q}}, \hat{4}_q)^\dagger) \\
 & \quad \left. -2\text{Re}(\mathcal{M}_5^0(1_Q, (\tilde{i}j)_g, \hat{4}_q; \hat{3}_{\bar{q}}, (\tilde{2}i)_{\bar{Q}})\mathcal{M}_5^0(1_Q, (\tilde{i}j)_g, (\tilde{2}i)_{\bar{Q}}; \hat{3}_{\bar{q}}, \hat{4}_q)^\dagger) \right) J_3^{(3)}(k_1, k_{\tilde{2}i}, k_{\tilde{i}j}) \\
 & +d_3^0(4_q; i_g, j_g) \left(|\mathcal{M}_5^0(1_Q, 2_{\bar{Q}}; \hat{3}_{\bar{q}}, (\tilde{i}j)_g, \hat{4}_q)|^2 \right. \\
 & \quad +2\text{Re}(\mathcal{M}_5^0(1_Q, \hat{4}_q; \hat{3}_{\bar{q}}, (\tilde{i}j)_g, 2_{\bar{Q}})\mathcal{M}_5^0(1_Q, (\tilde{i}j)_g, 2_{\bar{Q}}; \hat{3}_{\bar{q}}, \hat{4}_q)^\dagger) \\
 & \quad -2\text{Re}(\mathcal{M}_5^0(1_Q, \hat{4}_q; \hat{3}_{\bar{q}}, (\tilde{i}j)_g, 2_{\bar{Q}})\mathcal{M}_5^0(1_Q, 2_{\bar{Q}}; \hat{3}_{\bar{q}}, (\tilde{i}j)_g, \hat{4}_q)^\dagger) \\
 & \quad -2\text{Re}(\mathcal{M}_5^0(1_Q, (\tilde{i}j)_g, \hat{4}_q; \hat{3}_{\bar{q}}, 2_{\bar{Q}})\mathcal{M}_5^0(1_Q, 2_{\bar{Q}}; \hat{3}_{\bar{q}}, (\tilde{i}j)_g, \hat{4}_q)^\dagger) \\
 & \quad \left. -2\text{Re}(\mathcal{M}_5^0(1_Q, (\tilde{i}j)_g, \hat{4}_q; \hat{3}_{\bar{q}}, 2_{\bar{Q}})\mathcal{M}_5^0(1_Q, (\tilde{i}j)_g, 2_{\bar{Q}}; \hat{3}_{\bar{q}}, \hat{4}_q)^\dagger) \right) J_3^{(3)}(k_1, k_2, k_{\tilde{i}j}) \\
 & +d_3^0(3_{\bar{q}}; i_g, j_g) \left(|\mathcal{M}_5^0(1_Q, 2_{\bar{Q}}; \hat{3}_{\bar{q}}, (\tilde{i}j)_g, \hat{4}_q)|^2 \right. \\
 & \quad +2\text{Re}(\mathcal{M}_5^0(1_Q, (\tilde{i}j)_g, \hat{4}_q; \hat{3}_{\bar{q}}, 2_{\bar{Q}})\mathcal{M}_5^0(1_Q, (\tilde{i}j)_g, 2_{\bar{Q}}; \hat{3}_{\bar{q}}, \hat{4}_q)^\dagger) \\
 & \quad -2\text{Re}(\mathcal{M}_5^0(1_Q, \hat{4}_q; \hat{3}_{\bar{q}}, (\tilde{i}j)_g, 2_{\bar{Q}})\mathcal{M}_5^0(1_Q, (\tilde{i}j)_g, 2_{\bar{Q}}; \hat{3}_{\bar{q}}, \hat{4}_q)^\dagger) \\
 & \quad -2\text{Re}(\mathcal{M}_5^0(1_Q, \hat{4}_q; \hat{3}_{\bar{q}}, (\tilde{i}j)_g, 2_{\bar{Q}})\mathcal{M}_5^0(1_Q, 2_{\bar{Q}}; \hat{3}_{\bar{q}}, (\tilde{i}j)_g, \hat{4}_q)^\dagger) \\
 & \quad \left. -2\text{Re}(\mathcal{M}_5^0(1_Q, (\tilde{i}j)_g, \hat{4}_q; \hat{3}_{\bar{q}}, 2_{\bar{Q}})\mathcal{M}_5^0(1_Q, 2_{\bar{Q}}; \hat{3}_{\bar{q}}, (\tilde{i}j)_g, \hat{4}_q)^\dagger) \right) J_3^{(3)}(k_1, k_2, k_{\tilde{i}j}) \\
 & +\frac{1}{N_c^2} \left[A_3^0(1_Q, i_g, 2_{\bar{Q}}) \left(2|\mathcal{M}_5^0((\tilde{1}i)_Q, (\tilde{2}i)_{\bar{Q}}, \hat{3}_{\bar{q}}, \hat{4}_q, j_\gamma)|^2 - |\mathcal{M}_5^0((\tilde{1}i)_Q, (\tilde{2}i)_{\bar{Q}}; \hat{3}_{\bar{q}}, j_g, \hat{4}_q)|^2 \right. \right. \\
 & \quad \left. \left. -|\mathcal{M}_5^0((\tilde{1}i)_Q, j_g, (\tilde{2}i)_{\bar{Q}}; \hat{3}_{\bar{q}}, \hat{4}_q)|^2 \right) J_3^{(3)}(k_{\tilde{1}i}, k_{\tilde{2}i}, k_j) \right. \\
 & +A_3^0(3_{\bar{q}}, 4_q; i_g) \left(2|\mathcal{M}_5^0(\tilde{1}_Q, \tilde{2}_{\bar{Q}}, \hat{3}_{\bar{q}}, \hat{4}_q, \tilde{j}_\gamma)|^2 - |\mathcal{M}_5^0(\tilde{1}_Q, \tilde{2}_{\bar{Q}}; \hat{3}_{\bar{q}}, \tilde{j}_g, \hat{4}_q)|^2 \right. \\
 & \quad \left. -|\mathcal{M}_5^0(\tilde{1}_Q, \tilde{j}_g, \tilde{2}_{\bar{Q}}; \hat{3}_{\bar{q}}, \hat{4}_q)|^2 \right) J_3^{(3)}(\tilde{k}_1, \tilde{k}_2, \tilde{k}_j) \\
 & +A_3^0(4_q; 1_Q, i_g) 2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, 2_{\bar{Q}}; \hat{3}_{\bar{q}}, j_g, \hat{4}_q)\mathcal{M}_5^0((\tilde{1}i)_Q, j_g, 2_{\bar{Q}}; \hat{3}_{\bar{q}}, \hat{4}_q)^\dagger) J_3^{(3)}(k_{\tilde{1}i}, k_2, k_j) \\
 & +A_3^0(3_{\bar{q}}; 2_{\bar{Q}}, i_g) 2\text{Re}(\mathcal{M}_5^0(1_Q, (\tilde{2}i)_{\bar{Q}}; \hat{3}_{\bar{q}}, j_g, \hat{4}_q)\mathcal{M}_5^0(1_Q, j_g, (\tilde{2}i)_{\bar{Q}}; \hat{3}_{\bar{q}}, \hat{4}_q)^\dagger) J_3^{(3)}(k_1, k_{\tilde{2}i}, k_j) \\
 & -A_3^0(3_{\bar{q}}; 1_Q, i_g) 2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, 2_{\bar{Q}}; \hat{3}_{\bar{q}}, j_g, \hat{4}_q)\mathcal{M}_5^0((\tilde{1}i)_Q, j_g, 2_{\bar{Q}}; \hat{3}_{\bar{q}}, \hat{4}_q)^\dagger) J_3^{(3)}(k_{\tilde{1}i}, k_2, k_j) \\
 & \left. -A_3^0(4_q; 2_{\bar{Q}}, i_g) 2\text{Re}(\mathcal{M}_5^0(1_Q, (\tilde{2}i)_{\bar{Q}}; \hat{3}_{\bar{q}}, j_g, \hat{4}_q)\mathcal{M}_5^0(1_Q, j_g, (\tilde{2}i)_{\bar{Q}}; \hat{3}_{\bar{q}}, \hat{4}_q)^\dagger) J_3^{(3)}(k_1, k_{\tilde{2}i}, k_j) \right] \Big\}
 \end{aligned}$$

When 4_g and 6_g are crossed to the initial state in eq. (7.46), the squared matrix element for $qg \rightarrow Q\bar{Q}qg$ in terms of colour ordered subamplitudes is obtained. The subtraction term in this case reads,

