The heat kernel on AdS3 and its applications

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The heat kernel on AdS_3 and its applications

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Abstract: We derive the heat kernel for arbitrary tensor fields on S^3 and (Euclidean) AdS$_3$ using a group theoretic approach. We use these results to also obtain the heat kernel on certain quotients of these spaces. In particular, we give a simple, explicit expression for the one loop determinant for a field of arbitrary spin s in thermal AdS$_3$. We apply this to the calculation of the one loop partition function of $\mathcal{N} = 1$ supergravity on AdS$_3$. We find that the answer factorizes into left- and right-moving super Virasoro characters built on the SL$(2, \mathbb{C})$ invariant vacuum, as argued by Maloney and Witten on general grounds.

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1 Introduction

In studying the quantization of field theories on a general spacetime an important tool which captures the leading quantum properties of the theory is the heat kernel of the Laplacian. Even if the full quantum theory is ill-defined or ill-understood (as is the case for theories of gravity), this leading one loop behaviour is typically well defined and often under analytic control. Knowing the heat kernel enables one to compute, for instance, the one loop determinants that contribute to the free energy. The heat kernel also contains the information about the propagator and other important one loop effects such as the anomalies of the quantum theory.

In these notes we will study the heat kernel on (Euclidean) AdS_3 spacetime for particles of arbitrary spin s. In studying the leading quantum effects for pure gravity or supergravity on AdS_3 one needs to compute the heat kernel for particles with spin less than or equal to two. More generally, for a string theory on AdS_3 one would need the heat kernel for particles of arbitrary spin s. With a view to some of these potential applications we obtain expressions for the heat kernel of the Laplacian $\Delta_{(s)}$ acting on tensor fields (transverse and traceless of arbitrary spin s). We will give answers for the cases of S^3 and some simple quotients as well as for Euclidean AdS_3 (i.e. H_3^+) and its thermal quotient. In particular, we obtain explicit expressions for the heat kernel for coincident points whose integral over proper time gives the one loop determinant.

As an immediate application of these results we are able to evaluate the one loop contribution from the physical spin $\frac{3}{2}$ gravitino in, for example, $\mathcal{N}=1$ supergravity on thermal AdS_3. This one loop result together with the answer for the spin two graviton combines into left- and right-moving super-Virasoro characters for the identity representation

$$Z_{1-\mathrm{loop}} = \prod_{n=2}^{\infty} \frac{|1+q^{n-\frac{1}{2}}|^2}{|1-q^n|^2}\,, \tag{1.1}$$

where $q = e^{i\tau}$ parametrizes the boundary T^2 of the thermal AdS_3. This agrees with the general argument given by Maloney and Witten [1] which was based on an extension of the results of Brown and Henneaux [2]. Maloney and Witten in fact also argued that (in an appropriate choice of scheme) this result was perturbatively one loop exact. The bosonic version of this argument for pure gravity (the denominator term in (1.1)) has been checked by the computation of Giombi *et.al.* [3] who have explicitly evaluated the heat kernel for transverse vectors and spin two fields. Our results for the supergravity case complete this check of the Maloney-Witten argument.

We now give a broad overview of our methods. As mentioned above, the heat kernel for AdS_3 and its thermal quotient have been explicitly evaluated for transverse vectors and spin two tensors [3]. The method of evaluation employed there is however fairly cumbersome to generalise to arbitrary spin. We will instead adopt a more geometric approach. We will exploit the fact that $S^3 = \mathrm{SU}(2) = (\mathrm{SU}(2)\times\mathrm{SU}(2))/\mathrm{SU}(2)$ and $H_3^+ = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ are homogeneous spaces. The fields of arbitrary spin s are therefore sections of what are known as homogeneous vector bundles on these coset spaces. This will allow us to use some well-known techniques of harmonic analysis to write down the eigenfunctions of the spin s Laplacian $\Delta_{(s)}$ in terms of matrix elements of representations of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and $\mathrm{SL}(2, \mathbb{C})$. These have, in fact, already appeared in the physics literature in a series of papers by Camporesi and Higuchi [4–9] (see also [10–14] for some related work). We will heavily draw upon these methods and adapt them to obtain the expressions of interest to us.

Given the eigenfunctions of the Laplace operator we can evaluate the heat kernel as

$$K_{ab}^{(s)}(x, y; t) = \langle y, b | e^{t\Delta_{(s)}} | x, a \rangle = \sum_n \psi_{n,a}^{(s)}(x)\, \psi_{n,b}^{(s)}(y)^*\, e^{t\lambda_n^{(s)}} \tag{1.2}$$

for arbitrary pairs of points (x, y) on the space in question (S^3 or H_3^+). Here a, b are labels for the $2s + 1$ dimensional representation for spin s. The eigenfunctions ψ_n have been labelled by n, which will denote a multi-index, while $\lambda_n^{(s)}$ is the corresponding eigenvalue. Using the group theoretic origin of the wave functions $\psi_{n,a}^{(s)}(x)$ we can carry out partial sums over degenerate eigenstates (those having the same eigenvalue $\lambda_n^{(s)}$). This manifests itself as a generalised version of the addition theorems that make their appearance in special function theory.

Given the heat kernel one can compute the one loop determinant, for instance, by considering the coincident limit of the heat kernel

$$\ln\det(-\Delta_{(s)}) = \mathrm{Tr}\ln(-\Delta_{(s)}) = -\int_0^\infty \frac{dt}{t} \int \sqrt{g}\, d^3x\, K_{aa}^{(s)}(x, x; t)\ . \tag{1.3}$$

To compute the heat kernel, as well as one loop determinants, on quotients of S^3 or H_3^+ we can use the method of images. The basic quotients we will study are Lens space quotients of S^3 while the analogous quotient in H_3^+ is the one giving Euclidean thermal AdS.[1]

We will describe the S^3 case (and its quotient) in great detail in sections 2, 3 and 4, both because it is compact and because many of the group theoretic features use only familiar facts about representations of SU(2). In section 2 we briefly summarize some of the relevant ideas from harmonic analysis which lead to the explicit forms of the eigenfunctions of the spin s Laplacian. We go on to give a number of different expressions for these eigenfunctions as well as their explicit form for low values of the spin. Section 3 uses these expressions and their group theoretic origin to write down the heat kernel for separated points. Once again a number of explicit expressions are worked out. Section 4 deals with a Lens space like quotient of S^3 and the method of images is applied to obtain the heat kernel.

The case of H_3^+ is more subtle since it involves harmonic analysis on a non-compact group. The relevant representations are infinite dimensional, and the discrete sums in (1.2) become continuous integrals with an appropriate measure. While these are relatively well understood in the case of interest to us, namely $\mathrm{SL}(2, \mathbb{C})$, we will practically implement the calculation by performing a suitable analytic continuation of the answers from S^3. Analytic continuation from compact to non-compact groups is often fraught with danger, and one needs to proceed with caution. In this case, however, it is known from works of Helgason [15] and Camporesi-Higuchi [7, 9] that analytic continuation works. In fact, S^3 and H_3^+ are among the simplest examples of 'dual spaces' on which harmonic analysis can

[1] In the case of H_3^+ and its thermal quotient the expression in (1.3) suffers from a trivial volume divergence which we will ignore; we shall concentrate on the finite piece which contains all the nontrivial q dependence.

be analytically continued. We will elaborate on this in section 5. In section 6, we extend this analytic continuation to thermal quotients of S^3 and H_3^+ and obtain an explicit and relatively simple expression for the (integrated and coincident) heat kernel (see eq. (6.9)). We check that this answer correctly reproduces all the previously known cases (i.e. spins $s = 0, 1, 2$).

Finally, in section 7, we use the results of section 6 to evaluate the one loop partition function of $\mathcal{N} = 1$ supergravity on AdS_3. This additionally requires a careful analysis of the physical quadratic fluctuations of the massless gravitino about the AdS_3 background. We carry this out and show that the final answer takes the expected form (1.1). Various additional details are relegated to the four appendices.

2 Construction of harmonics on S^3

We will be interested in the symmetric traceless divergence free (transverse) tensors of spin s on S^3. This is sufficient information to study fields in arbitrary representations.[2] To construct the heat kernel we need the complete set of eigenfunctions of the corresponding Laplacian $\Delta_{(s)}$. This can be explicitly studied using harmonic analysis on homogeneous vector bundles which applies directly to homogeneous spaces of the form G/H (see [9] for an accessible introduction for physicists). The harmonic wavefunctions can be expressed in terms of matrix elements of particular representations of G. We will start by considering the case where G is compact as exemplified by S^3 which can be thought of as the homogeneous space

$$S^3 \cong (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathrm{SU}(2)\,, \tag{2.1}$$

with the denominator acting diagonally on $(\mathrm{SU}(2) \times \mathrm{SU}(2))$, i.e.

$$(g_L, g_R) \mapsto (g_L \cdot h, g_R \cdot h)\ , \qquad h \in \mathrm{SU}(2)\ . \tag{2.2}$$

We can identify the quotient space, via the projection map π, with $\mathrm{SU}(2) = S^3$ itself,

$$\pi : \mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SU}(2)\,, \qquad (g_L, g_R) \mapsto g_L \cdot g_R^{-1}\ . \tag{2.3}$$

This map is evidently independent of the representative, i.e. it is invariant under replacing (g_L, g_R) by $(g_L \cdot h, g_R \cdot h)$.