$$\begin{aligned}
 d\hat{\sigma}_{qg \rightarrow Q\bar{Q}qg}^S &= g^8 (N_c^2 - 1) d\Phi_4(k_{1Q}, k_{2\bar{Q}}, k_{3q}, k_{5g}; p_{4q}, p_{6g}) \\
 &\left\{ N_c^2 \left[A_3^0(3_q, 5_g, 2_{\bar{Q}}) |\mathcal{M}_5^0(1_Q, \hat{6}_g, \hat{4}_q; ; (\widetilde{35})_q, (\widetilde{25})_{\bar{Q}})|^2 J_3^{(3)}(k_1, k_{\widetilde{25}}, k_{\widetilde{35}}) \right. \right. \\
 &\quad + A_3^0(4_q; 1_Q, 5_g) |\mathcal{M}_5^0((\widetilde{15})_Q, \hat{4}_q; ; 3_q, \hat{6}_g, 2_{\bar{Q}})|^2 J_3^{(3)}(k_{\widetilde{15}}, k_2, k_3) \\
 &\quad + \frac{1}{2} A_3^0(6_g; 1_Q, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0((\widetilde{12})_Q, 5_g, \hat{4}_q; ; 3_q, \hat{6}_g)|^2 + |\mathcal{M}_5^0((\widetilde{12})_Q, \hat{4}_q; ; 3_q, 5_g, \hat{6}_g)|^2 \right. \\
 &\quad \left. + |\mathcal{M}_5^0(\hat{6}_{\bar{Q}}, 5_g, \hat{4}_q; ; 3_q, (\widetilde{12})_{\bar{Q}})|^2 + |\mathcal{M}_5^0(\hat{6}_{\bar{Q}}, \hat{4}_q; ; 3_q, 5_g, (\widetilde{12})_{\bar{Q}})|^2 \right) J_3^{(3)}(k_{\widetilde{12}}, k_3, k_5) \\
 &\quad + A_3^0(4_q, 6_g; 3_q) \left(|\mathcal{M}_5^0(\widetilde{1}_Q, \hat{4}_q; ; \hat{6}_q, \tilde{5}_g, \tilde{2}_{\bar{Q}})|^2 + |\mathcal{M}_5^0(\widetilde{1}_Q, \tilde{5}_g, \hat{4}_q; ; \hat{6}_q, \tilde{2}_{\bar{Q}})|^2 \right) J_3^{(3)}(\tilde{k}_1, \tilde{k}_2, \tilde{k}_5) \\
 &\quad + D_3^0(6_g; 5_g, 1_Q) |\mathcal{M}_5^0((\widetilde{15})_Q, \hat{6}_g, \hat{4}_q; ; 3_q, 2_{\bar{Q}})|^2 J_3^{(3)}(k_{\widetilde{15}}, k_2, k_3) \\
 &\quad + D_3^0(6_g; 5_g, 2_{\bar{Q}}) |\mathcal{M}_5^0(1_Q, \hat{4}_q; ; 3_q, \hat{6}_g, (\widetilde{25})_{\bar{Q}})|^2 J_3^{(3)}(k_1, k_{\widetilde{25}}, k_3) \\
 &\quad + D_3^0(6_g; 5_g, 3_q) |\mathcal{M}_5^0(1_q, \hat{4}_q; ; (\widetilde{35})_q, \hat{6}_g, 2_{\bar{Q}})|^2 J_3^{(3)}(k_1, k_2, k_{\widetilde{35}}) \\
 &\quad + D_3^0(4_q, 6_g; 5_g) |\mathcal{M}_5^0(\widetilde{1}_Q, \hat{6}_g, \hat{4}_q; ; \tilde{3}_q, \tilde{2}_{\bar{Q}})|^2 J_3^{(3)}(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3) \\
 &\quad + \frac{1}{2} E_3^0(4_q; 3_q, 1_Q) \left(|\mathcal{M}_5^0((\widetilde{13})_Q, 5_g, \hat{6}_g, \hat{4}_g, 2_{\bar{Q}})|^2 + |\mathcal{M}_5^0((\widetilde{13})_Q, 5_g, \hat{4}_g, \hat{6}_g, 2_{\bar{Q}})|^2 \right. \\
 &\quad + |\mathcal{M}_5^0((\widetilde{13})_Q, \hat{4}_g, 5_g, \hat{6}_g, 2_{\bar{Q}})|^2 + |\mathcal{M}_5^0((\widetilde{13})_Q, \hat{6}_g, 5_g, \hat{4}_g, 2_{\bar{Q}})|^2 \\
 &\quad + |\mathcal{M}_5^0((\widetilde{13})_Q, \hat{6}_g, \hat{4}_g, 5_g, 2_{\bar{Q}})|^2 + |\mathcal{M}_5^0((\widetilde{13})_Q, \hat{4}_g, \hat{6}_g, 5_g, 2_{\bar{Q}})|^2 \left. \right) J_3^{(3)}(k_{\widetilde{13}}, k_2, k_5) \\
 &\quad + \frac{1}{2} E_3^0(4_q; 3_q, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0(1_Q, 5_g, \hat{6}_g, \hat{4}_g, (\widetilde{23})_{\bar{Q}})|^2 + |\mathcal{M}_5^0(1_Q, 5_g, \hat{4}_g, \hat{6}_g, (\widetilde{23})_{\bar{Q}})|^2 \right. \\
 &\quad + |\mathcal{M}_5^0(1_Q, \hat{6}_g, 5_g, \hat{4}_g, (\widetilde{23})_{\bar{Q}})|^2 + |\mathcal{M}_5^0(1_Q, \hat{4}_g, 5_g, \hat{6}_g, (\widetilde{23})_{\bar{Q}})|^2 \\
 &\quad \left. + |\mathcal{M}_5^0(1_Q, \hat{6}_g, \hat{4}_g, 5_g, (\widetilde{23})_{\bar{Q}})|^2 + |\mathcal{M}_5^0(1_Q, \hat{4}_g, \hat{6}_g, 5_g, (\widetilde{23})_{\bar{Q}})|^2 \right) J_3^{(3)}(k_1, k_{\widetilde{23}}, k_5) \left. \right] \\
 &+ A_3^0(1_Q, 5_g, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0((\widetilde{15})_Q, (\widetilde{25})_{\bar{Q}}; ; 3_q, \hat{6}_g, \hat{4}_g)|^2 \right. \\
 &\quad + 2\text{Re}(\mathcal{M}_5^0((\widetilde{15})_Q, \hat{4}_q; ; 3_q, \hat{6}_g, (\widetilde{25})_{\bar{Q}}) \mathcal{M}_5^0((\widetilde{15})_Q, \hat{6}_g, \hat{4}_q; ; 3_q, (\widetilde{25})_{\bar{Q}})^\dagger) \\
 &\quad - 2\text{Re}(\mathcal{M}_5^0((\widetilde{15})_Q, \hat{4}_q; ; 3_q, \hat{6}_g, (\widetilde{25})_{\bar{Q}}) \mathcal{M}_5^0((\widetilde{15})_Q, (\widetilde{25})_{\bar{Q}}; ; 3_q, \hat{6}_g, \hat{4}_g)^\dagger) \\
 &\quad \left. - 2\text{Re}(\mathcal{M}_5^0((\widetilde{15})_Q, \hat{6}_g, \hat{4}_q; ; 3_q, (\widetilde{25})_{\bar{Q}}) \mathcal{M}_5^0((\widetilde{15})_Q, (\widetilde{25})_{\bar{Q}}; ; 3_q, \hat{6}_g, \hat{4}_g)^\dagger) \right) J_3^{(3)}(k_{\widetilde{15}}, k_{\widetilde{25}}, k_3) \\
 &+ A_3^0(1_Q, 5_g, 3_q) \left(2\text{Re}(\mathcal{M}_5^0((\widetilde{15})_Q, \hat{4}_q; ; (\widetilde{35})_q, \hat{6}_g, 2_{\bar{Q}}) \mathcal{M}_5^0((\widetilde{15})_Q, \hat{6}_g, 2_{\bar{Q}}; ; (\widetilde{35})_q, \hat{4}_g)^\dagger) \right. \\
 &\quad + 2\text{Re}(\mathcal{M}_5^0((\widetilde{15})_Q, \hat{6}_g, \hat{4}_q; ; (\widetilde{35})_q, 2_{\bar{Q}}) \mathcal{M}_5^0((\widetilde{15})_Q, 2_{\bar{Q}}; ; (\widetilde{35})_q, \hat{6}_g, \hat{4}_g)^\dagger) \\
 &\quad \left. - 2\text{Re}(\mathcal{M}_5^0((\widetilde{15})_Q, \hat{4}_q; ; (\widetilde{35})_q, \hat{6}_g, 2_{\bar{Q}}) \mathcal{M}_5^0((\widetilde{15})_Q, \hat{6}_g, \hat{4}_q; ; (\widetilde{35})_q, 2_{\bar{Q}})^\dagger) \right) J_3^{(0)}(k_{\widetilde{15}}, k_2, k_{\widetilde{35}}) \\
 &- A_3^0(3_q, 5_g, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0(1_Q, \hat{4}_q; ; (\widetilde{35})_q, \hat{6}_g, (\widetilde{25})_{\bar{Q}})|^2 \right. \\
 &\quad + |\mathcal{M}_5^0(1_Q, \hat{6}_g, \hat{4}_q; ; (\widetilde{35})_q, (\widetilde{25})_{\bar{Q}})|^2 \\
 &\quad + 2\text{Re}(\mathcal{M}_5^0(1_Q, \hat{6}_g, \hat{4}_q; ; (\widetilde{35})_q, (\widetilde{25})_{\bar{Q}}) \mathcal{M}_5^0(1_Q, (\widetilde{25})_{\bar{Q}}; ; (\widetilde{35})_q, \hat{6}_g, \hat{4}_g)^\dagger) \\
 &\quad + 2\text{Re}(\mathcal{M}_5^0(1_Q, \hat{6}_g, \hat{4}_q; ; (\widetilde{35})_q, (\widetilde{25})_{\bar{Q}}) \mathcal{M}_5^0(1_Q, \hat{6}_g, (\widetilde{25})_{\bar{Q}}; ; (\widetilde{35})_q, \hat{4}_g)^\dagger) \left. \right) J_3^{(3)}(k_1, k_{\widetilde{25}}, k_{\widetilde{35}}) \\
 &- A_3^0(4_q; 1_Q, 5_g) \left(|\mathcal{M}_5^0((\widetilde{15})_Q, \hat{6}_g, \hat{4}_q; ; 3_q, 2_{\bar{Q}})|^2 \right. \\
 &\quad + |\mathcal{M}_5^0((\widetilde{15})_Q, \hat{4}_q; ; 3_q, \hat{6}_g, 2_{\bar{Q}})|^2 \\
 &\quad + 2\text{Re}(\mathcal{M}_5^0((\widetilde{15})_Q, \hat{4}_q; ; 3_q, \hat{6}_g, 2_{\bar{Q}}) \mathcal{M}_5^0((\widetilde{15})_Q, 2_{\bar{Q}}; ; 3_q, \hat{6}_g, \hat{4}_g)^\dagger) \\
 &\quad \left. + 2\text{Re}(\mathcal{M}_5^0((\widetilde{15})_Q, \hat{4}_q; ; 3_q, \hat{6}_g, 2_{\bar{Q}}) \mathcal{M}_5^0((\widetilde{15})_Q, \hat{6}_g, 2_{\bar{Q}}; ; 3_q, \hat{4}_g)^\dagger) \right) J_3^{(3)}(k_{\widetilde{15}}, k_2, k_3)
 \end{aligned}$$

$$\begin{aligned}
 & +A_3^0(4_q; 2_{\bar{Q}}, 5_g) \left(2\text{Re}(\mathcal{M}_5^0(1_Q, \hat{4}_q; ; 3_q, \hat{6}_g, (\widetilde{25})_{\bar{Q}})\mathcal{M}_5^0(1_Q, (\widetilde{25})_{\bar{Q}}; ; 3_q, \hat{6}_g, \hat{4}_q)^\dagger) \right. \\
 & \quad +2\text{Re}(\mathcal{M}_5^0(1_Q, \hat{6}_g, \hat{4}_q; ; 3_q, (\widetilde{25})_{\bar{Q}})\mathcal{M}_5^0(1_Q, \hat{6}_g, (\widetilde{25})_{\bar{Q}}; ; 3_q, \hat{4}_q)^\dagger) \\
 & \quad \left. -2\text{Re}(\mathcal{M}_5^0(1_Q, \hat{4}_q; ; 3_q, \hat{6}_g, (\widetilde{25})_{\bar{Q}})\mathcal{M}_5^0(1_Q, \hat{6}_g, \hat{4}_q; ; 3_q, (\widetilde{25})_{\bar{Q}})^\dagger) \right) J_3^{(3)}(k_1, k_{\overline{25}}, k_3) \\
 & +A_3^0(4_q; 3_q, 5_g) \left(|\mathcal{M}_5^0(1_Q, \hat{6}_g, 2_{\bar{Q}}; (\widetilde{35})_q, \hat{4}_q)|^2 \right. \\
 & \quad +2\text{Re}(\mathcal{M}_5^0(1_Q, \hat{4}_q; ; (\widetilde{35})_q, \hat{6}_g, 2_{\bar{Q}})\mathcal{M}_5^0(1_Q, \hat{6}_g, \hat{4}_q; ; (\widetilde{35})_q, 2_{\bar{Q}})^\dagger) \\
 & \quad -2\text{Re}(\mathcal{M}_5^0(1_Q, \hat{4}_q; ; (\widetilde{35})_q, \hat{6}_g, 2_{\bar{Q}})\mathcal{M}_5^0(1_Q, \hat{6}_g, 2_{\bar{Q}}; ; (\widetilde{35})_q, \hat{4}_q)^\dagger) \\
 & \quad \left. -2\text{Re}(\mathcal{M}_5^0(1_Q, \hat{6}_g, \hat{4}_q; ; (\widetilde{35})_q, 2_{\bar{Q}})\mathcal{M}_5^0(1_Q, \hat{6}_g, 2_{\bar{Q}}; ; (\widetilde{35})_q, \hat{4}_q)^\dagger) \right) J_3^{(3)}(k_1, k_2, k_{\overline{35}}) \\
 & +\frac{1}{2}A_3^0(6_g; 1_Q, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0((\overline{12})_Q, \hat{6}_Q; ; 3_q, 5_g, \hat{4}_q)|^2 + |\mathcal{M}_5^0((\overline{12})_Q, 5_g, \hat{6}_Q; ; 3_q, \hat{4}_q)|^2 \right. \\
 & \quad +|\mathcal{M}_5^0(\hat{6}_{\bar{Q}}, (\overline{12})_{\bar{Q}}; ; 3_q, 5_g, \hat{4}_q)|^2 + |\mathcal{M}_5^0(\hat{6}_{\bar{Q}}, 5_g, (\overline{12})_{\bar{Q}}; ; 3_q, \hat{4}_q)|^2 \\
 & \quad -|\mathcal{M}_5^0((\overline{12})_Q, \hat{4}_q; ; 3_q, 5_g, \hat{6}_Q)|^2 - |\mathcal{M}_5^0((\overline{12})_Q, 5_g, \hat{4}_q; ; 3_q, \hat{6}_Q)|^2 \\
 & \quad -|\mathcal{M}_5^0(\hat{6}_{\bar{Q}}, \hat{4}_q; ; 3_q, 5_g, (\overline{12})_{\bar{Q}})|^2 - |\mathcal{M}_5^0(\hat{6}_{\bar{Q}}, 5_g, \hat{4}_q; ; 3_q, (\overline{12})_{\bar{Q}})|^2 \\
 & \quad \left. -2|\mathcal{M}_5^0((\overline{12})_Q, \hat{6}_Q, 3_q, \hat{4}_q, 5_\gamma)|^2 - 2|\mathcal{M}_5^0(\hat{6}_{\bar{Q}}, (\overline{12})_{\bar{Q}}, 3_q, \hat{4}_q, 5_\gamma)|^2 \right) J_3^{(3)}(k_{\overline{12}}, k_3, k_5) \\
 & +A_3^0(4_q, 6_g; 3_q) \left(|\mathcal{M}_5^0(\overline{1}_Q, \tilde{5}_g, \tilde{2}_{\bar{Q}}; ; \hat{6}_{\bar{q}}, \hat{4}_q)|^2 + |\mathcal{M}_5^0(\overline{1}_Q, \tilde{2}_{\bar{Q}}; ; \hat{6}_{\bar{q}}, \tilde{5}_g, \hat{4}_q)|^2 \right. \\
 & \quad -|\mathcal{M}_5^0(\overline{1}_Q, \tilde{5}_g, \hat{4}_q; ; \hat{6}_{\bar{q}}, \tilde{2}_{\bar{Q}})|^2 - |\mathcal{M}_5^0(\overline{1}_Q, \hat{4}_q; ; \hat{6}_{\bar{q}}, \tilde{5}_g, \tilde{2}_{\bar{Q}})|^2 \\
 & \quad \left. -2|\mathcal{M}_5^0(\overline{1}_Q, \tilde{2}_{\bar{Q}}, \hat{6}_{\bar{q}}, \hat{4}_q, \tilde{5}_\gamma)|^2 \right) J_3^0(\tilde{k}_1, \tilde{k}_2, \tilde{k}_5) \\
 & +D_3^0(6_g; 5_g, 1_Q) \left(|\mathcal{M}_5^0((\overline{15})_Q, \hat{6}_g, 2_{\bar{Q}}; ; 3_q, \hat{4}_q)|^2 \right. \\
 & \quad +2\text{Re}(\mathcal{M}_5^0((\overline{15})_Q, \hat{4}_q; ; 3_q, \hat{6}_g, 2_{\bar{Q}})\mathcal{M}_5^0((\overline{15})_Q, 2_{\bar{Q}}; ; 3_q, \hat{6}_g, \hat{4}_q)^\dagger) \\
 & \quad -2\text{Re}(\mathcal{M}_5^0((\overline{15})_Q, \hat{4}_q; ; 3_q, \hat{6}_g, 2_{\bar{Q}})\mathcal{M}_5^0((\overline{15})_Q, \hat{6}_g, 2_{\bar{Q}}; ; 3_q, \hat{4}_q)^\dagger) \\
 & \quad -2\text{Re}(\mathcal{M}_5^0((\overline{15})_Q, \hat{6}_g, \hat{4}_q; ; 3_q, 2_{\bar{Q}})\mathcal{M}_5^0((\overline{15})_Q, \hat{6}_g, 2_{\bar{Q}}; ; 3_q, \hat{4}_q)^\dagger) \\
 & \quad \left. -2\text{Re}(\mathcal{M}_5^0((\overline{15})_Q, \hat{6}_g, \hat{4}_q; ; 3_q, 2_{\bar{Q}})\mathcal{M}_5^0((\overline{15})_Q, 2_{\bar{Q}}; ; 3_q, \hat{6}_g, \hat{4}_q)^\dagger) \right) J_3^{(3)}(k_{\overline{15}}, k_2, k_3) \\
 & +D_3^0(6_g; 5_g, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0(1_q, \hat{6}_g, (\widetilde{25})_{\bar{Q}}; ; 3_q, \hat{4}_q)|^2 \right. \\
 & \quad +2\text{Re}(\mathcal{M}_5^0(1_Q, \hat{6}_g, \hat{4}_q; ; 3_q, (\widetilde{25})_{\bar{Q}})\mathcal{M}_5^0(1_Q, (\widetilde{25})_{\bar{Q}}; ; 3_q, \hat{6}_g, \hat{4}_q)^\dagger) \\
 & \quad -2\text{Re}(\mathcal{M}_5^0(1_Q, \hat{4}_q; ; 3_q, \hat{6}_g, (\widetilde{25})_{\bar{Q}})\mathcal{M}_5^0(1_Q, (\widetilde{25})_{\bar{Q}}; ; 3_q, \hat{6}_g, \hat{4}_q)^\dagger) \\
 & \quad -2\text{Re}(\mathcal{M}_5^0(1_Q, \hat{4}_q; ; 3_q, \hat{6}_g, (\widetilde{25})_{\bar{Q}})\mathcal{M}_5^0(1_Q, \hat{6}_g, (\widetilde{25})_{\bar{Q}}; ; 3_q, \hat{4}_q)^\dagger) \\
 & \quad \left. -2\text{Re}(\mathcal{M}_5^0(1_Q, \hat{6}_g, \hat{4}_q; ; 3_q, (\widetilde{25})_{\bar{Q}})\mathcal{M}_5^0(1_Q, \hat{6}_g, (\widetilde{25})_{\bar{Q}}; ; 3_q, \hat{4}_q)^\dagger) \right) J_3^{(3)}(k_1, k_{\overline{25}}, k_3) \\
 & +D_3^0(6_g; 5_g, 3_q) \left(|\mathcal{M}_5^0(1_Q, 2_{\bar{Q}}; ; (\widetilde{35})_q, \hat{6}_g, \hat{4}_q)|^2 \right. \\
 & \quad +2\text{Re}(\mathcal{M}_5^0(1_q, \hat{6}_g, \hat{4}_q; ; (\widetilde{35})_q, 2_{\bar{Q}})\mathcal{M}_5^0(1_Q, \hat{6}_g, 2_{\bar{Q}}; ; (\widetilde{35})_q, \hat{4}_q)^\dagger) \\
 & \quad -2\text{Re}(\mathcal{M}_5^0(1_q, \hat{4}_q; ; (\widetilde{35})_q, \hat{6}_g, 2_{\bar{Q}})\mathcal{M}_5^0(1_Q, 2_{\bar{Q}}; ; (\widetilde{35})_q, \hat{6}_g, \hat{4}_q)^\dagger) \\
 & \quad -2\text{Re}(\mathcal{M}_5^0(1_q, \hat{4}_q; ; (\widetilde{35})_q, \hat{6}_g, 2_{\bar{Q}})\mathcal{M}_5^0(1_Q, \hat{6}_g, 2_{\bar{Q}}; ; (\widetilde{35})_q, \hat{4}_q)^\dagger) \\
 & \quad \left. -2\text{Re}(\mathcal{M}_5^0(1_q, \hat{6}_g, \hat{4}_q; ; (\widetilde{35})_q, 2_{\bar{Q}})\mathcal{M}_5^0(1_Q, 2_{\bar{Q}}; ; (\widetilde{35})_q, \hat{6}_g, \hat{4}_q)^\dagger) \right) J_3^{(3)}(k_1, k_2, k_{\overline{35}}) \\
 & +D_3^0(4_q, 6_g; 5_g) \left(|\mathcal{M}_5^0(\overline{1}_Q, \tilde{2}_{\bar{Q}}; ; \tilde{3}_q, \hat{6}_g, \hat{4}_q)|^2 \right. \\
 & \quad +2\text{Re}(\mathcal{M}_5^0(\overline{1}_Q, \hat{4}_q; ; \tilde{3}_q, \hat{6}_g, \tilde{2}_{\bar{Q}})\mathcal{M}_5^0(\overline{1}_Q, \hat{6}_g, \tilde{2}_{\bar{Q}}; ; \tilde{3}_q, \hat{4}_q)^\dagger) \\
 & \quad -2\text{Re}(\mathcal{M}_5^0(\overline{1}_Q, \hat{4}_q; ; \tilde{3}_q, \hat{6}_g, \tilde{2}_{\bar{Q}})\mathcal{M}_5^0(\overline{1}_Q, \tilde{2}_{\bar{Q}}; ; \tilde{3}_q, \hat{6}_g, \hat{4}_q)^\dagger) \\
 & \quad -2\text{Re}(\mathcal{M}_5^0(\overline{1}_Q, \hat{6}_g, \hat{4}_q; ; \tilde{3}_q, \tilde{2}_{\bar{Q}})\mathcal{M}_5^0(\overline{1}_Q, \tilde{2}_{\bar{Q}}; ; \tilde{3}_q, \hat{6}_g, \hat{4}_q)^\dagger) \\
 & \quad \left. -2\text{Re}(\mathcal{M}_5^0(\overline{1}_Q, \hat{6}_g, \hat{4}_q; ; \tilde{3}_q, \tilde{2}_{\bar{Q}})\mathcal{M}_5^0(\overline{1}_Q, 2_{\bar{Q}}; ; \tilde{3}_q, \hat{6}_g, \hat{4}_q)^\dagger) \right) J_3^{(3)}(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3)
 \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{2}E_3^0(4_q; 3_q, 1_Q) \left(|\mathcal{M}_5^0((\widetilde{13})_Q, 5_g, \hat{6}_g, \hat{4}_\gamma, 2_{\bar{Q}})|^2 + |\mathcal{M}_5^0((\widetilde{13})_Q, \hat{6}_g, 5_g, \hat{4}_\gamma, 2_{\bar{Q}})|^2 \right. \\
 & \quad + |\mathcal{M}_5^0((\widetilde{13})_Q, 5_g, \hat{4}_g, \hat{6}_\gamma, 2_{\bar{Q}})|^2 + |\mathcal{M}_5^0((\widetilde{13})_Q, \hat{4}_g, 5_g, \hat{6}_\gamma, 2_{\bar{Q}})|^2 \\
 & \quad \left. + |\mathcal{M}_5^0((\widetilde{13})_Q, \hat{6}_g, \hat{4}_g, 5_\gamma, 2_{\bar{Q}})|^2 + |\mathcal{M}_5^0((\widetilde{13})_Q, \hat{4}_g, \hat{6}_g, 5_\gamma, 2_{\bar{Q}})|^2 \right) J_3^{(3)}(k_{\widetilde{13}}, k_2, k_5) \\
 & -\frac{1}{2}E_3^0(4_q; 3_q, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0(1_Q, 5_g, \hat{6}_g, \hat{4}_\gamma, (\widetilde{23})_{\bar{Q}})|^2 + |\mathcal{M}_5^0(1_Q, \hat{6}_g, 5_g, \hat{4}_\gamma, (\widetilde{23})_{\bar{Q}})|^2 \right. \\
 & \quad + |\mathcal{M}_5^0(1_Q, 5_g, \hat{4}_g, \hat{6}_\gamma, (\widetilde{23})_{\bar{Q}})|^2 + |\mathcal{M}_5^0(1_Q, \hat{4}_g, 5_g, \hat{6}_\gamma, (\widetilde{23})_{\bar{Q}})|^2 \\
 & \quad \left. + |\mathcal{M}_5^0(1_Q, \hat{6}_g, \hat{4}_g, 5_\gamma, (\widetilde{23})_{\bar{Q}})|^2 + |\mathcal{M}_5^0(1_Q, \hat{4}_g, \hat{6}_g, 5_\gamma, (\widetilde{23})_{\bar{Q}})|^2 \right) J_3^{(3)}(k_1, k_{\widetilde{23}}, k_5) \\
 & + \frac{1}{N_c^2} \left[A_3^0(1_Q, 5_g, 2_{\bar{Q}}) \left(2|\mathcal{M}_5^0((\widetilde{15})_Q, (\widetilde{25})_{\bar{Q}}, 3_q, \hat{4}_q, \hat{6}_\gamma)|^2 \right. \right. \\
 & \quad \left. - |\mathcal{M}_5^0(\widetilde{15})_Q, (\widetilde{25})_{\bar{Q}}; ; 3_q, \hat{6}_g, \hat{4}_q)|^2 - |\mathcal{M}_5^0(\widetilde{15})_Q, \hat{6}_g, (\widetilde{25})_{\bar{Q}}; ; 3_q, \hat{4}_q)|^2 \right) J_3^{(3)}(k_{\widetilde{15}}, k_{\widetilde{25}}, k_3) \\
 & \quad + A_3^0(3_q, 5_g, 2_{\bar{Q}}) 2\text{Re}(\mathcal{M}_5^0(1_Q, (\widetilde{25})_{\bar{Q}}; ; (\widetilde{35})_q, \hat{6}_g, \hat{4}_q) \mathcal{M}_5^0(1_Q, \hat{6}_g, (\widetilde{25})_{\bar{Q}}; ; (\widetilde{35})_q, \hat{4}_q)^\dagger) J_3^{(3)}(k_1, k_{\widetilde{25}}, k_{\widetilde{35}}) \\
 & \quad - A_3^0(1_Q, 5_g, 3_q) 2\text{Re}(\mathcal{M}_5^0((\widetilde{15})_Q, 2_{\bar{Q}}; ; (\widetilde{35})_q, \hat{6}_g, \hat{4}_q) \mathcal{M}_5^0((\widetilde{15})_Q, \hat{6}_g, 2_{\bar{Q}}; ; (\widetilde{35})_q, \hat{4}_q)^\dagger) J_3^{(3)}(k_{\widetilde{15}}, k_2, k_{\widetilde{35}}) \\
 & \quad + A_3^0(4_q; 1_Q, 5_g) 2\text{Re}(\mathcal{M}_5^0((\widetilde{15})_Q, 2_{\bar{Q}}; ; 3_q, \hat{6}_g, \hat{4}_q) \mathcal{M}_5^0((\widetilde{15})_Q, \hat{6}_g, 2_{\bar{Q}}; ; 3_q, \hat{4}_q)^\dagger) J_3^{(3)}(k_{\widetilde{15}}, k_2, k_3) \\
 & \quad - A_3^0(4_q; 2_{\bar{Q}}, 5_g) 2\text{Re}(\mathcal{M}_5^0(1_Q, (\widetilde{25})_{\bar{Q}}; ; 3_q, \hat{6}_g, \hat{4}_q) \mathcal{M}_5^0(1_Q, \hat{6}_g, (\widetilde{25})_{\bar{Q}}; ; 3_q, \hat{4}_q)^\dagger) J_3^{(3)}(k_1, k_{\widetilde{25}}, k_3) \\
 & \quad \left. + A_3^0(4_q; 3_q, 5_g) \left(2|\mathcal{M}_5^0(1_Q, 2_{\bar{Q}}, (\widetilde{35})_q, \hat{4}_q, \hat{6}_\gamma)|^2 \right. \right. \\
 & \quad \left. - |\mathcal{M}_5^0(1_Q, \hat{6}_g, 2_{\bar{Q}}; (\widetilde{35})_q, \hat{4}_q)|^2 - |\mathcal{M}_5^0(1_Q, 2_{\bar{Q}}; ; (\widetilde{35})_q, \hat{6}_g, \hat{4}_q)|^2 \right) J_3^{(3)}(k_1, k_2, k_{\widetilde{35}}) \\
 & \quad + \frac{1}{2}A_3^0(6_g; 1_Q, 2_{\bar{Q}}) \left(2|\mathcal{M}_5^0((\widetilde{12})_Q, \hat{6}_Q, 3_q, \hat{4}_q, 5_\gamma)|^2 + 2|\mathcal{M}_5^0(\hat{6}_{\bar{Q}}(\widetilde{12})_{\bar{Q}}, 3_q, \hat{4}_q, 5_\gamma)|^2 \right. \\
 & \quad - |\mathcal{M}_5^0((\widetilde{12})_Q, 5_g, \hat{6}_Q; ; 3_q, \hat{4}_q)|^2 - |\mathcal{M}_5^0(\hat{6}_{\bar{Q}}, 5_g, (\widetilde{12})_{\bar{Q}}; ; 3_q, \hat{4}_q)|^2 \\
 & \quad \left. - |\mathcal{M}_5^0((\widetilde{12})_Q, \hat{6}_Q; ; 3_q, 5_g, \hat{4}_q)|^2 - |\mathcal{M}_5^0(\hat{6}_{\bar{Q}}, (\widetilde{12})_{\bar{Q}}; ; 3_q, 5_g, \hat{4}_q)|^2 \right) J_3^{(3)}(k_{\widetilde{12}}, k_3, k_5) \\
 & \quad + A_3^0(4_q, 6_g; 3_q) \left(2|\mathcal{M}_5^0(\widetilde{1}_Q, \widetilde{2}_{\bar{Q}}, \hat{6}_{\bar{q}}, \hat{4}_q, \hat{5}_\gamma)|^2 \right. \\
 & \quad \left. - |\mathcal{M}_5^0(\widetilde{1}_Q, \hat{5}_\gamma, \widetilde{2}_{\bar{Q}}; ; \hat{6}_{\bar{q}}, \hat{4}_q)|^2 - |\mathcal{M}_5^0(\widetilde{1}_Q, \widetilde{2}_{\bar{Q}}; ; \hat{6}_{\bar{q}}, \hat{5}_\gamma, \hat{4}_q)|^2 \right) J_3^{(3)}(\tilde{k}_1, \tilde{k}_2, \tilde{k}_5) \\
 & \quad + \frac{1}{2}E_3^0(4_q; 3_q, 1_Q) |\mathcal{M}_5^0((\widetilde{13})_Q, 2_{\bar{Q}}, \hat{4}_\gamma, 5_\gamma, \hat{6}_\gamma)|^2 J_3^{(3)}(k_{\widetilde{13}}, k_2, k_5) \\
 & \quad \left. + \frac{1}{2}E_3^0(4_q; 3_q, 2_{\bar{Q}}) |\mathcal{M}_5^0(1_Q, (\widetilde{25})_{\bar{Q}}, \hat{4}_\gamma, 5_\gamma, \hat{6}_\gamma)|^2 J_3^{(3)}(k_1, k_{\widetilde{23}}, k_5) \right] \}
 \end{aligned}$$