Below we will describe the corresponding tensor harmonics on S^3 in terms of matrix elements of $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

To write explicit expressions we will also need to choose definite coordinates on S^3. The most common set of coordinates is the spherical system parametrized by (χ, θ, ϕ) in which the metric of S^3 reads

$$ds^2 = d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\ . \tag{2.4}$$

[2] Note that since we are working in three dimensions there are no non-trivial antisymmetric representations that need to be considered: the two form is dual to a vector and the three form to a scalar.

The corresponding group element in SU(2) is parametrized by

$$g(\chi,\theta,\phi)=\begin{pmatrix}\cos\chi+i\sin\chi\cos\theta & i\sin\chi\,\sin\theta\,e^{i\phi}\\ i\sin\chi\,\sin\theta\,e^{-i\phi} & \cos\chi-i\sin\chi\cos\theta\end{pmatrix}\,. \tag{2.5}$$

This will be useful for comparing some of the results to known expressions in the literature.

However, for performing the thermal quotient it will be most convenient to use double polar coordinates (ψ,η,φ) in terms of which the metric reads

$$ds^2=d\psi^2+\cos^2\psi\,d\eta^2+\sin^2\psi\,d\varphi^2\,. \tag{2.6}$$

In terms of these coordinates the elements of SU(2) are given by

$$g(\psi,\eta,\varphi)=\begin{pmatrix}e^{-i\eta}\cos\psi & ie^{i\varphi}\sin\psi\\ ie^{-i\varphi}\sin\psi & e^{i\eta}\cos\psi\end{pmatrix}\,. \tag{2.7}$$

2.1 Tensor harmonics and representation theory

The nature of S^3 as a homogeneous space allows one to choose tensor harmonics with respect to a basis which reflects this homogeneity (see below). Though focussing on S^3 (and later H_3^+) many of the ideas are general and we will often indicate the generalization to general homogeneous spaces. We refer to [9] for a more comprehensive discussion.

An important role will be played by sections $\sigma(x)$ of the principal bundle $\mathrm{SU}(2)\times\mathrm{SU}(2)$ over the base SU(2) (being parametrized by x). That is

$$\sigma:\mathrm{SU}(2)\to\mathrm{SU}(2)\times\mathrm{SU}(2)\,,\qquad \text{such that}\qquad \pi\circ\sigma=\mathrm{id}_{\mathrm{SU}(2)}\,. \tag{2.8}$$

Obviously, there is no canonical choice of a section. In particular, for any given σ, we can define $\hat\sigma$ via

$$\hat\sigma=\sigma\cdot(h(x),h(x))\,, \tag{2.9}$$

where $h(x)$ is any map from $\mathrm{SU}(2)\to\mathrm{SU}(2)$. From the definition of the quotient action (2.2), it is clear that any two sections are related in this manner.

Any given section $\sigma(x)$ actually also determines a natural choice for a basis of tensor valued functions. Define $\mathbf{v}_a$ $(a=1\ldots 2s+1)$ as a basis for a spin s representation of SU(2) at the origin (of S^3 viewed as a group). Then a basis of sections of the spin s tensor bundle can be defined via

$$\theta_a(x)=\sigma(x)\mathbf{v}_a\,. \tag{2.10}$$

For the case of spin $s=1$, $\mathbf{v}_a$ can be thought of as a vector in the tangent space of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ at the identity, and the action of $\sigma(x)\in\mathrm{SU}(2)\times\mathrm{SU}(2)$ is the usual push-forward. The form of the resulting vielbein basis, for some of the sections that we will use, is summarized in appendix B. The generalization to arbitrary spin s is then straightforward.

We will expand our tensor harmonics in this basis.

$$\Psi(x)=\sum_a\Psi_a(x)\theta_a(x)\,. \tag{2.11}$$

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In other words, it is the components $\Psi_a(x)$ (with respect to the basis $\theta_a(x)$) which will be the eigenfunctions of the Laplace operator $\Delta_{(s)}$. The arbitrariness we saw above in the choice of the section reflects a freedom in the choice of basis (see appendix A for more details). We will see below that this freedom will be reduced in the presence of quotients. The tensor harmonics that will be explicitly given below are always defined with respect to some basis $\{\theta_\alpha(x)\}$ determined by a particular choice of section.

Having identified the basis of tensors, we can now give explicit formulae for the component tensor harmonics [9]. Here we will describe the approach for a general compact homogeneous space. Geometrically, the tensors we are considering are sections of homogeneous vector bundles E_ρ associated to the principal bundle G over the homogeneous space G/H, with structure group H and transforming under some particular representation ρ of H. The harmonic analysis of such vector bundles is an extension of the usual harmonic analysis for scalars.

The crucial point we shall use is that there is a natural embedding of the space of sections of these bundles into the space of functions on G. We can make this correspondence one to one if we restrict ourselves to the functions $\psi_a(g)$ on G,[3] that are *equivariant* with respect to H. These functions obey

$$\psi_a(gh) = \rho(h^{-1})_a^b\, \psi_b(g) \tag{2.12}$$

for any $g \in G$ and $h \in H$, where $\rho(h)$ is the representation of H acting on the fibres of the vector bundle. We can thus think of the $\psi_a(g)$ as components of a vector which lie in the vector space of a typical fibre (e.g. at the origin with respect to a basis $\{\mathbf{v}_a\}$ in our case) of the associated vector bundle.

Now we can use the section $\sigma(x)$ of the principal fibre bundle G to construct tensor valued component functions on G/H (with respect to the basis $\theta_a(x)$ arising from the section $\sigma(x)$ as in (2.10)) via

$$\Psi_a(x) = \psi_a(\sigma(x)) \ . \tag{2.13}$$

In our case, with $g \in G = \mathrm{SU}(2) \times \mathrm{SU}(2)$, is not difficult to see that the functions

$$\psi_a^{(\lambda;I)}(g) = U^\lambda(g^{-1})_a^I\,, \tag{2.14}$$

are equivariant with respect to $H = \mathrm{SU}(2)$. Here λ denotes a representation of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ which contains the spin s representation under the diagonal action of SU(2). The label a takes values in the spin s representation that is contained in λ under the diagonal action, while I labels the different states in the representation λ. Finally, U^λ denotes the matrix elements of the unitary representation λ. We shall exhibit this formula more explicitly below, see (2.19) and (2.20). There is also an obvious generalization of this for arbitrary G and H.

For each such choice of λ, we can thus write down, using the above correspondence (2.13), the components of a tensor section as

$$\Psi_a^{(\lambda;I)}(x) = U^\lambda(\sigma(x)^{-1})_a^I \ . \tag{2.15}$$

[3]Technically, this is the statement that $L^2(G)$ decomposes into a union (over representations ρ, with some multiplicity) of the spaces $L^2(G/H, E_\rho)$. This is familiar to physicists in the study of monopole harmonics on S^2 $(G = \mathrm{SU}(2), H = \mathrm{U}(1))$ all of which arise from (equivariant) functions on S^3.

In fact, these components of (2.14) are actually eigenfunctions of the spin s Laplacian (with the conventional spin connection in the covariant derivative) for each state in λ (labelled by I) [9]. These constitute a complete set of rank s tensor harmonics, whose components (with respect to the basis (2.10)) are described by the index a. In order to describe the *transverse and traceless* tensors of spin s the representations λ must be taken to be of the form [4, 13]

$$\lambda_+ = \left(\frac{n}{2}+s, \frac{n}{2}\right) \qquad \text{or} \qquad \lambda_- = \left(\frac{n}{2}, \frac{n}{2}+s\right) , \tag{2.16}$$

where $n = 0, 1, \ldots$. It is clear that these representations contain the spin s representation in their diagonal. The eigenvalue of the tensor harmonics only depends on λ (or equivalently n), and for λ of the form (2.16) is given by [8]

$$-E_n^{(s)} = 2\left[C_2\left(\frac{n}{2}+s\right) + C_2\left(\frac{n}{2}\right)\right] - C_2(s) = (s+n)(s+n+2) - s\,, \tag{2.17}$$

where $C_2(j) = j(j+1)$ is the usual second order Casimir for the SU(2) representation labelled by j.

For each such λ (or n), the label I takes $(n+2s+1)\cdot(n+1)$ different values; for $s > 0$ there are then $2\cdot(n+2s+1)\cdot(n+1)$ different transverse and traceless rank s tensor harmonics with the same eigenvalue $E_n^{(s)}$, whereas for $s = 0$ (scalar harmonics), the two choices $\lambda_\pm$ coincide, and the degeneracy is $(n+1)^2$, as is familiar from the description of the hydrogen atom. In the following we shall only be considering the transverse and traceless tensor harmonics corresponding to the representations (2.16).