Finally, when gluons 5_g and 6_g are crossed to the initial state in eq. (7.46), the squared matrix element for $gg \rightarrow Q\bar{Q}q\bar{q}$ is obtained. This partonic process only contains collinear and quasi-collinear limits but no soft limits as no gluons are present in the final state. Using the decoupling identities given in eqs. (7.6), (7.12), (7.13), the reduced matrix elements multiplying each antenna function can be rewritten in a fairly compact form with no interference terms left. After doing this simplification the subtraction term for this process reads,

$$\begin{aligned}
 d\hat{\sigma}_{gg \rightarrow Q\bar{Q}q\bar{q}}^S &= g^8 N_F (N_c^2 - 1) d\Phi_4(k_{1Q}, k_{2\bar{Q}}, k_{3q}, k_{4\bar{q}}; p_{5g}, p_{6g}) \\
 &\times \left\{ \sum_{(i,j) \in P(5,6)} \left[\right. \right. \\
 & \quad N_c^2 \left[\frac{1}{2}A_3^0(i_g; 1_Q, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0((\widetilde{12})_Q, 4_{\bar{q}}; ; 3_q, \hat{j}_g, \hat{i}_Q)|^2 + |\mathcal{M}_5^0(\hat{i}_{\bar{Q}}, 4_{\bar{q}}; ; 3_q, \hat{j}_g, (\widetilde{12})_{\bar{Q}})|^2 \right. \right. \right. \\
 & \quad \left. \left. + |\mathcal{M}_5^0((\widetilde{12})_Q, \hat{j}_g, 4_{\bar{q}}; ; 3_q, \hat{i}_Q)|^2 + |\mathcal{M}_5^0(\hat{i}_{\bar{Q}}, \hat{j}_g, 4_{\bar{q}}; ; 3_q, (\widetilde{12})_{\bar{Q}})|^2 \right) J_3^{(3)}(k_{\widetilde{12}}, k_3, k_4) \right. \\
 & \quad \left. \left. + \frac{1}{2}A_3^0(i_g; 3_q, 4_{\bar{q}}) \left(|\mathcal{M}_5^0(1_Q, \hat{i}_g; ; (\widetilde{34})_q, \hat{j}_g, 2_{\bar{Q}})|^2 + |\mathcal{M}_5^0(1_Q, (\widetilde{34})_{\bar{q}}; ; \hat{i}_{\bar{q}}, \hat{j}_g, 2_{\bar{Q}})|^2 \right. \right. \right.
 \end{aligned}$$

$$\begin{aligned}
 & + |\mathcal{M}_5^0(1_Q, \hat{j}_g, \hat{i}_q; ; (\widetilde{34})_q, 2_{\bar{Q}})|^2 + |\mathcal{M}_5^0(1_Q, \hat{j}_g, (\widetilde{34})_{\bar{q}}; ; \hat{i}_{\bar{q}}, 2_{\bar{Q}})|^2 \Big) J_3^{(3)}(k_1, k_2, k_{\widetilde{34}}) \\
 & + \frac{1}{2} E_3^0(1_Q, 3_q, 4_{\bar{q}}) \left(|\mathcal{M}_5^0((\widetilde{13})_Q, \hat{i}_g, \hat{j}_g, (\widetilde{34})_g, 2_{\bar{Q}})|^2 + |\mathcal{M}_5^0((\widetilde{13})_Q, \hat{i}_g, (\widetilde{34})_g, \hat{j}_g, 2_{\bar{Q}})|^2 \right. \\
 & \quad \left. + |\mathcal{M}_5^0((\widetilde{13})_Q, (\widetilde{34})_g, \hat{i}_g, \hat{j}_g, 2_{\bar{Q}})|^2 \right) J_3^{(3)}(k_{\widetilde{13}}, k_2, k_{\widetilde{34}}) \\
 & + \frac{1}{2} E_3^0(2_{\bar{Q}}, 3_q, 4_{\bar{q}}) \left(|\mathcal{M}_5^0(1_Q, \hat{i}_g, \hat{j}_g, (\widetilde{34})_g, (\widetilde{23})_{\bar{Q}})|^2 + |\mathcal{M}_5^0(1_Q, \hat{i}_g, (\widetilde{34})_g, \hat{j}_g, (\widetilde{23})_{\bar{Q}})|^2 \right. \\
 & \quad \left. + |\mathcal{M}_5^0(1_Q, (\widetilde{34})_g, \hat{i}_g, \hat{j}_g, (\widetilde{23})_{\bar{Q}})|^2 \right) J_3^{(3)}(k_1, k_{\widetilde{23}}, k_{\widetilde{34}}) \\
 & + \frac{1}{2} A_3^0(i_g; 1_Q, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0((\widetilde{12})_Q, \hat{i}_Q; ; 3_q, \hat{j}_g, 4_{\bar{q}})|^2 + |\mathcal{M}_5^0(\hat{i}_{\bar{Q}}, (\widetilde{12})_{\bar{Q}}; ; 3_q, \hat{j}_g, 4_{\bar{q}})|^2 \right. \\
 & \quad + |\mathcal{M}_5^0((\widetilde{12})_Q, \hat{j}_g, \hat{i}_Q; ; 3_q, 4_{\bar{q}})|^2 + |\mathcal{M}_5^0(\hat{i}_{\bar{Q}}, \hat{j}_g, (\widetilde{12})_{\bar{Q}}; ; 3_q, 4_{\bar{q}})|^2 \\
 & \quad - |\mathcal{M}_5^0((\widetilde{12})_Q, 4_{\bar{q}}; ; 3_q, \hat{j}_g, \hat{i}_Q)|^2 - |\mathcal{M}_5^0(\hat{i}_{\bar{Q}}, 4_{\bar{q}}; ; 3_q, \hat{j}_g, (\widetilde{12})_{\bar{Q}})|^2 \\
 & \quad - |\mathcal{M}_5^0((\widetilde{12})_Q, \hat{j}_g, 4_{\bar{q}}; ; 3_q, \hat{i}_Q)|^2 - |\mathcal{M}_5^0(\hat{i}_{\bar{Q}}, \hat{j}_g, 4_{\bar{q}}; ; 3_q, (\widetilde{12})_{\bar{Q}})|^2 \\
 & \quad \left. - 2|\mathcal{M}_5^0((\widetilde{12})_Q, \hat{i}_Q, 3_q, 4_{\bar{q}}, \hat{j}_\gamma)|^2 - 2|\mathcal{M}_5^0(\hat{i}_{\bar{Q}}, (\widetilde{12})_{\bar{Q}}, 3_q, 4_{\bar{q}}, \hat{j}_\gamma)|^2 \right) J_3^{(3)}(k_{\widetilde{12}}, k_3, k_4) \\
 & + \frac{1}{2} A_3^0(i_g; 3_q, 4_{\bar{q}}) \left(|\mathcal{M}_5^0(1_Q, 2_{\bar{Q}}; ; (\widetilde{34})_q, \hat{j}_g, \hat{i}_q)|^2 + |\mathcal{M}_5^0(1_Q, 2_{\bar{Q}}; ; \hat{i}_{\bar{q}}, \hat{j}_g, (\widetilde{34})_{\bar{q}})|^2 \right. \\
 & \quad + |\mathcal{M}_5^0(1_Q, \hat{j}_g, 2_{\bar{Q}}; ; (\widetilde{34})_q, \hat{i}_q)|^2 + |\mathcal{M}_5^0(1_Q, \hat{j}_g, 2_{\bar{Q}}; ; \hat{i}_{\bar{q}}, (\widetilde{34})_{\bar{q}})|^2 \\
 & \quad - |\mathcal{M}_5^0(1_Q, \hat{i}_q; ; (\widetilde{34})_q, \hat{j}_g, 2_{\bar{Q}})|^2 - |\mathcal{M}_5^0(1_Q, (\widetilde{34})_{\bar{q}}; ; \hat{i}_{\bar{q}}, \hat{j}_g, 2_{\bar{Q}})|^2 \\
 & \quad - |\mathcal{M}_5^0(1_Q, \hat{j}_g, \hat{i}_q; ; (\widetilde{34})_q, 2_{\bar{Q}})|^2 - |\mathcal{M}_5^0(1_Q, \hat{j}_g, (\widetilde{34})_{\bar{q}}; ; \hat{i}_{\bar{q}}, 2_{\bar{Q}})|^2 \\
 & \quad \left. - 2|\mathcal{M}_5^0(1_Q, 2_{\bar{Q}}, (\widetilde{34})_q, \hat{i}_q, \hat{j}_\gamma)|^2 - 2|\mathcal{M}_5^0(1_Q, 2_{\bar{Q}}, \hat{i}_{\bar{q}}, (\widetilde{34})_{\bar{q}}, \hat{j}_\gamma)|^2 \right) J_3^{(3)}(k_1, k_2, k_{\widetilde{34}}) \\
 & - \frac{1}{2} E_3^0(1_Q, 3_q, 4_{\bar{q}}) \left(|\mathcal{M}_5^0((\widetilde{13})_Q, \hat{i}_g, \hat{j}_g, (\widetilde{34})_\gamma, 2_{\bar{Q}})|^2 + |\mathcal{M}_5^0((\widetilde{13})_Q, \hat{i}_g, (\widetilde{34})_g, \hat{j}_\gamma, 2_{\bar{Q}})|^2 \right. \\
 & \quad \left. + |\mathcal{M}_5^0((\widetilde{13})_Q, (\widetilde{34})_g, \hat{i}_g, \hat{j}_\gamma, 2_{\bar{Q}})|^2 \right) J_3^{(3)}(k_{\widetilde{13}}, k_2, k_{\widetilde{34}}) \\
 & - \frac{1}{2} E_3^0(2_{\bar{Q}}, 3_q, 4_{\bar{q}}) \left(|\mathcal{M}_5^0(1_Q, \hat{i}_g, \hat{j}_g, (\widetilde{34})_\gamma, (\widetilde{23})_{\bar{Q}})|^2 + |\mathcal{M}_5^0(1_Q, \hat{i}_g, (\widetilde{34})_g, \hat{j}_\gamma, (\widetilde{23})_{\bar{Q}})|^2 \right. \\
 & \quad \left. + |\mathcal{M}_5^0(1_Q, (\widetilde{34})_g, \hat{i}_g, \hat{j}_\gamma, (\widetilde{23})_{\bar{Q}})|^2 \right) J_3^{(3)}(k_1, k_{\widetilde{23}}, k_{\widetilde{34}}) \\
 & - \frac{1}{2N_c^2} \left[A_3^0(i_g; 1_Q, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0((\widetilde{12})_Q, \hat{i}_Q; ; 3_q, \hat{j}_g, 4_{\bar{q}})|^2 + |\mathcal{M}_5^0(\hat{i}_{\bar{Q}}, (\widetilde{12})_{\bar{Q}}; ; 3_q, \hat{j}_g, 4_{\bar{q}})|^2 \right. \right. \\
 & \quad + |\mathcal{M}_5^0((\widetilde{12})_Q, \hat{j}_g, \hat{i}_Q; ; 3_q, 4_{\bar{q}})|^2 + |\mathcal{M}_5^0(\hat{i}_{\bar{Q}}, \hat{j}_g, (\widetilde{12})_{\bar{Q}}; ; 3_q, 4_{\bar{q}})|^2 \\
 & \quad \left. - 2|\mathcal{M}_5^0((\widetilde{12})_Q, \hat{i}_Q, 3_q, 4_{\bar{q}}, \hat{j}_\gamma)|^2 - 2|\mathcal{M}_5^0(\hat{i}_{\bar{Q}}, (\widetilde{12})_{\bar{Q}}, 3_q, 4_{\bar{q}}, \hat{j}_\gamma)|^2 \right) J_3^{(3)}(k_{\widetilde{12}}, k_3, k_4) \\
 & + A_3^0(i_g; 3_q, 4_{\bar{q}}) \left(|\mathcal{M}_5^0(1_Q, 2_{\bar{Q}}; ; (\widetilde{34})_q, \hat{j}_g, \hat{i}_q)|^2 + |\mathcal{M}_5^0(1_Q, 2_{\bar{Q}}; ; \hat{i}_{\bar{q}}, \hat{j}_g, (\widetilde{34})_{\bar{q}})|^2 \right. \\
 & \quad + |\mathcal{M}_5^0(1_Q, \hat{j}_g, 2_{\bar{Q}}; ; (\widetilde{34})_q, \hat{i}_q)|^2 + |\mathcal{M}_5^0(1_Q, \hat{j}_g, 2_{\bar{Q}}; ; \hat{i}_{\bar{q}}, (\widetilde{34})_{\bar{q}})|^2 \\
 & \quad \left. - 2|\mathcal{M}_5^0(1_Q, 2_{\bar{Q}}, (\widetilde{34})_q, \hat{i}_q, \hat{j}_\gamma)|^2 - 2|\mathcal{M}_5^0(1_Q, 2_{\bar{Q}}, \hat{i}_{\bar{q}}, (\widetilde{34})_{\bar{q}}, \hat{j}_\gamma)|^2 \right) J_3^{(3)}(k_1, k_2, k_{\widetilde{34}}) \\
 & - \frac{N_c^2 - 1}{4N_c^2} \left(E_3^0(1_Q, 3_q, 4_{\bar{q}}) |\mathcal{M}_5^0((\widetilde{13})_Q, (\widetilde{34})_\gamma, \hat{i}_\gamma, \hat{j}_\gamma, 2_{\bar{Q}})|^2 J_3^{(3)}(k_{\widetilde{13}}, k_2, k_{\widetilde{34}}) \right. \\
 & \quad \left. + E_3^0(2_{\bar{Q}}, 3_q, 4_{\bar{q}}) |\mathcal{M}_5^0(1_Q, (\widetilde{34})_\gamma, \hat{i}_\gamma, \hat{j}_\gamma, (\widetilde{23})_{\bar{Q}})|^2 J_3^{(3)}(k_1, k_{\widetilde{23}}, k_{\widetilde{34}}) \right) \Big] \Big\}.
 \end{aligned}
 \tag{7.47}$$