To write out (2.14) more explicitly, we specify a section as

$$\sigma(x) = (g_L(x), g_R(x))\,, \qquad \text{where} \quad g_L(x)\cdot g_R^{-1}(x) = x\ . \tag{2.18}$$

The tensor harmonics for $\lambda = \lambda_+ = (\frac{n}{2}+s, \frac{n}{2})$ are then explicitly

$$\Psi_a^{(s)(n+;m_1,m_2)}(x) = \sum_{k_1,k_2} \langle s, a|\frac{n}{2}+s, k_1; \frac{n}{2}, k_2\rangle\, D_{k_1,m_1}^{(\frac{n}{2}+s)}(g_L^{-1}(x))\, D_{k_2,m_2}^{(\frac{n}{2})}(g_R^{-1}(x))\,, \tag{2.19}$$

while for $\lambda = \lambda_- = (\frac{n}{2}, \frac{n}{2}+s)$ we have instead

$$\Psi_a^{(s)(n-;m_1,m_2)}(x) = \sum_{k_1,k_2} \langle s, a|\frac{n}{2}, k_1; \frac{n}{2}+s, k_2\rangle\, D_{k_1,m_1}^{(\frac{n}{2})}(g_L^{-1}(x))\, D_{k_2,m_2}^{(\frac{n}{2}+s)}(g_R^{-1}(x))\ . \tag{2.20}$$

In either case $I = (m_1, m_2)$ labels the different states in λ and thus denotes different tensor harmonics. Concentrating for definiteness on $\lambda = \lambda_+$, $\langle s, a|\frac{n}{2}+s, k_1; \frac{n}{2}, k_2\rangle$ is the Clebsch-Gordon coefficient describing the decomposition of the tensor product $(\frac{n}{2}+s)\otimes(\frac{n}{2})$ into spin s, while $D_{m,n}^{(j)}(g)$ is the (m,n)-matrix element of the SU(2) rotation g in the representation j. The above wavefunctions are normalized so that

$$\sum_a \int d\mu(x)\, \Psi_a^{(s)(n+;m_1,m_2)}(x)^*\, \Psi_a^{(s)(n+;m_1',m_2')}(x) = \frac{2\pi^2(2s+1)}{(n+2s+1)(n+1)}\, \delta^{m_1,m_1'}\, \delta^{m_2,m_2'}\,, \tag{2.21}$$

where $d\mu(x)$ is the Haar measure on $S^3 = \mathrm{SU}(2)$, normalized so that the volume of S^3 is $2\pi^2$.

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2.2 Choice of section

The formula (2.19) (or (2.20)) obviously depends on the choice of a section $\sigma(x)$ or, in other words, of $(g_L(x), g_R(x))$. We will now concentrate, for reasons that will become clearer later, on two out of infinitely many choices of sections.

The first, which we call the 'canonical section' is in some sense the most obvious choice:

$$\sigma_{can}(x) = (g_L(x), g_R(x)) = (e, x^{-1}) \; . \tag{2.22}$$

With respect to the induced basis of tensor functions, the tensor harmonics labelled by $(n; m_1, m_2)$ in (2.19) are given as

$$\begin{aligned} \Psi^{(s)(n+;m_1,m_2)}_{a(can)}(x) &= \sum_{l_1,l_2} \langle s, a | \frac{n}{2} + s, l_1; \frac{n}{2}, l_2 \rangle \, D^{(\frac{n}{2}+s)}_{l_1,m_1}(e) \, D^{(\frac{n}{2})}_{l_2,m_2}(x) \\ &= \langle s, a | \frac{n}{2} + s, m_1; \frac{n}{2}, a - m_1 \rangle \, D^{(\frac{n}{2})}_{a-m_1,m_2}(x) \; . \end{aligned} \tag{2.23}$$

This answer is simple in some respects, being given purely in terms of single SU(2) rotation matrix elements.

The second choice of section we will consider is a so-called 'thermal section' because it respects the thermal quotient symmetry. As we shall explain in more detail below, the thermal quotient is obtained by the group action

$$x \mapsto A \, x \, B^{-1} \, , \tag{2.24}$$

where A and B are fixed elements of SU(2). Given any such group action, there are special sections that respect this symmetry. By this one means that the quotient acts on the principal bundle $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ in a way which commutes with the *right* action by $H = \mathrm{SU}(2)$. This is achieved by having the quotient act by a *left* action on G. Not all sections of the principal bundle will be compatible with this left action in the sense of obeying

$$(g_L(AxB^{-1}), g_R(AxB^{-1})) = \sigma\left(A \, x \, B^{-1}\right) = (A, B) \cdot \sigma(x) = (A \cdot g_L(x), B \cdot g_R(x)) \; . \tag{2.25}$$

The thermal section will turn out to obey this relation in the case of thermal quotients.

In terms of the spherical coordinates of (2.4) and (2.5), a thermal section is given by

$$g_L(\chi, \theta, \phi) = \begin{pmatrix} \cos\frac{\theta}{2} \, e^{i(\phi+\chi)/2} & -\sin\frac{\theta}{2} \, e^{i(\phi-\chi)/2} \\ \sin\frac{\theta}{2} \, e^{-i(\phi-\chi)/2} & \cos\frac{\theta}{2} \, e^{-i(\phi+\chi)/2} \end{pmatrix} , \tag{2.26}$$

and

$$g_R(\chi, \theta, \phi) = \begin{pmatrix} \cos\frac{\theta}{2} \, e^{i(\phi-\chi)/2} & -\sin\frac{\theta}{2} \, e^{i(\phi+\chi)/2} \\ \sin\frac{\theta}{2} \, e^{-i(\phi+\chi)/2} & \cos\frac{\theta}{2} \, e^{-i(\phi-\chi)/2} \end{pmatrix} . \tag{2.27}$$

For the following it will be important that these group elements factorize as

$$g_L(x) = \mathrm{U}(\hat{n}) e^{i\frac{\chi}{2}\sigma_3} \, , \qquad g_R(x) = \mathrm{U}(\hat{n}) e^{-i\frac{\chi}{2}\sigma_3} \, , \tag{2.28}$$

where σ_3 is the usual Pauli matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \text{and} \qquad \mathrm{U}(\hat{n}) = \begin{pmatrix} \cos\frac{\theta}{2} e^{i\frac{\phi}{2}} & -\sin\frac{\theta}{2} e^{i\frac{\phi}{2}} \\ \sin\frac{\theta}{2} e^{-i\frac{\phi}{2}} & \cos\frac{\theta}{2} e^{-i\frac{\phi}{2}} \end{pmatrix}. \tag{2.29}$$

Note that $\mathrm{U}(\hat{n})$ can be viewed as a (local) section for the principal U(1) (Hopf) bundle S^3 over the base S^2. This section is well defined except at the poles $\theta = 0, \pi$.

Later on we shall also need the thermal section in the double polar coordinates (2.6), for which it takes the form

$$g_L(\psi, \eta, \varphi) = \begin{pmatrix} e^{i(\varphi-\eta)/2} \cos\frac{\psi}{2} & i e^{i(\varphi-\eta)/2} \sin\frac{\psi}{2} \\ i e^{-i(\varphi-\eta)/2} \sin\frac{\psi}{2} & e^{-i(\varphi-\eta)/2} \cos\frac{\psi}{2} \end{pmatrix}, \tag{2.30}$$

and

$$g_R(\psi, \eta, \varphi) = \begin{pmatrix} e^{i(\varphi+\eta)/2} \cos\frac{\psi}{2} & -i e^{i(\varphi+\eta)/2} \sin\frac{\psi}{2} \\ -i e^{-i(\varphi+\eta)/2} \sin\frac{\psi}{2} & e^{-i(\varphi+\eta)/2} \cos\frac{\psi}{2} \end{pmatrix}. \tag{2.31}$$

Note that in these coordinates we can write

$$g_L(\psi, \eta, \varphi) = e^{i\frac{(\varphi-\eta)}{2}\sigma_3} V(\psi), \qquad g_R(\psi, \eta, \varphi) = e^{i\frac{(\varphi+\eta)}{2}\sigma_3} V(\psi)^{-1}, \tag{2.32}$$

where

$$V(\psi) = \begin{pmatrix} \cos\frac{\psi}{2} & i\sin\frac{\psi}{2} \\ i\sin\frac{\psi}{2} & \cos\frac{\psi}{2} \end{pmatrix}. \tag{2.33}$$

It is straightforward to check that with both sets of coordinates we have indeed $g_L(x) g_R^{-1}(x) = x$, where x is of the form (2.5) and (2.7), respectively. The expression for the components of the tensor harmonics are then given by (2.19) with $g_L(x)$, $g_R(x)$ as above. There is no immediate simplification (see however section 2.3.2 below), and the expressions are more complicated than (2.23).

2.3 Explicit formulae

In order to illustrate the general construction from above we shall now exhibit some explicit solutions. This will also allow us to connect our formulae to existing results in the literature. The reader who is not interested in this detailed comparison may proceed directly to section 3.

2.3.1 The scalar case

The scalar case ($s = 0$) is the simplest since the answer will be independent of the choice of section, as we shall verify momentarily. In fact, using the general formula (2.19) for $s = a = 0$ we get (recall that $\lambda_+ = \lambda_-$ in this case)

$$\Psi^{(n;m_1,m_2)}(x) = \sum_m \langle 0, 0 | \frac{n}{2}, -m; \frac{n}{2}, m \rangle \, D^{(\frac{n}{2})}_{-m,m_1}\big(g_L(x)^{-1}\big) \, D^{(\frac{n}{2})}_{m,m_2}\big(g_R(x)^{-1}\big), \tag{2.34}$$

where $g_L(x)$ and $g_R(x)$ are any section, i.e. satisfy $g_L(x) \cdot g_R(x)^{-1} = x$. Using

$$\langle 0,0|\frac{n}{2},-m;\frac{n}{2},m\rangle = \frac{(-1)^{\frac{n}{2}-m}}{\sqrt{n+1}}\,, \tag{2.35}$$

as well as the fact that $D^{(j)}_{-m,m_1}(g_L^{-1}) = (-1)^{m_1+m} D^{(j)}_{-m_1,m}(g_L)$ we can do the sum over m in (2.34) explicitly, and we obtain

$$\Psi^{(n;m_1,m_2)}(x) = \frac{(-1)^{\frac{n}{2}+m_1}}{\sqrt{n+1}}\, D^{(\frac{n}{2})}_{-m_1,m_2}(x)\ . \tag{2.36}$$

This is evidently independent of the chosen section. All these functions have eigenvalue $\lambda_n = -n(n+2)$. Since m_1, m_2 each range over $(n+1)$ values, we have a total degeneracy of $(n+1)^2$. The answer (2.36) is also familiar from the Peter-Weyl theorem as forming a complete, orthonormal basis for functions on S^3.

2.3.2 Factorization

In the spherical coordinates of (2.4) the sphere S^3 is parametrized in terms of the angles (θ,ϕ) defining an S^2, times a radial coordinate χ. Typical results for tensor harmonics available in the literature (e.g. [8, 9]) are usually given in a factorized form in terms of these coordinates. However, our group theoretic basis of eigenfunctions (2.19), (2.20) with the thermal section (2.26), (2.27) does not exhibit such a factorization. To compare with the results in the literature we will consider particular linear combinations of the group theoretic eigenfunctions which exhibit this factorization.

For example, for the scalar harmonics, we define

$$\Phi_{n\,l\,m}(x) = \frac{n+1}{\sqrt{2\pi^2}} \sum_{m_1,m_2} \langle \frac{n}{2},m_1;\frac{n}{2},m_2|l,m\rangle \Psi^{(n;m_1,m_2)}(x)\,, \tag{2.37}$$

where $l = 0,1\ldots n$, and m runs over the $(2l+1)$ values $m = -l\ldots l$, thus accounting again for the $(n+1)^2$ fold degeneracy of the scalar harmonics with eigenvalue

$$\Delta_{(0)}\Phi_{n\,l\,m} = -n(n+2)\Phi_{n\,l\,m}\ . \tag{2.38}$$

A straightforward computation then exhibits the factorized form

$$\Phi_{n\,l\,m}(\chi,\theta,\phi) = C_{n\,l}\frac{1}{(\sin\chi)^{1/2}} P^{-l-1/2}_{n+1/2}(\cos\chi)\, Y^{lm}(\theta,\phi)\,, \tag{2.39}$$

were $C_{n\,l} = \sqrt{(n+1)\frac{(n+l+1)!}{(n-l)!}}$ and $P^{-l-1/2}_{n+1/2}$ is the associated Legendre function of the first kind, which can be expressed either in terms of hypergeometric functions or Jacobi Polynomials (see [16])

$$\begin{aligned} P^{-l-1/2}_{n+1/2}(\cos\chi) &= \frac{1}{\Gamma(l+3/2)}\left(\frac{\sin\chi/2}{\cos\chi/2}\right)^{l+1/2} F(-n-1/2, n+3/2, l+3/2; \sin^2\chi/2) \\ &= 2^{-l-\frac{1}{2}}\frac{(n-l)!}{\Gamma(n+\frac{3}{2})}\sin^{l+\frac{1}{2}}\chi\, P^{(l+\frac{1}{2},l+\frac{1}{2})}_{n-l}(\cos\chi)\ . \end{aligned} \tag{2.40}$$

$Y^{lm}(\theta,\phi)$ are the normalized (scalar) spherical harmonics on S^2.
For the general case of spin s, we define, using the thermal section,

$$\begin{aligned}
\Phi^{+(s)}_{a,n\,l\,m}(\chi,\theta,\phi) &= \sum_{m_1,m_2} \langle \frac{n}{2}+s, m_1; \frac{n}{2}, m_2 | l, m\rangle \Psi^{(s)(n+;m_1,m_2)}_{a(therm)}(x) \\
\Phi^{-(s)}_{a,n\,l\,m}(\chi,\theta,\phi) &= \sum_{m_1,m_2} \langle \frac{n}{2}, m_1; \frac{n}{2}+s, m_2 | l, m\rangle \Psi^{(s)(n-;m_1,m_2)}_{a(therm)}(x)\,,
\end{aligned}$$

where l runs over the values

$$l = s, s+1, \dots, s+n\,, \tag{2.41}$$

while m takes the $(2l+1)$ values $m=-l,-l+1,\dots,l-1,l$; altogether we thus have again

$$2\cdot\sum_{l=s}^{s+n}(2l+1) = 2(n+1)(2s+n+1) \tag{2.42}$$

different solutions. To see that these solutions are again in factorized form we insert the definition of $\Psi^{(s)(n\pm;m_1,m_2)}_{a(therm)}(x)$ from (2.19) and (2.20) into (2.41), and use (2.28) as well as (A.1). A straightforward computation then shows that

$$\Phi^{\pm(s)}_{a,n\,l\,m}(\chi,\theta,\phi) = Q^{\pm(s)}_{a,n\,l}(\chi)\, D^{(l)}_{a,m}(U^\dagger(\hat{n}))\,, \tag{2.43}$$

where

$$\begin{aligned}
Q^{+(s)}_{a,n\,l}(\chi) &= \sum_k \langle s, a | \frac{n}{2}+s, k; \frac{n}{2}, a-k\rangle\, e^{-i\chi(2k-a)} \langle \frac{n}{2}+s, k; \frac{n}{2}, a-k | l, a\rangle \\
Q^{-(s)}_{a,n\,l}(\chi) &= \sum_k \langle s, a | \frac{n}{2}, k; \frac{n}{2}+s, a-k\rangle\, e^{-i\chi(2k-a)} \langle \frac{n}{2}, k; \frac{n}{2}+s, a-k | l, a\rangle\ .
\end{aligned} \tag{2.44}$$

Since $\mathrm{U}(\hat{n})$ is only a function of (θ,ϕ), (2.43) thus gives a formula for the harmonics in factorized form. In fact, the $D^{(l)}_{a,m}(U^\dagger(\hat{n}))$ are equivariant functions on S^2 under the U(1) action of the principal U(1) bundle over S^2. Thus they correspond to different tensor harmonics on S^2. They are the same as the usual spin-weighted spherical harmonics of Newman and Penrose, and essentially the same as the familiar monopole harmonics [17].

For the spinor case, $s=\frac{1}{2}$, we have checked that the resulting harmonics agree precisely with the explicit formulae given in [9]. Actually, these functions are also eigenfunctions of the Dirac operator $\nabla\!\!\!\!/$ with eigenvalues $\pm i(n+\frac{3}{2})$, and thus the eigenvalue with respect to $\nabla\!\!\!\!/^2$ is $-(n+\frac{3}{2})^2$. This differs from $E^{(1/2)}_n$ in (2.17) by a constant (independent of n) whose origin lies in the non-trivial curvature of S^3.

We have also worked out (2.41) for the vector harmonics $s=1$, and compared them to the explicit formulae of [8]. In identifying these solutions with each other one has to take into account, as mentioned in section 2.1, that the components of the harmonics in the thermal section are defined with respect to the standard vielbein on S^3, see eq. (B.9). On the other hand, the vector harmonics of [8] are given with respect to a coordinate basis. It follows from (B.9) that the dictionary between the two bases is

$$\Psi_{\pm1} = \frac{1}{\sqrt{2}\sin\chi}\Big[\frac{1}{\sin\theta}\Psi_\phi \mp i\Psi_\theta\Big]\,, \qquad \Psi_0 = \Psi_\chi\,, \tag{2.45}$$

where we have suppressed the $[\pm,(n,l,m)]$ labels that are common on both sides. Once this is taken into account, the above group theory solutions $\Phi^{\pm(1)}_{a,n\,l\,m}$ agree precisely with (linear combinations) of the harmonics given in [8].

3 Heat kernel on S^3

With this detailed understanding of the spin s harmonics we can now calculate the spin s heat kernel as per (1.2)

$$K^{(s)}_{ab}(x,y;t) = \sum_{(n\pm;m_1,m_2)} a^{(s)}_n\, \Psi^{(s)(n\pm;m_1,m_2)}_a(x) \left(\Psi^{(s)(n\pm;m_1,m_2)}_b(y)\right)^* e^{E^{(s)}_n t}\,, \tag{3.1}$$

where x and y are two points of S^3, and the sum runs over all spin s harmonics labelled by $(n\pm;m_1,m_2)$ as above. Furthermore, $E^{(s)}_n$ is defined in (2.17), while the normalisation constant $a^{(s)}_n$ equals

$$a^{(s)}_n = \frac{1}{2\pi^2}\,\frac{(n+2s+1)\,(n+1)}{(2s+1)}\,. \tag{3.2}$$

This normalizes the heat kernel so that, using (2.21), we get

$$\sum_a \int d\mu(x)\, K^{(s)}_{aa}(x,x;t) = \sum_{n=0}^{\infty} d^{(s)}_n\, e^{E^{(s)}_n t}\,, \tag{3.3}$$

where

$$d^{(s)}_n = (2-\delta_{s,0})\,(n+1)\,(n+2s+1) \tag{3.4}$$

is the total multiplicity of transverse spinor harmonics of eigenvalue $E^{(s)}_n$. (The prefactor $(2-\delta_{s,0})$ takes into account that for $s>0$ there are two sets of harmonics for each n, while for $s=0$ there is only one.) Note that (3.3) is the 'trace' over the heat kernel that is important for the calculation of the one-loop determinant.