For all three crossings considered, we have performed the consistency check explained in section 7.1.3. We have checked that in all collinear or quasi-collinear limits contained in these subtraction terms, the sum of the terms contributing to a given limit collapses to

the appropriate colour factor multiplying a given massless or massive splitting function, corresponding to the given collinear splitting, times the appropriate non-colour ordered matrix-element squared.

7.2.4 Partonic process $gg \rightarrow Q\bar{Q}gg$

The colour decomposition for the unphysical process $0 \rightarrow Q\bar{Q}gggg$ is

$$M_6^0(1_Q, 2_{\bar{Q}}, 3_g, 4_g, 5_g, 6_g) = 4g^4 \sum_{(i,j,k,l) \in P(3,4,5,6)} (T^{a_i} T^{a_j} T^{a_k} T^{a_l})_{i_1, i_2} \mathcal{M}(i, j, k, l), \quad (7.48)$$

where we used $\mathcal{M}(i, j, k, l) = \mathcal{M}_6^0(1_Q, i_g, j_g, k_g, l_g, 2_{\bar{Q}})$ for simplification of the formulae. Squaring gives

$$\begin{aligned} |M_6^0(1_Q, 2_{\bar{Q}}, 3_g, 4_g, 5_g, 6_g)|^2 &= \frac{g^8(N_c^2 - 1)}{N_c^3} \\ &\times \left\{ \sum_{(i,j,k,l) \in P(3,4,5,6)} \left[N_c^6 |\mathcal{M}(i, j, k, l)|^2 - N_c^4 |\mathcal{M}(i, j, k; ; l)|^2 + \frac{N_c^2}{2!} |\mathcal{M}(i, j; ; k, l)|^2 \right. \right. \\ &\quad - N_c^4 \text{Re} \left((\mathcal{M}(j, i, l, k) + \mathcal{M}(j, l, i, k) + \mathcal{M}(j, l, k, i) \right. \\ &\quad \quad + \mathcal{M}(k, i, l, j) + \mathcal{M}(k, j, l, i) + \mathcal{M}(l, i, k, j) \\ &\quad \quad \quad \left. \left. + \mathcal{M}(l, j, i, k) + \mathcal{M}(l, k, j, i) \right) \times \mathcal{M}(i, j, k, l)^\dagger \right) \\ &\quad \left. + \frac{(N_c^4 - 3N_c^2 - 1)}{4!} |\bar{\mathcal{M}}(i, j, k, l)|^2 \right\}, \quad (7.49) \end{aligned}$$

where when the gluon labelled with l , when the gluons labelled with l and with k or when all four gluons decouple we have respectively,

$$\begin{aligned} \mathcal{M}(i, j, k; ; l) &= \mathcal{M}_6^0(1_Q, i_g, j_g, k_g, l_\gamma, 2_{\bar{Q}}) \\ &= \mathcal{M}(i, j, k, l) + \mathcal{M}(i, j, l, k) + \mathcal{M}(i, l, j, k) + \mathcal{M}(l, i, j, k) \end{aligned} \quad (7.50)$$

$$\begin{aligned} \mathcal{M}(i, j; ; k, l) &= \mathcal{M}_6^0(1_Q, i_g, j_g, k_\gamma, l_\gamma, 2_{\bar{Q}}) \\ &= \mathcal{M}(i, j, k, l) + \mathcal{M}(i, j, l, k) + \mathcal{M}(i, k, j, l) + \mathcal{M}(i, l, j, k) \\ &\quad + \mathcal{M}(i, k, l, j) + \mathcal{M}(i, l, k, j) + \mathcal{M}(k, i, j, l) + \mathcal{M}(l, i, j, k) \\ &\quad + \mathcal{M}(k, i, l, j) + \mathcal{M}(l, i, k, j) + \mathcal{M}(k, l, i, j) + \mathcal{M}(l, k, i, j) \end{aligned} \quad (7.51)$$

$$\begin{aligned} \bar{\mathcal{M}}(3, 4, 5, 6) &= \mathcal{M}_6^0(1_Q, 3_\gamma, 4_\gamma, 5_\gamma, 6_\gamma, 2_{\bar{Q}}) \\ &= \sum_{(i,j,k,l) \in P(3,4,5,6)} \mathcal{M}_6^0(1_Q, i_g, j_g, k_g, l_g, 2_{\bar{Q}}) \end{aligned} \quad (7.52)$$

This colour-ordered decomposition for the squared amplitude for $gg \rightarrow Q\bar{Q}gg$ has also been derived in [112]. Our result is in complete agreement with it.

Crossing gluons 5_g and 6_g in eq. (7.49) gives the squared matrix element for $gg \rightarrow Q\bar{Q}gg$ reading,

$$|M_6^0(1_Q, 2_{\bar{Q}}, 3_g, 4_g, \hat{5}_g, \hat{6}_g)|^2 = \frac{g^8(N_c^2 - 1)}{N_c^3}$$