Inserting our general formula for the harmonics, see eq. (2.19), the heat kernel becomes

$$\begin{aligned} K^{(s)}_{ab}(x,y;t) = \sum_{l_1,l_2;m_1,m_2}\ \sum_{p_1,p_2;q_1,q_2} & a^{(s)}_n\, \langle s,a|l_1,p_1;l_2,p_2\rangle\, \langle l_1,q_1;l_2,q_2|s,b\rangle\, e^{E^{(s)}_n t} \\ & \times D^{(l_1)}_{p_1,m_1}(g_L(x)^{-1}) \left(D^{(l_1)}_{q_1,m_1}(g_L(y)^{-1})\right)^* \\ & \times D^{(l_2)}_{p_2,m_2}(g_R(x)^{-1}) \left(D^{(l_2)}_{q_2,m_2}(g_R(y)^{-1})\right)^*\,, \end{aligned} \tag{3.5}$$

where (l_1,l_2) runs over all pairs of representations of the form $(\frac{n}{2}+s,\frac{n}{2})$ or $(\frac{n}{2},\frac{n}{2}+s)$, and $E^{(s)}_n$, expressed in terms of (l_1,l_2), equals

$$E^{(s)}_n = -(s+n)(s+n+2)+s = -2\Big[l_1(l_1+1)+l_2(l_2+1)\Big]+s(s+1)\,. \tag{3.6}$$

Since the representations are unitary we have

$$\left(D^{(l_1)}_{q_1,m_1}(g_L(y)^{-1})\right)^* = D^{(l_1)}_{m_1,q_1}(g_L(y))\,, \qquad \left(D^{(l_2)}_{q_2,m_2}(g_R(y)^{-1})\right)^* = D^{(l_2)}_{m_2,q_2}(g_R(y))\,. \tag{3.7}$$

Thus we can perform the sum over m_1 and m_2 and obtain

$$K_{ab}^{(s)}(x,y;t) = \sum_{l_1,l_2} \sum_{p_1,p_2;q_1,q_2} a_n^{(s)} \langle s,a|l_1,p_1;l_2,p_2\rangle \langle l_1,q_1;l_2,q_2|s,b\rangle\, e^{E_n^{(s)}t} \times D^{(l_1)}_{p_1,q_1}\Big(g_L(x)^{-1}g_L(y)\Big)\, D^{(l_2)}_{p_2,q_2}\Big(g_R(x)^{-1}g_R(y)\Big)\,. \tag{3.8}$$

Written in terms of the more abstract description of the tensor harmonics, eq. (2.15), this formula takes the form

$$K_{ab}^{(s)}(x,y;t) = \sum_{\lambda} a_n^{(s)}\, U^{\lambda}(\sigma(x)^{-1}\sigma(y))_{ab}\, e^{E_n^{(s)}t}\,, \tag{3.9}$$

where λ runs over all the representations of the form (2.16), and $a_n^{(s)}$ and $E_n^{(s)}$ are as defined in (3.2) and (2.17), respectively. Furthermore, the matrix elements are taken in the spin s subrepresentation with respect to the diagonal SU(2). Finally, we can also use (A.1) to rewrite (3.8) as

$$\begin{aligned} K_{ab}^{(s)}(x,y;t) = &\sum_{a',b'} D^{(s)}_{aa'}(g_L(x)^{-1})\, D^{(s)}_{b'b}(g_L(y)) \\ &\times \sum_{l_1,l_2} \sum_{p_1,p_2;q_2} a_n^{(s)} \langle s,a'|l_1,p_1;l_2,p_2\rangle \langle l_1,p_1;l_2,q_2|s,b'\rangle\, e^{E_n^{(s)}t} D^{(l_2)}_{p_2,q_2}(x\,y^{-1})\,, \end{aligned} \tag{3.10}$$

where we have used that $g_L(y)g_R(y)^{-1} = y$ and similarly for x.

The un-integrated heat kernel (3.8) and (3.10) obviously depends in general on the choice of section, as is clear, for instance, from the first line of (3.10). Indeed this dependence just reflects the way the components of the harmonics themselves depend on the choice of section, see (A.2). For the case of the scalar, this ambiguity is not present and one can write the final answer explicitly, which we do in the next subsection. For higher spin, the expression cannot be simplified further unless one makes a specific choice of section (as also coordinates). We exhibit the answer for the thermal section in section 3.2.

3.1 The scalar case

In the scalar case, $s = 0$, the representation labels a and b are trivial, and so is the first line of (3.10). The scalar heat kernel is then of the form

$$\begin{aligned} K^{(0)}(x,y;t) &= \frac{1}{2\pi^2} \sum_{n=0}^{\infty} \sum_m (n+1)^2 |\langle \tfrac{n}{2}, m; \tfrac{n}{2}, -m|0,0\rangle|^2\, e^{-n(n+2)t}\, D^{(\frac{n}{2})}_{m,m}(y\,x^{-1}) \\ &= \frac{1}{2\pi^2} \sum_{n=0}^{\infty} (n+1) e^{-n(n+2)t}\ \mathrm{Tr}_{\frac{n}{2}}(y\,x^{-1})\,, \end{aligned} \tag{3.11}$$

where we have used (2.35). Since

$$\mathrm{Tr}_{\frac{n}{2}}(y\,x^{-1}) = \frac{\sin(n+1)\rho}{\sin\rho}\,, \tag{3.12}$$

where ρ is the geodesic distance between x and y, we can rewrite the scalar heat kernel as

$$K^{(0)}(\rho;t) = \frac{1}{2\pi^2} \sum_{n=0}^{\infty} (n+1) \frac{\sin(n+1)\rho}{\sin\rho} e^{-n(n+2)t}\,. \tag{3.13}$$

This reproduces the answer given, for example, in [5].

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3.2 Higher spin

As mentioned above, for larger s, (3.10) does not simplify further, unless we make some specific choices. In the following we shall use the spherical coordinates (2.4), and consider the thermal section (2.26) and (2.27).

Since S^3 is a homogeneous space, we may, without loss of generality, assume the point y to be at the 'origin', i.e. to be represented by the identity matrix

$$g_L(y) = g_R(y) = e \ . \tag{3.14}$$

The thermal section for the other point x is then described by (2.28). Then we can write (3.8) as

$$\begin{aligned}
K^{(s)}_{ab}(x,e;t) &= \sum_{l_1,l_2}\sum_{p_1,p_2;q_1,q_2} a^{(s)}_n \langle s,a|l_1,p_1;l_2,p_2\rangle \langle l_1,q_1;l_2,q_2|s,b\rangle\, e^{E^{(s)}_n t} \\
&\qquad \times D^{(l_1)}_{p_1,q_1}\left(e^{-i\frac{\chi}{2}\sigma_3}U^\dagger(\hat n)\right) D^{(l_2)}_{p_2,q_2}\left(e^{i\frac{\chi}{2}\sigma_3}U^\dagger(\hat n)\right) \\
&= \sum_{l_1,l_2}\sum_{p_1,p_2;q_1,q_2} a^{(s)}_n \langle s,a|l_1,p_1;l_2,p_2\rangle \langle l_1,q_1;l_2,q_2|s,b\rangle\, e^{E^{(s)}_n t} \\
&\qquad \times e^{i(p_2-p_1)\chi} D^{(l_1)}_{p_1,q_1}(U^\dagger(\hat n))\, D^{(l_2)}_{p_2,q_2}(U^\dagger(\hat n)) \\
&= \sum_{b'} D^{(s)}_{b'b}(U^\dagger(\hat n)) \sum_{l_1,l_2} a^{(s)}_n\, e^{E^{(s)}_n t} \\
&\qquad \times \sum_{p_1,p_2} \langle s,a|l_1,p_1;l_2,p_2\rangle \langle l_1,p_1;l_2,p_2|s,b'\rangle\, e^{i(p_2-p_1)\chi} \\
&\equiv D^{(s)}_{ab}(U^\dagger(\hat n))\, K^{(s)}_a(\chi,0;t) \ .
\end{aligned} \tag{3.15}$$

In the penultimate line we have employed the identity (A.1), and in the last line we have used that the Clebsch Gordan coefficents vanish unless $b' = a$. Finally, we have defined

$$K^{(s)}_a(\chi,0;t) \ = \sum_{l_1,l_2}\sum_{p_1,p_2} a^{(s)}_n\, |\langle l_1,p_1;l_2,p_2|s,a\rangle|^2\, e^{E^{(s)}_n t}\, e^{i\chi(p_2-p_1)} \ . \tag{3.16}$$

We should mention in passing that this form of the heat kernel in spherical coordinates can also be deduced from the alternative factorized form of the eigenfunctions $\Phi^{\pm(s)}_{a,nlm}$ that we obtained in (2.43).

The radial part of the heat kernel $K^{(s)}_a(\chi,0;t)$ can be evaluated using the explicit form of the Clebsch-Gordan coefficents appearing in (3.16); this is carried out in appendix C. The final answer is

$$K^{(s)}_a(\chi,0;t) = \frac{1}{2\pi^2}\,\frac{1}{(2s+1)} \sum_{n=0}^{\infty} \frac{(n+1)!(2s+1)!}{(n+2s)!}\, K^{(s)}_{a;n}(\chi)\, e^{E^{(s)}_n t}\,, \tag{3.17}$$

where $K^{(s)}_{a;n}(\chi)$ is given in terms of Gegenbauer polynomials in (C.7). It follows from the explicit formula for $K^{(s)}_{a;n}(\chi)$ that

$$K^{(s)}_{a;n}(\chi=0) = (2-\delta_{s,0})\,\frac{(n+2s+1)!}{n!(2s+1)!}\,, \tag{3.18}$$

and thus for $\chi = 0$ the complete heat kernel simplifies to

$$K^{(s)}_{ab}((\chi=0,\theta,\phi),e;t) = D^{(s)}_{ab}(U^\dagger(\hat{n}))\,\frac{1}{2\pi^2}\,\frac{1}{(2s+1)}\sum_{n=0}^{\infty} d^{(s)}_n\, e^{E^{(s)}_n t}\,, \tag{3.19}$$

where $\mathrm{U}(\hat{n})$ was defined in terms of (θ,ϕ) in (2.29), and the mutliplicity $d^{(s)}_n$ was introducted in (3.4).

3.2.1 The spinor case

As a cross check we can compare with some of the existing results in the literature. We have already evaluated the scalar case. The next simplest case is then the spinor case $(s=\frac{1}{2})$. This has been obtained explicitly in, for instance [9]. The only small difference is that they evaluate the heat kernel for the operator $\nabla\!\!\!\!/^{\,2}$ rather than the spinor Laplacian. The eigenvalues of the former are $-(n+\frac{3}{2})^2$ while that of the latter are $-(n+\frac{3}{2})^2+\frac{3}{2}$. Taking this shift into account, the result given there (see e.g. eq. (3.4) of the published version of [9] or eq. (4.12) of the `arXiv` version) is

$$K^{(\frac{1}{2})}_{ab}((\chi,0,0),e;t) = \delta_{ab}\left[\frac{1}{2\pi^2}\sum_{n=0}^{\infty}(n+1)(n+2)\phi_n(\chi)\,e^{-t(n+\frac{3}{2})^2+\frac{3}{2}t}\right]\,, \tag{3.20}$$

where $\phi_n(\chi)$ is given in terms of Jacobi polynomials as

$$\phi_n(\chi) = \frac{n!\,\Gamma(\frac{3}{2})}{\Gamma(n+\frac{3}{2})}\cos\frac{\chi}{2}\,P^{(\frac{1}{2},\frac{3}{2})}_n(\cos\chi)\ . \tag{3.21}$$

Using the recursion

$$P^{(\frac{1}{2},\frac{3}{2})}_n(\cos\chi) = P^{(\frac{1}{2},\frac{1}{2})}_n(\cos\chi) - \sin^2\frac{\chi}{2}P^{(\frac{3}{2},\frac{3}{2})}_{n+}(\cos\chi)\,, \tag{3.22}$$

and the relation of the Jacobi polynomials $P^{(m,m)}_n$ to the Gegenbauer polynomials we find

$$\phi_n(\chi) = \frac{2}{(n+1)(n+2)}\cos\frac{\chi}{2}\left[C^2_n(\cos\chi) - C^2_{n-1}(\cos\chi)\right]\ . \tag{3.23}$$

Putting this back in (3.20), we find that it agrees precisely with the general expression in (3.17) for the special case of $(s=\frac{1}{2})$.

3.2.2 The vector case

As a last example we write the answer for the vector case $(s=1)$ in full detail. We again consider the heat kernel for the points between the north pole e, and the point (χ,θ,ϕ) on S^3. The heat kernel is obtained from (3.15) and (3.17)

$$K^{(1)}_{ab}((\chi,\theta,\phi),e;t) = D^{(1)}_{ab}(U^\dagger(\hat{n}))\,\frac{1}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(n+2)}K^{(1)}_{a;n}(\chi)\,e^{-t((n+1)(n+3)-1)}\,, \tag{3.24}$$

and (C.7) implies that the explicit expressions for $K^{(1)}_{a;n}(\chi)$ are

$$K^{(1)}_{1;n}(\chi) = K^{(1)}_{-1;n}(\chi) = 2\,\left[\cos\chi\, C^3_n(\cos\chi) - 2\,C_{n-1}(\cos\chi) + \cos\chi\, C^3_{n-2}(\cos\chi)\right]\,, \tag{3.25}$$

and

$$K^{(1)}_{0;n}(\chi) = 2\,C^2_n(\cos\chi)\ . \tag{3.26}$$

It is also useful to rewrite this expressions in terms of trignometric functions. Using (C.9) and the recursion relations satisfied by the Gegenbauer polynomials we find that

$$\begin{aligned}
K^{(1)}_{0;n}(\chi) &= \frac{1}{2\sin^3\chi}\Big((n+3)\sin(n+1)\chi-(n+1)\sin(n+3)\chi\Big)\,, \qquad (3.27)\\
K^{(1)}_{1;n}(\chi) &= K^{(1)}_{-1;n}(\chi) = -\frac{1}{8\sin^3\chi}\Big[(2+n)(3+n)\sin n\chi-2(2+4n+n^2)\sin(n+2)\chi \\
&\qquad +(n+2)(n+1)\sin(n+4)\chi\Big]\ .
\end{aligned}$$

The above form of the radial heat kernel is suitable for analytical continuation to AdS_3 (see section 5.3.3).

4 Heat kernel on thermal S^3

In perparation for the calculation on thermal H_3^+ we now want to study the heat kernel on the thermal quotient of S^3, i.e. on the manifold S^3/Γ, where Γ describes a specific group of identifications. These identifications are most easily described in the double polar coordinates (2.6), where the action of the generator γ of Γ, is given by

$$\gamma:\quad \eta\mapsto\eta+\beta\,,\qquad \varphi\mapsto\varphi+\vartheta\ . \tag{4.1}$$

In order for this group action to be globally well-defined, we should take Γ to be of finite order, $\Gamma\cong\mathbb{Z}_N$, i.e. $\gamma^N=1$. This corresponds to a Lens space quotient of S^3. The generator γ acts on the group element g in (2.7) as

$$g\mapsto\tilde g=\begin{pmatrix} e^{i\frac{\tau}{2}} & 0\\ 0 & e^{-i\frac{\tau}{2}}\end{pmatrix} g \begin{pmatrix} e^{-i\frac{\bar\tau}{2}} & 0\\ 0 & e^{i\frac{\bar\tau}{2}}\end{pmatrix} = A\,g\,\bar A^{-1}\,, \tag{4.2}$$

where

$$\tau\equiv\tau_1-\tau_2=\vartheta-\beta;\qquad \bar\tau\equiv\tau_1+\tau_2=\vartheta+\beta \tag{4.3}$$

and

$$A=\begin{pmatrix} e^{i\frac{\tau}{2}} & 0\\ 0 & e^{-i\frac{\tau}{2}}\end{pmatrix},\qquad \bar A=\begin{pmatrix} e^{i\frac{\bar\tau}{2}} & 0\\ 0 & e^{-i\frac{\bar\tau}{2}}\end{pmatrix}\ . \tag{4.4}$$

The section that is compatible with this group action must satisfy (compare (2.25))

$$\sigma(\gamma(x)) = (A,\bar A)\cdot\sigma(x)\ . \tag{4.5}$$

As explained above (2.25), such a choice of section is necessary for the compatibility of the thermal quotient with the coset space identification on the principal bundle G. Another way to understand this requirement is as follows. The group action (4.2) induces a natural map (via push forward) relating the tangent basis at g to that at $\tilde g$. On the other hand, the choice of section specifies a vielbein (see (2.10)) for all $g\in G$. The condition (2.25) implies that the vielbein at $\tilde g$ agrees precisely with the push-forward via (4.2) of the vielbein at g.

Obviously, (4.5) is not satisfied by every section; in particular, it is not true for the 'canonical' section (2.22). On the other hand, one easily checks that it is satisfied by the thermal section (2.30) and (2.31).

4.1 Method of images

The heat kernel on the quotient space can be calculated from that on S^3 by the method of images. We can fix one of the points (say x) and sum over the images of the second one (y). This is to say, we have

$$\sum_{m\in\mathbb{Z}_N} K^{(s)}_{ab}(x,\gamma^m(y);t)\,, \tag{4.6}$$

where N is the order of γ. We will be interested in obtaining the determinant of $\Delta_{(s)}$ on S^3/Γ, which means that we need to find the integrated traced heat kernel for coincident points on the orbifolded space, i.e.

$$\sum_{m\in\mathbb{Z}_N}\sum_a\int_{S^3/\Gamma} d\mu(x)\, K^{(s)}_{aa}(x,\gamma^m(x);t)\ . \tag{4.7}$$

Here we have traced over the group theory indices a,b with a simple Kronecker delta since we are working in a tangent space basis (such as the usual vielbein basis for the $s=1$ case). If we were working in a coordinate basis, then the expression would be more complicated, involving a Jacobian factor such as $\frac{\partial\gamma(x)^\mu}{\partial x^\nu}$ [3].

Since we are considering the identification (4.1), we need to understand the heat kernel evaluated at two points x and $y=\gamma^m(x)$ that have the same value for the ψ-component (and only differ in their η- and φ-component). In this case it follows from (2.32) that

$$g_L(x)^{-1}g_L(y) = V(\psi)^{-1}\,U_1\,V(\psi)\,,\qquad g_R(x)^{-1}g_R(y) = V(\psi)\,U_2\,V(\psi)^{-1}\,, \tag{4.8}$$

where $V(\psi)$ is defined in (2.33) with $\psi=\psi(x)=\psi(y)$, and U_1 and U_2 are of the form

$$U_1 = \exp\left(i\frac{(\Delta\varphi-\Delta\eta)}{2}\sigma_3\right) = e^{im\frac{\tau}{2}\sigma_3}\,,\quad U_2 = \exp\left(i\frac{(\Delta\varphi+\Delta\eta)}{2}\sigma_3\right) = e^{im\frac{\bar\tau}{2}\sigma_3} \tag{4.9}$$

with $\Delta\varphi=\varphi(y)-\varphi(x)=m\vartheta$ and $\Delta\eta=\eta(y)-\eta(x)=m\beta$, and additionally using the definition (4.3). With these conventions (3.8) for the particular case of $y=\gamma^m(x)$ becomes

$$\begin{aligned}K^{(s)}_{ab}(x,y=\gamma^m(x);t) = \sum_{l_1,l_2}\sum_{p_1,p_2;q_1,q_2} & a^{(s)}_n\,\langle s,a|l_1,p_1;l_2,p_2\rangle\,\langle l_1,q_1;l_2,q_2|s,b\rangle\, e^{E^{(s)}_n t}\\ &\times D^{(l_1)}_{p_1,q_1}\Big(V(\psi)^{-1}U_1V(\psi)\Big)\,D^{(l_2)}_{p_2,q_2}\Big(V(\psi)\,U_2V(\psi)^{-1}\Big)\ .\end{aligned}$$