$$\begin{aligned}
 & \times \left\{ \sum_{(i,j) \in P(3,4), (k,l) \in P(5,6)} \left[N_c^6 \left(|\mathcal{M}(i, j, \hat{k}, \hat{l})|^2 + |\mathcal{M}(i, \hat{k}, j, \hat{l})|^2 + |\mathcal{M}(i, \hat{k}, \hat{l}, j)|^2 \right. \right. \\
 & \quad \left. \left. + |\mathcal{M}(\hat{k}, i, j, \hat{l})|^2 + |\mathcal{M}(\hat{k}, i, \hat{l}, j)|^2 + |\mathcal{M}(\hat{k}, \hat{l}, i, j)|^2 \right) \right. \\
 & - N_c^4 \left(|\mathcal{M}(i, j, \hat{k}; ; \hat{l})|^2 + |\mathcal{M}(i, \hat{k}, j; ; \hat{l})|^2 + |\mathcal{M}(i, \hat{k}, \hat{l}; ; j)|^2 \right. \\
 & \quad \left. + |\mathcal{M}(\hat{k}, i, j; ; \hat{l})|^2 + |\mathcal{M}(\hat{k}, i, \hat{l}; ; j)|^2 + |\mathcal{M}(\hat{k}, \hat{l}, i; ; j)|^2 \right) \\
 & - N_c^4 \left[\text{Re} \left((\mathcal{M}(j, i, \hat{l}, \hat{k}) + \mathcal{M}(j, \hat{l}, i, \hat{k}) + \mathcal{M}(j, \hat{l}, \hat{k}, i) + \mathcal{M}(\hat{k}, i, \hat{l}, j) \right. \right. \\
 & \quad \left. \left. + \mathcal{M}(\hat{k}, j, \hat{l}, i) + \mathcal{M}(\hat{l}, i, \hat{k}, j) + \mathcal{M}(\hat{l}, j, i, \hat{k}) + \mathcal{M}(\hat{l}, \hat{k}, j, i) \right) \times \mathcal{M}(i, j, \hat{k}, \hat{l})^\dagger \right) \\
 & + \text{Re} \left((\mathcal{M}(\hat{k}, i, \hat{l}, j) + \mathcal{M}(\hat{k}, \hat{l}, i, j) + \mathcal{M}(\hat{k}, \hat{l}, j, i) + \mathcal{M}(j, i, \hat{l}, \hat{k}) \right. \\
 & \quad \left. + \mathcal{M}(j, \hat{k}, \hat{l}, i) + \mathcal{M}(\hat{l}, i, j, \hat{k}) + \mathcal{M}(\hat{l}, \hat{k}, i, j) + \mathcal{M}(\hat{l}, j, \hat{k}, i) \right) \times \mathcal{M}(i, \hat{k}, j, \hat{l})^\dagger \right) \\
 & + \text{Re} \left((\mathcal{M}(\hat{k}, i, j, \hat{l}) + \mathcal{M}(\hat{k}, j, i, \hat{l}) + \mathcal{M}(\hat{k}, j, \hat{l}, i) + \mathcal{M}(\hat{l}, i, j, \hat{k}) \right. \\
 & \quad \left. + \mathcal{M}(\hat{l}, \hat{k}, j, i) + \mathcal{M}(j, i, \hat{l}, \hat{k}) + \mathcal{M}(j, \hat{k}, i, \hat{l}) + \mathcal{M}(j, \hat{l}, \hat{k}, i) \right) \times \mathcal{M}(i, \hat{k}, \hat{l}, j)^\dagger \right) \\
 & + \text{Re} \left((\mathcal{M}(i, \hat{k}, \hat{l}, j) + \mathcal{M}(i, \hat{l}, \hat{k}, j) + \mathcal{M}(i, \hat{l}, j, \hat{k}) + \mathcal{M}(j, \hat{k}, \hat{l}, i) \right. \\
 & \quad \left. + \mathcal{M}(j, i, \hat{l}, \hat{k}) + \mathcal{M}(\hat{l}, \hat{k}, j, i) + \mathcal{M}(\hat{l}, i, \hat{k}, j) + \mathcal{M}(\hat{l}, j, i, \hat{k}) \right) \times \mathcal{M}(\hat{k}, i, j, \hat{l})^\dagger \right) \\
 & + \text{Re} \left((\mathcal{M}(i, \hat{k}, j, \hat{l}) + \mathcal{M}(i, j, \hat{k}, \hat{l}) + \mathcal{M}(i, j, \hat{l}, \hat{k}) + \mathcal{M}(\hat{l}, \hat{k}, j, i) \right. \\
 & \quad \left. + \mathcal{M}(\hat{l}, i, j, \hat{k}) + \mathcal{M}(j, \hat{k}, \hat{l}, i) + \mathcal{M}(j, i, \hat{k}, \hat{l}) + \mathcal{M}(j, \hat{l}, i, \hat{k}) \right) \times \mathcal{M}(\hat{k}, i, \hat{l}, j)^\dagger \right) \\
 & + \text{Re} \left((\mathcal{M}(\hat{l}, \hat{k}, j, i) + \mathcal{M}(\hat{l}, j, \hat{k}, i) + \mathcal{M}(\hat{l}, j, i, \hat{k}) + \mathcal{M}(i, \hat{k}, j, \hat{l}) \right. \\
 & \quad \left. + \mathcal{M}(i, \hat{l}, j, \hat{k}) + \mathcal{M}(j, \hat{k}, i, \hat{l}) + \mathcal{M}(j, \hat{l}, \hat{k}, i) + \mathcal{M}(j, i, \hat{l}, \hat{k}) \right) \times \mathcal{M}(\hat{k}, \hat{l}, i, j)^\dagger \right) \\
 & \left. + N_c^2 \left[\frac{1}{2} |\mathcal{M}(i, j; ; \hat{k}, \hat{l})|^2 + |\mathcal{M}(i, \hat{k}; ; j, \hat{l})|^2 + |\mathcal{M}(\hat{k}, i; ; j, \hat{l})|^2 + \frac{1}{2} |\mathcal{M}(\hat{k}, \hat{l}; ; i, k)|^2 \right] \right. \\
 & \left. + \frac{(N_c^4 - 3N_c^2 - 1)}{4} |\bar{\mathcal{M}}(i, j, \hat{k}, \hat{l})|^2 \right\}, \tag{7.53}
 \end{aligned}$$

The expression above is obtained as follows. We start by expanding the sum over permutations for all four gluons present in the final state in the colour decomposition for the unphysical process $0 \rightarrow Q\bar{Q}g g g g$, given in eq. (7.49). Then, gluons 5_g and 6_g are crossed to the initial state. Finally, the terms are recombined in a double sum over permutations for the

two initial state and the two final state gluons. The corresponding subtraction term reads,