We can write the trace over $a=b$ more abstractly as

$$\begin{aligned}&\sum_a K^{(s)}_{aa}(x,\gamma^m(x);t)\\ &\quad=\sum_{l_1,l_2} a^{(s)}_n\, e^{E^{(s)}_n t}\ \mathrm{Tr}_{\,s}\left[\Big(V(\psi)^{-1}U_1V(\psi)\Big)^{(l_1)}\otimes\Big(V(\psi)\,U_2V(\psi)^{-1}\Big)^{(l_2)}\right]\,,\end{aligned} \tag{4.10}$$

where the trace is only taken over the spin s subrepresentation in the tensor product $(l_1\otimes l_2)$. Conjugation with the operator $V(\psi)\otimes V(\psi)$ does not modify the trace (since the subpresentation s is invariant under the action of $g\otimes g$), and thus (4.10) can be rewritten as

$$\sum_a K^{(s)}_{aa}(x,y;t) = \sum_{l_1,l_2} a^{(s)}_n\, e^{E^{(s)}_n t}\ \mathrm{Tr}_{\,s}\Big[U_1^{(l_1)}\otimes\big(V(\psi)^2U_2V(\psi)^{-2}\big)^{(l_2)}\Big]\ . \tag{4.11}$$

Let us denote a general diagonal group element by

$$D(\alpha) = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} . \tag{4.12}$$

Since both U_1 and U_2 are diagonal, it follows that

$$D(\alpha)U_1 D(\alpha)^{-1} = U_1 \,, \qquad D(\beta)U_2 D(\beta)^{-1} = U_2 \,. \tag{4.13}$$

Taking $\alpha = -(\varphi-\eta)/2$ and $\beta = -(\varphi+\eta)/2$, and using the same argument as in going to (4.11), we then obtain

$$\sum_a K^{(s)}_{aa}(x,y;t) = \sum_{l_1,l_2} a^{(s)}_n \, e^{E^{(s)}_n t} \, \mathrm{Tr}_{\,s}\Big[U_1^{(l_1)} \otimes \big(g\, U_2\, g^{-1}\big)^{(l_2)}\Big] \,, \tag{4.14}$$

where

$$g = D\Big((\varphi-\eta)/2\Big) V(\psi)^2 D\Big(-(\varphi+\eta)/2\Big) = \begin{pmatrix} e^{-i\eta}\cos\psi & ie^{i\varphi}\sin\psi \\ ie^{-i\varphi}\sin\psi & e^{i\eta}\cos\psi \end{pmatrix} = g(\psi,\eta,\varphi) \,, \tag{4.15}$$

and $g(\psi,\eta,\varphi)$ is defined in (2.7). Next we perform the integral over S^3/Γ in (4.7). This amounts to integrating (4.14) over ψ in the fundamental domain of S^3/Γ. Equivalently, we may integrate ψ over the full range $\psi \in [0, \frac{\pi}{2}]$, and divide by the appropriate volume factor. In addition, since (4.14) is actually independent of η and φ — this is obvious from (4.11) — we may also integrate $\eta, \varphi \in [0, 2\pi]$. But then the second group element in (4.14) equals

$$\int_{S^3} dg \, \big(g\, U_2\, g^{-1}\big)^{(l_2)} = \frac{2\pi^2}{\dim(l_2)} \, \mathrm{Tr}_{\,(l_2)}(U_2)\, \mathbf{1}_{l_2} \,, \tag{4.16}$$

where we have used Schur's lemma, observing that the operator on the left hand side commutes with all group elements. Thus the integrated heat kernel becomes

$$\begin{aligned} &\int_{S^3/\Gamma} d\mu(x) \sum_a K^{(s)}_{aa}(x,\gamma^m(x);t) \\ &\qquad = \pi\tau_2 \sum_{l_1,l_2} \frac{a^{(s)}_n}{\dim(l_2)} \, \mathrm{Tr}_{\,(l_2)}(U_2) e^{E^{(s)}_n t} \, \mathrm{Tr}_{\,s}\Big[U_1^{(l_1)} \otimes \mathbf{1}^{(l_2)}\Big] \,, \end{aligned} \tag{4.17}$$

where the prefactor $\pi\tau_2 = 2\pi^2 \frac{\tau_2}{2\pi}$ comes from the relative volume of S^3/Γ to S^3. The final trace can now be easily done (for example using similar arguments as above), and it equals

$$\mathrm{Tr}_{\,s}\Big[U_1^{(l_1)} \otimes \mathbf{1}^{(l_2)}\Big] = \mathrm{Tr}_{\,(l_1)}(U_1) \, \frac{2s+1}{\dim(l_1)} \,. \tag{4.18}$$

Plugging this back into (4.17) we therefore obtain

$$\int_{S^3/\Gamma} d\mu(x) \sum_a K^{(s)}_{aa}(x,\gamma^m(x);t) = \frac{\pi\tau_2}{2\pi^2} \sum_{l_1,l_2} \mathrm{Tr}_{\,(l_1)}(U_1) \, \mathrm{Tr}_{\,(l_2)}(U_2)\, e^{E^{(s)}_n t} \,, \tag{4.19}$$

where we have used the formula for $a_n^{(s)}$ from (3.2). Finally, doing the sum over m leads to

$$\begin{aligned}\sum_{m\in\mathbb{Z}_N}\sum_a\int_{S^3/\Gamma} & d\mu(x)K_{aa}^{(s)}(x,\gamma^m(x);t)\\ &=\frac{\tau_2}{2\pi}\sum_{m\in\mathbb{Z}_N}\sum_{n=0}^{\infty}\Big[\chi_{(\frac{n}{2})}(m\tau)\chi_{(\frac{n}{2}+s)}(m\bar\tau)+\chi_{(\frac{n}{2}+s)}(m\tau)\chi_{(\frac{n}{2})}(m\bar\tau)\Big]\,e^{E_n^{(s)}t}\\ &\equiv K^{(s)}(\tau,\bar\tau,t)\,,\end{aligned}\tag{4.20}$$

where we have assumed that $s>0$; otherwise the second term in the middle line of (4.20) is absent. We have also used the notation

$$\chi_{(l)}(\tau)=\mathrm{Tr}_{(l)}(e^{i\frac{\tau}{2}\sigma_3})=\frac{\sin\frac{(2l+1)\tau}{2}}{\sin\frac{\tau}{2}}\tag{4.21}$$

for the SU(2) character in the representation l.

5 Heat kernel on AdS_3

Having derived the heat kernel for an arbitrary tensor Laplacian on S^3 as well as on its 'thermal' quotient, we will now extend the analysis to the case of H_3^+; the thermal quotient of H_3^+ will be discussed in the next section. As mentioned in the introduction, this is simplest done by performing a suitable analytic continuation to H_3^+ (and its thermal quotient). Since this is, in general, a tricky procedure we will motivate and describe in some detail how it is to be carried out. As will become clear, for the particular case of H_3^+, the central ingredients in our calculation (such as the eigenfunctions, eigenvalues and their measure) have been independently computed and checked to obey the analytic continuation from their S^3 counterparts, see in particular the series of papers by Camporesi and Higuchi [7–9]. These explicit results can be taken as the ultimate justification for our use of the analytic continuation procedure.

5.1 Preliminaries

Euclidean AdS_3 is the hyperbolic space H_3^+ which can be thought of as the homogeneous space

$$H_3^+\cong \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)\,,\tag{5.1}$$

where the quotienting is done by the usual right action. We can view $\mathrm{SL}(2,\mathbb{C})$ as an analytic continuation of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ in a way which will be made explicit below.

As in the case of S^3 we will need to choose coordinates for explicit expressions. Corresponding to the spherical coordinates on S^3 (2.4) we have now

$$ds^2=dy^2+\sinh y^2\,(d\theta^2+\sin^2\theta\,d\phi^2)\,,\tag{5.2}$$

which is obtained by the continuation $\chi\to -iy$ and $ds^2\to -ds^2$, of (2.4).

The coset space representative of SL$(2,\mathbb{C})$/SU(2) (for a given (y,θ,ϕ)) can be taken to be the continuation of (2.5)

$$\tilde{g}(y,\theta,\phi)=\begin{pmatrix}\cosh y+\sinh y\,\cos\theta & \sinh y\,\sin\theta\,e^{i\phi}\\ \sinh y\,\sin\theta\,e^{-i\phi} & \cosh y-\sinh y\,\cos\theta\end{pmatrix}\ . \tag{5.3}$$

For the thermal quotient it will be convenient to work in the double polar coordinate analogue of (2.6), i.e. to use the metric

$$ds^2=d\rho^2+\cosh^2\rho\,(dt)^2+\sinh^2\rho\,(d\varphi)^2\ . \tag{5.4}$$

This is related to (2.6) by the continuation $\psi\to-i\rho$, $\eta\to it$ and $ds^2\to-ds^2$. Therefore corresponding to (2.7) we now have the coset space element

$$\tilde{g}(\rho,t,\varphi)=\begin{pmatrix}e^t\cosh\rho & e^{i\varphi}\sinh\rho\\ e^{-i\varphi}\sinh\rho & e^{-t}\cosh\rho\end{pmatrix}\ . \tag{5.5}$$

To carry through the construction of eigenfunctions as described in section 2, we will first need an appropriate choice of section. As is familiar from the analysis of the Lorentz group in four dimensions, the representations of SL$(2,\mathbb{C})$ are most easily described in terms of SU(2) $\times$ SU(2). The Lie algebra of the former is a complexified version of the latter. More precisely, if we write the Lie algebra of SO(4) as $so(4)\simeq su(2)\oplus su(2)$ with generators $a^{(1)}$ and $a^{(2)}$, respectively, then the diagonal SU(2) by which we quotient SO(4) to obtain S^3 is generated by $h=a^{(1)}+a^{(2)}$. Defining $k=a^{(1)}-a^{(2)}$, the complexification $k\to-ik$ describes then the continuation from S^3 to H_3^+. This is equivalent to the continuation $\chi\to-iy$ described above.