$$\begin{aligned}
 d\hat{\sigma}_{gg \rightarrow Q\bar{Q}gg}^S &= g^8(N_c^2 - 1)d\Phi_4(k_{1Q}, k_{2\bar{Q}}, k_{3g}, k_{4g}; p_{5g}, p_{6g}) \\
 &\times \sum_{(i,j) \in P(3,4), (k,l) \in P(5,6)} \left\{ \right. \\
 &N_c^3 \left[\frac{1}{2}A_3^0(l_g; 1_Q, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0((\tilde{1}2)_Q, i_g, j_g, \hat{k}_g, \hat{l}_Q)|^2 + |\mathcal{M}_5^0(\hat{l}_{\bar{Q}}, i_g, j_g, \hat{k}_g, (\tilde{1}2)_{\bar{Q}})|^2 \right. \right. \\
 &\quad + |\mathcal{M}_5^0((\tilde{1}2)_Q, i_g, \hat{k}_g, j_g, \hat{l}_Q)|^2 + |\mathcal{M}_5^0(\hat{l}_{\bar{Q}}, i_g, \hat{k}_g, j_g, (\tilde{1}2)_{\bar{Q}})|^2 \\
 &\quad \left. \left. + |\mathcal{M}_5^0((\tilde{1}2)_Q, \hat{k}_g, i_g, j_g, \hat{l}_Q)|^2 + |\mathcal{M}_5^0(\hat{l}_{\bar{Q}}, \hat{k}_g, i_g, j_g, (\tilde{1}2)_{\bar{Q}})|^2 \right) J_3^{(3)}(k_{\tilde{1}2}, k_3, k_4) \right. \\
 &\quad + d_3^0(1_Q, i_g, j_g) |\mathcal{M}_5^0((\tilde{1}i)_Q, (\tilde{i}j)_g, \hat{k}_g, \hat{l}_g, 2_{\bar{Q}})|^2 J_3^{(3)}(k_{\tilde{1}i}, k_2, k_{\tilde{i}j}) \\
 &\quad + d_3^0(2_{\bar{Q}}, i_g, j_g) |\mathcal{M}_5^0(1_Q, \hat{k}_g, \hat{l}_g, (\tilde{i}j)_g, (\tilde{2}i)_{\bar{Q}})|^2 J_3^{(3)}(k_1, k_{\tilde{2}i}, k_{\tilde{i}j}) \\
 &\quad + D_3^0(l_g; i_g, 1_Q) \left(|\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{l}_g, j_g, \hat{k}_g, 2_{\bar{Q}})|^2 \right. \\
 &\quad \quad \left. + |\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{l}_g, \hat{k}_g, j_g, 2_{\bar{Q}}) \right|^2 J_3^{(3)}(k_{\tilde{1}i}, k_2, k_j) \right. \\
 &\quad + D_3^0(l_g; i_g, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0(1_Q, j_g, \hat{k}_g, \hat{l}_g, (\tilde{2}i)_{\bar{Q}})|^2 \right. \\
 &\quad \quad \left. + |\mathcal{M}_5^0(1_Q, \hat{k}_g, j_g, \hat{l}_g, (\tilde{2}i)_{\bar{Q}})|^2 \right) J_3^{(3)}(k_1, k_{\tilde{2}i}, k_j) \\
 &\quad + f_3^0(l_g; i_g, j_g) \left(|\mathcal{M}_5^0(1_Q, (\tilde{i}j)_g, \hat{l}_g, \hat{k}_g, 2_{\bar{Q}})|^2 + |\mathcal{M}_5^0(1_Q, \hat{l}_g, (\tilde{i}j)_g, \hat{k}_g, 2_{\bar{Q}})|^2 \right. \\
 &\quad \quad \left. + |\mathcal{M}_5^0(1_Q, \hat{k}_g, (\tilde{i}j)_g, \hat{l}_g, 2_{\bar{Q}})|^2 + |\mathcal{M}_5^0(1_Q, \hat{k}_g, \hat{l}_g, (\tilde{i}j)_g, 2_{\bar{Q}})|^2 \right) J_3^{(3)}(k_1, k_2, k_{\tilde{i}j}) \\
 &\quad \left. + F_3^0(k_g, l_g; i_g) \left(|\mathcal{M}_5^0(\tilde{1}Q, \hat{k}_g, \hat{l}_g, \tilde{j}g, \tilde{2}\bar{Q})|^2 + |\mathcal{M}_5^0(\tilde{1}Q, \tilde{j}g, \hat{k}_g, \hat{l}_g, \tilde{2}\bar{Q})|^2 \right) J_3^{(3)}(\tilde{k}_1, \tilde{k}_2, \tilde{k}_j) \right] \\
 &- N_c \left[A_3^0(1_Q, i_g, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0((\tilde{1}i)_Q, j_g, \hat{k}_g, \hat{l}_g, (\tilde{2}i)_{\bar{Q}})|^2 - (1/2)|\mathcal{M}_5^0((\tilde{1}i)_Q, j_\gamma, \hat{k}_\gamma, \hat{l}_\gamma, (\tilde{2}i)_{\bar{Q}})|^2 \right. \right. \\
 &\quad + |\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{k}_g, j_g, \hat{l}_g, (\tilde{2}i)_{\bar{Q}})|^2 + |\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{k}_g, \hat{l}_g, j_g, (\tilde{2}i)_{\bar{Q}})|^2 \\
 &\quad + \text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, j_g, \hat{k}_g, \hat{l}_g, (\tilde{2}i)_{\bar{Q}})\mathcal{M}_5^0((\tilde{1}i)_Q, j_g, \hat{l}_g, \hat{k}_g, (\tilde{2}i)_{\bar{Q}})^\dagger) \\
 &\quad + \text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{k}_g, j_g, \hat{l}_g, (\tilde{2}i)_{\bar{Q}})\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{l}_g, j_g, \hat{k}_g, (\tilde{2}i)_{\bar{Q}})^\dagger) \\
 &\quad + \text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{k}_g, \hat{l}_g, j_g, (\tilde{2}i)_{\bar{Q}})\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{l}_g, \hat{k}_g, j_g, (\tilde{2}i)_{\bar{Q}})^\dagger) \\
 &\quad + 2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, j_g, \hat{k}_g, \hat{l}_g, (\tilde{2}i)_{\bar{Q}})\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{k}_g, j_g, \hat{l}_g, (\tilde{2}i)_{\bar{Q}})^\dagger) \\
 &\quad + 2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, j_g, \hat{k}_g, \hat{l}_g, (\tilde{2}i)_{\bar{Q}})\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{l}_g, \hat{k}_g, j_g, (\tilde{2}i)_{\bar{Q}})^\dagger) \\
 &\quad \left. \left. + 2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{k}_g, j_g, \hat{l}_g, (\tilde{2}i)_{\bar{Q}})\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{k}_g, \hat{l}_g, j_g, (\tilde{2}i)_{\bar{Q}})^\dagger) \right) J_3^{(3)}(k_{\tilde{1}i}, k_{\tilde{2}i}, k_j) \right. \\
 &\quad \left. + \frac{1}{2}A_3^0(l_g; 1_Q, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0((\tilde{1}2)_Q, i_g, j_g, \hat{k}_g, \hat{l}_Q)|^2 + |\mathcal{M}_5^0(\hat{l}_{\bar{Q}}, i_g, j_g, \hat{k}_g, (\tilde{1}2)_{\bar{Q}})|^2 \right. \right. \\
 &\quad + |\mathcal{M}_5^0((\tilde{1}2)_Q, i_g, \hat{k}_g, j_g, \hat{l}_Q)|^2 + |\mathcal{M}_5^0(\hat{l}_{\bar{Q}}, i_g, \hat{k}_g, j_g, (\tilde{1}2)_{\bar{Q}})|^2 \\
 &\quad + |\mathcal{M}_5^0((\tilde{1}2)_Q, \hat{k}_g, i_g, j_g, \hat{l}_Q)|^2 + |\mathcal{M}_5^0(\hat{l}_{\bar{Q}}, \hat{k}_g, i_g, j_g, (\tilde{1}2)_{\bar{Q}})|^2 \\
 &\quad + |\mathcal{M}_5^0((\tilde{1}2)_Q, i_g, j_g, \hat{k}_\gamma, \hat{l}_Q)|^2 + |\mathcal{M}_5^0(\hat{l}_{\bar{Q}}, i_g, j_g, \hat{k}_\gamma, (\tilde{1}2)_{\bar{Q}})|^2 \\
 &\quad \left. \left. + |\mathcal{M}_5^0((\tilde{1}2)_Q, i_g, \hat{k}_g, j_\gamma, \hat{l}_Q)|^2 + |\mathcal{M}_5^0(\hat{l}_{\bar{Q}}, i_g, \hat{k}_g, j_\gamma, (\tilde{1}2)_{\bar{Q}})|^2 \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 & + |\mathcal{M}_5^0((\tilde{1}2)_Q, \hat{k}_g, i_g, j_\gamma, \hat{l}_Q)|^2 + |\mathcal{M}_5^0(\hat{l}_Q, \hat{k}_g, i_g, j_\gamma, (\tilde{1}2)_{\bar{Q}})|^2 \\
 & - (1/2) |\mathcal{M}_5^0((\tilde{1}2)_Q, i_\gamma, j_\gamma, \hat{k}_\gamma, \hat{l}_Q)|^2 \\
 & - (1/2) |\mathcal{M}_5^0(\hat{l}_Q, i_\gamma, j_\gamma, \hat{k}_\gamma, (\tilde{1}2)_{\bar{Q}})|^2 \Big) J_3^{(3)}(k_{\tilde{1}2}, k_i, k_j) \\
 & + d_3^0(1_Q, i_g, j_g) \left(|\mathcal{M}_5^0((\tilde{1}i)_Q, (\tilde{i}j)_g, \hat{k}_g, \hat{l}_\gamma, 2_{\bar{Q}})|^2 \right. \\
 & \quad + \text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{k}_g, (\tilde{i}j)_g, \hat{l}_\gamma, 2_{\bar{Q}}) \mathcal{M}_5^0((\tilde{1}i)_Q, \hat{l}_\gamma, (\tilde{i}j)_g, \hat{k}_g, 2_{\bar{Q}})^\dagger) \\
 & \quad - \text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, (\tilde{i}j)_g, \hat{k}_g, \hat{l}_\gamma, 2_{\bar{Q}}) \mathcal{M}_5^0((\tilde{1}i)_Q, (\tilde{i}j)_g, \hat{l}_\gamma, \hat{k}_g, 2_{\bar{Q}})^\dagger) \\
 & \quad - \text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{k}_g, \hat{l}_\gamma, (\tilde{i}j)_g, 2_{\bar{Q}}) \mathcal{M}_5^0((\tilde{1}i)_Q, \hat{l}_\gamma, \hat{k}_g, (\tilde{i}j)_g, 2_{\bar{Q}})^\dagger) \\
 & \quad + 2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{k}_g, \hat{l}_\gamma, (\tilde{i}j)_g, 2_{\bar{Q}}) \mathcal{M}_5^0((\tilde{1}i)_Q, \hat{l}_\gamma, (\tilde{i}j)_g, \hat{k}_g, 2_{\bar{Q}})^\dagger) \\
 & \quad - 2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, (\tilde{i}j)_g, \hat{k}_g, \hat{l}_\gamma, 2_{\bar{Q}}) \mathcal{M}_5^0((\tilde{1}i)_Q, \hat{l}_\gamma, (\tilde{i}j)_g, \hat{k}_g, 2_{\bar{Q}})^\dagger) \\
 & \quad \left. - 2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, (\tilde{i}j)_g, \hat{k}_g, \hat{l}_\gamma, 2_{\bar{Q}}) \mathcal{M}_5^0((\tilde{1}i)_Q, \hat{l}_\gamma, \hat{k}_g, (\tilde{i}j)_g, 2_{\bar{Q}})^\dagger) \right) J_3^{(3)}(k_{\tilde{1}i}, k_2, k_{\tilde{i}j}) \\
 & + d_3^0(2_{\bar{Q}}, i_g, j_g) \left(|\mathcal{M}_5^0(1_Q, \hat{k}_g, (\tilde{i}j)_g, \hat{l}_\gamma, (\tilde{2}i)_{\bar{Q}})|^2 \right. \\
 & \quad + \text{Re}(\mathcal{M}_5^0(1_Q, \hat{k}_g, (\tilde{i}j)_g, \hat{l}_\gamma, (\tilde{2}i)_{\bar{Q}}) \mathcal{M}_5^0(1_Q, \hat{l}_\gamma, (\tilde{i}j)_g, \hat{k}_g, (\tilde{2}i)_{\bar{Q}})^\dagger) \\
 & \quad - \text{Re}(\mathcal{M}_5^0(1_Q, (\tilde{i}j)_g, \hat{k}_g, \hat{l}_\gamma, (\tilde{2}i)_{\bar{Q}}) \mathcal{M}_5^0(1_Q, (\tilde{i}j)_g, \hat{l}_\gamma, \hat{k}_g, (\tilde{2}i)_{\bar{Q}})^\dagger) \\
 & \quad - \text{Re}(\mathcal{M}_5^0(1_Q, \hat{k}_g, \hat{l}_\gamma, (\tilde{i}j)_g, (\tilde{2}i)_{\bar{Q}}) \mathcal{M}_5^0(1_Q, \hat{l}_\gamma, \hat{k}_g, (\tilde{i}j)_g, (\tilde{2}i)_{\bar{Q}})^\dagger) \\
 & \quad + 2\text{Re}(\mathcal{M}_5^0(1_Q, (\tilde{i}j)_g, \hat{k}_g, \hat{l}_\gamma, (\tilde{2}i)_{\bar{Q}}) \mathcal{M}_5^0(1_Q, \hat{l}_\gamma, (\tilde{i}j)_g, \hat{k}_g, (\tilde{2}i)_{\bar{Q}})^\dagger) \\
 & \quad - 2\text{Re}(\mathcal{M}_5^0(1_Q, \hat{k}_g, \hat{l}_\gamma, (\tilde{i}j)_g, (\tilde{2}i)_{\bar{Q}}) \mathcal{M}_5^0(1_Q, \hat{l}_\gamma, (\tilde{i}j)_g, \hat{k}_g, (\tilde{2}i)_{\bar{Q}})^\dagger) \\
 & \quad \left. - 2\text{Re}(\mathcal{M}_5^0(1_Q, (\tilde{i}j)_g, \hat{k}_g, \hat{l}_\gamma, (\tilde{2}i)_{\bar{Q}}) \mathcal{M}_5^0(1_Q, \hat{l}_\gamma, \hat{k}_g, (\tilde{i}j)_g, (\tilde{2}i)_{\bar{Q}})^\dagger) \right) J_3^{(3)}(k_1, k_{\tilde{2}i}, k_{\tilde{i}j}) \\
 & + D_3^0(l_g; i_g, 1_Q) \left(|\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{l}_g, j_g, \hat{k}_\gamma, 2_{\bar{Q}})|^2 + |\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{l}_g, \hat{k}_g, j_\gamma, 2_{\bar{Q}})|^2 \right. \\
 & \quad + 2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, j_g, \hat{l}_g, \hat{k}_g, 2_{\bar{Q}}) \mathcal{M}_5^0((\tilde{1}i)_Q, \hat{k}_g, j_g, \hat{l}_g, 2_{\bar{Q}})^\dagger) \\
 & \quad + 2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, j_g, \hat{l}_g, \hat{k}_g, 2_{\bar{Q}}) \mathcal{M}_5^0((\tilde{1}i)_Q, \hat{k}_g, \hat{l}_g, j_g, 2_{\bar{Q}})^\dagger) \\
 & \quad + 2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, j_g, \hat{k}_g, \hat{l}_g, 2_{\bar{Q}}) \mathcal{M}_5^0((\tilde{1}i)_Q, \hat{k}_g, \hat{l}_g, j_g, 2_{\bar{Q}})^\dagger) \\
 & \quad - 2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, j_g, \hat{l}_g, \hat{k}_g, 2_{\bar{Q}}) \mathcal{M}_5^0((\tilde{1}i)_Q, \hat{l}_g, \hat{k}_g, j_g, 2_{\bar{Q}})^\dagger) \\
 & \quad - 2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, j_g, \hat{k}_g, \hat{l}_g, 2_{\bar{Q}}) \mathcal{M}_5^0((\tilde{1}i)_Q, \hat{l}_g, \hat{k}_g, j_g, 2_{\bar{Q}})^\dagger) \\
 & \quad - 2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{l}_g, j_g, \hat{k}_g, 2_{\bar{Q}}) \mathcal{M}_5^0((\tilde{1}i)_Q, \hat{l}_g, \hat{k}_g, j_g, 2_{\bar{Q}})^\dagger) \\
 & \quad - 2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, j_g, \hat{k}_g, \hat{l}_g, 2_{\bar{Q}}) \mathcal{M}_5^0((\tilde{1}i)_Q, \hat{k}_g, j_g, \hat{l}_g, 2_{\bar{Q}})^\dagger) \\
 & \quad - 2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{l}_g, j_g, \hat{k}_g, 2_{\bar{Q}}) \mathcal{M}_5^0((\tilde{1}i)_Q, \hat{k}_g, j_g, \hat{l}_g, 2_{\bar{Q}})^\dagger) \\
 & \quad \left. - 2\text{Re}(\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{l}_g, j_g, \hat{k}_g, 2_{\bar{Q}}) \mathcal{M}_5^0((\tilde{1}i)_Q, \hat{k}_g, \hat{l}_g, j_g, 2_{\bar{Q}})^\dagger) \right) |^2 J_3^{(3)}(k_{\tilde{1}i}, k_2, k_j) \\
 & + D_3^0(l_g; i_g, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0(1_Q, j_g, \hat{l}_g, \hat{k}_\gamma, (\tilde{2}i)_{\bar{Q}})|^2 + |\mathcal{M}_5^0(1_Q, \hat{k}_g, \hat{l}_g, j_\gamma, (\tilde{2}i)_{\bar{Q}})|^2 \right. \\
 & \quad + 2\text{Re}(\mathcal{M}_5^0(1_Q, j_g, \hat{l}_g, \hat{k}_g, (\tilde{2}i)_{\bar{Q}}) \mathcal{M}_5^0(1_Q, \hat{l}_g, \hat{k}_g, j_g, (\tilde{2}i)_{\bar{Q}})^\dagger) \\
 & \quad + 2\text{Re}(\mathcal{M}_5^0(1_Q, j_g, \hat{l}_g, \hat{k}_g, (\tilde{2}i)_{\bar{Q}}) \mathcal{M}_5^0(1_Q, \hat{k}_g, \hat{l}_g, j_g, (\tilde{2}i)_{\bar{Q}})^\dagger) \\
 & \quad \left. + 2\text{Re}(\mathcal{M}_5^0(1_Q, \hat{l}_g, j_g, \hat{k}_g, (\tilde{2}i)_{\bar{Q}}) \mathcal{M}_5^0(1_Q, \hat{k}_g, \hat{l}_g, j_g, (\tilde{2}i)_{\bar{Q}})^\dagger) \right)
 \end{aligned}$$

$$\begin{aligned}
 & -2\text{Re}(\mathcal{M}_5^0(1_Q, j_g, \hat{k}_g, \hat{l}_g, (\tilde{2}i)_{\bar{Q}})\mathcal{M}_5^0(1_Q, \hat{l}_g, \hat{k}_g, j_g, (\tilde{2}i)_{\bar{Q}})^\dagger) \\
 & -2\text{Re}(\mathcal{M}_5^0(1_Q, \hat{l}_g, j_g, \hat{k}_g, (\tilde{2}i)_{\bar{Q}})\mathcal{M}_5^0(1_Q, \hat{l}_g, \hat{k}_g, j_g, (\tilde{2}i)_{\bar{Q}})^\dagger) \\
 & -2\text{Re}(\mathcal{M}_5^0(1_Q, j_g, \hat{l}_g, \hat{k}_g, (\tilde{2}i)_{\bar{Q}})\mathcal{M}_5^0(1_Q, \hat{k}_g, j_g, \hat{l}_g, (\tilde{2}i)_{\bar{Q}})^\dagger) \\
 & -2\text{Re}(\mathcal{M}_5^0(1_Q, j_g, \hat{k}_g, \hat{l}_g, (\tilde{2}i)_{\bar{Q}})\mathcal{M}_5^0(1_Q, \hat{k}_g, j_g, \hat{l}_g, (\tilde{2}i)_{\bar{Q}})^\dagger) \\
 & -2\text{Re}(\mathcal{M}_5^0(1_Q, \hat{l}_g, j_g, \hat{k}_g, (\tilde{2}i)_{\bar{Q}})\mathcal{M}_5^0(1_Q, \hat{k}_g, j_g, \hat{l}_g, (\tilde{2}i)_{\bar{Q}})^\dagger) \\
 & -2\text{Re}(\mathcal{M}_5^0(1_Q, j_g, \hat{k}_g, \hat{l}_g, (\tilde{2}i)_{\bar{Q}})\mathcal{M}_5^0(1_Q, \hat{k}_g, \hat{l}_g, j_g, (\tilde{2}i)_{\bar{Q}})^\dagger) \Big) J_3^{(3)}(k_1, k_{\tilde{2}i}, k_j) \\
 & + f_3^0(l_g; i_g, j_g) \left(|\mathcal{M}_5^0(1_Q, (\tilde{i}j)_g, \hat{l}_g, \hat{k}_\gamma, 2_{\bar{Q}})|^2 + |\mathcal{M}_5^0(1_Q, \hat{l}_g, (\tilde{i}j)_g, \hat{k}_\gamma, 2_{\bar{Q}})|^2 \right. \\
 & + 4\text{Re}(\mathcal{M}_5^0(1_Q, (\tilde{i}j)_g, \hat{k}_g, \hat{l}_g, 2_{\bar{Q}})\mathcal{M}_5^0(1_Q, \hat{l}_g, \hat{k}_g, (\tilde{i}j)_g, 2_{\bar{Q}})^\dagger) \\
 & - 4\text{Re}(\mathcal{M}_5^0(1_Q, (\tilde{i}j)_g, \hat{l}_g, \hat{k}_g, 2_{\bar{Q}})\mathcal{M}_5^0(1_Q, \hat{k}_g, (\tilde{i}j)_g, \hat{l}_g, 2_{\bar{Q}})^\dagger) \\
 & - 4\text{Re}(\mathcal{M}_5^0(1_Q, \hat{l}_g, (\tilde{i}j)_g, \hat{k}_g, 2_{\bar{Q}})\mathcal{M}_5^0(1_Q, \hat{k}_g, (\tilde{i}j)_g, \hat{l}_g, 2_{\bar{Q}})^\dagger) \\
 & - 4\text{Re}(\mathcal{M}_5^0(1_Q, (\tilde{i}j)_g, \hat{l}_g, \hat{k}_g, 2_{\bar{Q}})\mathcal{M}_5^0(1_Q, \hat{k}_g, \hat{l}_g, (\tilde{i}j)_g, 2_{\bar{Q}})^\dagger) \\
 & \left. - 4\text{Re}(\mathcal{M}_5^0(1_Q, \hat{l}_g, (\tilde{i}j)_g, \hat{k}_g, 2_{\bar{Q}})\mathcal{M}_5^0(1_Q, \hat{k}_g, \hat{l}_g, (\tilde{i}j)_g, 2_{\bar{Q}})^\dagger) \right) J_3^{(3)}(k_1, k_2, k_{\tilde{i}j}) \\
 & + F_3^0(k_g, l_g; i_g) \left(|\mathcal{M}_5^0(\tilde{1}_Q, \hat{k}_g, \hat{l}_g, \tilde{j}_\gamma, \tilde{2}_{\bar{Q}})|^2 \right. \\
 & + 2\text{Re}(\mathcal{M}_5^0(\tilde{1}_Q, \hat{k}_g, \tilde{j}_\gamma, \hat{l}_g, \tilde{2}_{\bar{Q}})\mathcal{M}_5^0(\tilde{1}_Q, \hat{l}_g, \tilde{j}_\gamma, \hat{k}_g, \tilde{2}_{\bar{Q}})^\dagger) \\
 & - 4\text{Re}(\mathcal{M}_5^0(\tilde{1}_Q, \tilde{j}_\gamma, \hat{k}_g, \hat{l}_g, \tilde{2}_{\bar{Q}})\mathcal{M}_5^0(\tilde{1}_Q, \hat{k}_g, \hat{l}_g, \tilde{j}_\gamma, \tilde{2}_{\bar{Q}})^\dagger) \\
 & \left. - 4\text{Re}(\mathcal{M}_5^0(\tilde{1}_Q, \tilde{j}_\gamma, \hat{k}_g, \hat{l}_g, \tilde{2}_{\bar{Q}})\mathcal{M}_5^0(\tilde{1}_Q, \hat{l}_g, \hat{k}_g, \tilde{j}_\gamma, \tilde{2}_{\bar{Q}})^\dagger) \right) J_3^{(3)}(\tilde{k}_1, \tilde{k}_2, \tilde{k}_j) \\
 & + \frac{1}{N_c} \left[A_3^0(1_Q, i_g, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0((\tilde{1}i)_Q, j_g, \hat{k}_g, \hat{l}_\gamma, (\tilde{2}i)_{\bar{Q}})|^2 + |\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{k}_g, j_g, \hat{l}_\gamma, (\tilde{2}i)_{\bar{Q}})|^2 \right. \right. \\
 & + |\mathcal{M}_5^0((\tilde{1}i)_Q, \hat{k}_g, \hat{l}_g, j_\gamma, (\tilde{2}i)_{\bar{Q}})|^2 \\
 & \left. \left. - (3/2)|\mathcal{M}_5^0((\tilde{1}i)_Q, j_\gamma, \hat{k}_\gamma, \hat{l}_\gamma, (\tilde{2}i)_{\bar{Q}})|^2 \right) J_3^{(3)}(k_{\tilde{1}i}, k_{\tilde{2}i}, k_j) \right. \\
 & + \frac{1}{2} A_3^0(l_g; 1_Q, 2_{\bar{Q}}) \left(|\mathcal{M}_5^0((\tilde{1}2)_Q, i_g, j_g, \hat{k}_\gamma, \hat{l}_Q)|^2 + |\mathcal{M}_5^0(\hat{l}_{\bar{Q}}, i_g, j_g, \hat{k}_\gamma, (\tilde{1}2)_{\bar{Q}})|^2 \right. \\
 & + |\mathcal{M}_5^0((\tilde{1}2)_Q, i_g, \hat{k}_g, j_\gamma, \hat{l}_Q)|^2 + |\mathcal{M}_5^0(\hat{l}_{\bar{Q}}, i_g, \hat{k}_g, j_\gamma, (\tilde{1}2)_{\bar{Q}})|^2 \\
 & \left. + |\mathcal{M}_5^0((\tilde{1}2)_Q, \hat{k}_g, i_g, j_\gamma, \hat{l}_Q)|^2 + |\mathcal{M}_5^0(\hat{l}_{\bar{Q}}, \hat{k}_g, i_g, j_\gamma, (\tilde{1}2)_{\bar{Q}})|^2 \right) J_3^{(3)}(k_{\tilde{1}2}, k_i, k_j) \\
 & + \frac{1}{2} d_3^0(1_Q, i_g, j_g) |\mathcal{M}_5^0((\tilde{1}i)_Q, (\tilde{i}j)_\gamma, \hat{k}_\gamma, \hat{l}_\gamma, 2_{\bar{Q}})|^2 J_3^{(3)}(k_{\tilde{1}i}, k_2, k_{\tilde{i}j}) \\
 & + \frac{1}{2} d_3^0(2_{\bar{Q}}, i_g, j_g) |\mathcal{M}_5^0(1_Q, (\tilde{i}j)_\gamma, \hat{k}_\gamma, \hat{l}_\gamma, (\tilde{2}i)_{\bar{Q}})|^2 J_3^{(3)}(k_1, k_{\tilde{2}i}, k_{\tilde{i}j}) \\
 & + D_3^0(l_g; i_g, 1_Q) |\mathcal{M}_5^0((\tilde{1}i)_Q, j_\gamma, \hat{k}_\gamma, \hat{l}_\gamma, 2_{\bar{Q}})|^2 J_3^{(3)}(k_{\tilde{1}i}, k_2, k_j) \\
 & \left. + D_3^0(l_g; i_g, 2_{\bar{Q}}) |\mathcal{M}_5^0(1_Q, j_\gamma, \hat{k}_\gamma, \hat{l}_\gamma, (\tilde{2}i)_{\bar{Q}})|^2 J_3^{(3)}(k_1, k_{\tilde{2}i}, k_j) \right] \\
 & - \frac{1}{2N_c^3} \left[A_3^0(1_Q, i_g, 2_{\bar{Q}}) |\mathcal{M}_5^0((\tilde{1}i)_Q, j_\gamma, \hat{k}_\gamma, \hat{l}_\gamma, (\tilde{2}i)_{\bar{Q}})|^2 J_3^{(3)}(k_{\tilde{1}i}, k_{\tilde{2}i}, k_j) \right.
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} A_3^0(l_g; 1_Q, 2_{\bar{Q}}) \left(|\mathcal{M}_3^0((\tilde{1}\tilde{2})_Q, i_\gamma, j_\gamma, \hat{k}_\gamma, \hat{l}_Q)|^2 \right. \\
 & \quad \left. + |\mathcal{M}_3^0(\hat{l}_{\bar{Q}}, i_\gamma, j_\gamma, \hat{k}_\gamma, (\tilde{1}\tilde{2})_{\bar{Q}})|^2 \right) J_3^{(3)}(k_{\tilde{1}\tilde{2}}, k_i, k_j) \Big] \Big\}
 \end{aligned}$$

For this subtraction term, we have also checked that in all its collinear and quasi-collinear limits, it reduces to the product of a Casimir factor multiplied by a splitting function corresponding to the given limit and the appropriate non-colour ordered matrix-element squared as explained in section 7.4.1, providing us with a powerful check on the correctness of our result.

8 Conclusions

We have presented the extension of the antenna formalism required for the calculation of hadronic processes involving massive final states in association with jets at the NLO level. The construction of massive subtraction terms with all its mass-dependent ingredients is presented in all required configurations (final-final, initial-final, initial-initial). The unknown massive antenna functions are derived, their limiting behaviour is presented and those are finally integrated over a factorised form of the massive phase space. Besides the massive extension of the flavour-conserving antennae, new massive flavour-violating antennae were derived in unintegrated and integrated forms. One of the integrated massive initial-final antenna, $\mathcal{A}_{g;Q\bar{Q}}$, can be directly related to the well-known heavy quark coefficient function. A special section is dedicated to this comparison and full agreement is found. In section 5, when all antennae are integrated over the appropriate factorised massive phase space, we showed that we can capture all poles of the massive integrated antennae in universal factors. Those poles are related either to massive colour ordered $\mathbf{I}_{ij}^{(1)}$ -type operators or well-known splitting kernels $p_{(ij)}(x)$ associated to initial-final massless collinear singularities. The colour-ordered massive $\mathbf{I}_{ij}^{(1)}$ -type operators are here presented for the first time. As a first application of our massive extension of the antenna formalism, we constructed the colour ordered real contributions and subtraction terms for the production of a top quark pair and for the production of a top quark pair and a jet at NLO. In the second case, the presence of interference terms in the colour decomposition of the real matrix element squared renders the construction of the subtraction terms more involved. The treatment of those terms is explained in detail in the paper in section 7.2.1. All colour-ordered subtraction terms constructed have been checked in all soft, collinear and quasi-collinear limits of the real matrix element squared. Furthermore, for each subtraction term, it has been verified that in all collinear and/or quasi-collinear limits present in it, the sum of all terms contributing to a given limit add up to reproduce the product of the required splitting function with the corresponding Casimir factor multiplied with the non colour-ordered matrix element squared. This check is explicitly derived for the subtraction term related to the process $gg \rightarrow Q\bar{Q}g$ contributing to the process $pp \rightarrow t\bar{t}$ at NLO. This verification provides us with an extremely powerful test on the correctness of our results for the subtraction terms for $t\bar{t}$ and $t\bar{t} + 1$ jet production at NLO.

The results presented in this paper represent a substantial step towards the calculation of the NNLO corrections to top quark pair production within the antenna subtraction formalism. The decomposition of the real matrix elements into colour-ordered amplitudes squared and the identification of the different leading and subleading colour structures, including the treatment of interference terms, described in section 7.2.1, will allow the application of the NNLO antenna subtraction method to compute the double real radiation contributions to $pp \rightarrow t\bar{t}$ at NNLO. The NLO antenna subtraction terms for $pp \rightarrow t\bar{t} + \text{jet}$, provided in this paper, are already part of the NNLO corrections to $t\bar{t}$ production at hadron colliders.

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