Thus it will still be useful to describe the coset representative of SL$(2,\mathbb{C})$/SU(2) in terms of pairs of group elements $(\tilde{g}_L,\tilde{g}_R)$ that live in the appropriately complexified version of SU(2) $\times$ SU(2). The relevant expressions for the complexification are obtained from those on S^3 precisely by the analytic continuation of the coordinates described above. In particular, the analogue of the thermal section is now described by $(\tilde{g}_L(x),\tilde{g}_R(x))$, where in spherical coordinates we have (compare with (2.26) and (2.27))

$$\tilde{g}_L(y,\theta,\phi)=\begin{pmatrix}\cos\frac{\theta}{2}\,e^{i(\phi-iy)/2} & -\sin\frac{\theta}{2}\,e^{i(\phi+iy)/2}\\ \sin\frac{\theta}{2}\,e^{-i(\phi+iy)/2} & \cos\frac{\theta}{2}\,e^{-i(\phi-iy)/2}\end{pmatrix}=\mathrm{U}(\hat{n})e^{\frac{y}{2}\sigma_3} \tag{5.6}$$

and

$$\tilde{g}_R(y,\theta,\phi)=\begin{pmatrix}\cos\frac{\theta}{2}\,e^{i(\phi+iy)/2} & -\sin\frac{\theta}{2}\,e^{i(\phi-iy)/2}\\ \sin\frac{\theta}{2}\,e^{-i(\phi-iy)/2} & \cos\frac{\theta}{2}\,e^{-i(\phi+iy)/2}\end{pmatrix}=\mathrm{U}(\hat{n})e^{-\frac{y}{2}\sigma_3}. \tag{5.7}$$

In the double polar coordinates which we use for the quotienting, we have similarly (compare with (2.30) and (2.31))

$$\tilde{g}_L(\rho,t,\varphi)=\begin{pmatrix}e^{t/2}e^{i\varphi/2}\cosh\dfrac{\rho}{2} & e^{t/2}e^{i\varphi/2}\sinh\dfrac{\rho}{2}\\ e^{-t/2}e^{-i\varphi/2}\sinh\dfrac{\rho}{2} & e^{-t/2}e^{-i\varphi/2}\cosh\dfrac{\rho}{2}\end{pmatrix} \tag{5.8}$$

and

$$\tilde{g}_R(\rho,t,\varphi) = \begin{pmatrix} e^{-t/2}e^{i\varphi/2}\cosh\frac{\rho}{2} & -e^{-t/2}e^{i\varphi/2}\sinh\frac{\rho}{2} \\ -e^{t/2}e^{-i\varphi/2}\sinh\frac{\rho}{2} & e^{t/2}e^{-i\varphi/2}\cosh\frac{\rho}{2} \end{pmatrix} . \tag{5.9}$$

One can check that with both sets of coordinates we have indeed $\tilde{g}_L(x)\cdot\tilde{g}_R^{-1}(x) = \tilde{g}(x)$, where $\tilde{g}(x)$ is given in (5.3) and (5.5), respectively.

5.2 Harmonic analysis on H_3^+

As was described in section2.1, to obtain the eigenfunctions of the Laplacian $\Delta_{(s)}$ on G/H, we need facts from the harmonic analysis on G. For a general noncompact semi-simple G this is an intricate subject (see e.g. [18]). However, the results for $G = \mathrm{SL}(2,\mathbb{C})$ are relatively well known to physicists since $\mathrm{SL}(2,\mathbb{C})$ is the Lorentz group in four dimensions. Some useful general references on the subject, particularly for the infinite dimensional representations which we will need below, are [19, 20].

The component eigenfunctions of the tensor harmonics are given in terms of matrix elements of appropriate unitary representations of $\mathrm{SL}(2,\mathbb{C})$. One of the major differences between the compact and the noncompact cases is that the (nontrivial) unitary representations of the latter are necessarily infinite dimensional. Recall that the usual finite dimensional (and hence non-unitary) representations of $\mathrm{SL}(2,\mathbb{C})$ are labelled by (j_1,j_2), where j_1 and j_2 are the half-integer spin representations of the two SU(2)s. In fact, the most general representation (or the 'complete series') of $\mathrm{SL}(2,\mathbb{C})$, including the unitary representations, can also be labelled by (j_1,j_2), where j_1,j_2 are now complex but subject to some constraints such as $(j_i - j_2)$ being a half integer.

The unitary representations come in two series: the so-called 'principal series' and the 'complementary series'. However, only the principal series will play a role in what follows. This is because they are the only representations that arise in the decomposition of functions on $\mathrm{SL}(2,\mathbb{C})$ and therefore (see the discussion around (2.12)) for sections of bundles on $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$.[4] These correspond to j_1 and j_2 taking the values

$$2j_1 = s - 1 + i\lambda\,, \qquad 2j_2 = -s - 1 + i\lambda\,, \tag{5.10}$$

where $\lambda \in \mathbb{R}^+$ and s is half-integer, see for example [18, section II.4]. When restricted to the diagonal SU(2) subgroup, these representations decompose into an infinite number of SU(2) representations of spin $s, s+1, s+2, \ldots$ [19, 20]. Thus these representations play the role of the representations $(\frac{n}{2}+s,\frac{n}{2})$ in the S^3 case and will describe the transverse, traceless spin s tensors on H_3^+. Comparison to (5.10) suggests that the appropriate analytic continuation for n is [8]

$$n \mapsto -s - 1 + i\lambda\ . \tag{5.11}$$

Thus eigenfunctions of $\Delta_{(s)}$ are given (in the thermal section) by the matrix elements of the $\mathrm{SL}(2,\mathbb{C})$ element $(\tilde{g}_L(x),\tilde{g}_R(x))$ in these representations labelled by a continuous

[4] In general, additional (normalizable) representations - the 'discrete series' - could also appear when considering even dimensional hyperbolic spaces.

parameter $\lambda \in \mathbb{R}^+$ (for fixed s). Their eigenvalues are, up to a sign, given by the same analytic continuation (5.11) applied to (2.17),

$$E_\lambda^{(s)} = -(\lambda^2 + s + 1)\ . \tag{5.12}$$

The sign is a consequence of the fact that the metric has changed sign under the analytic continuation, $ds^2 \to -ds^2$. Thus the analytic continuation of (2.17) gives minus the eigenvalue of the Laplacian on H_3^+.

5.3 The heat kernel on H_3^+

In computing the heat kernel the sum over n in (1.2) is now to be replaced by an integral over λ. The measure for the integration is determined from the so-called Plancherel measure which describes the decomposition of the space of functions on G into its irreducible representations. We will continue to refer to the measure thus obtained for the decomposition of the sections on G/H with spin s (in the case of $G = \mathrm{SL}(2,\mathbb{C})$ and $H = \mathrm{SU}(2)$) as the Plancherel measure and denote it by $d\mu^{(s)}(\lambda)$.

This Plancherel measure for H_3^+ (or more generally, the hyperbolic spaces H_N) has been computed by Camporesi and Higuchi (see for example [7, 8]). The explicit expression is given by

$$d\mu^{(s)}(\lambda) = \frac{1}{2\pi^2}\,(2-\delta_{s,0})\,\frac{(\lambda^2+s^2)}{(2s+1)}\,d\lambda\,, \tag{5.13}$$

which is, up to a sign and the prefactor $(2-\delta_{s,0})$, precisely the analytic continuation of the S^3 normalisation constant $a_n^{(s)} = \frac{1}{2\pi^2}\frac{(n+2s+1)(n+1)}{(2s+1)}$ (see (3.2)) by our analytic continuation (5.11). (The origin of this sign is again the change of sign in the analytic continuation of the metric $ds^2 \to -ds^2$. The origin of the prefactor $(2-\delta_{s,0})$ is also the same as before, namely that there are two choices $\lambda_\pm$ for $s>0$ (see (2.16)), which fall together for $s=0$.)

The H_3^+ heat kernel for spin s fields then takes the form

$$K_{ab}^{(s)}(x,y;t) = \int_0^\infty d\mu^{(s)}(\lambda)\,\phi_{\lambda,ab}^{(s)}(x,y)\,e^{-t(\lambda^2+s+1)}\,, \tag{5.14}$$

where $\phi_{\lambda,ab}^{(s)}(x,y) = U_{ab}^{\lambda,s}(\sigma(x)^{-1}\sigma(y))$ are the matrix elements of the representation (λ,s) projected onto the spin s representation of the diagonal SU(2) (cf. (3.9)). In particular, the index a still labels the components of the spin s field and takes values from $-s$ to s. The functions $U_{ab}^{\lambda,s}(g)$ are sometimes known as generalised spherical functions (for spin s) and have many important properties. For example, they are determined completely by knowing the values on a maximal torus.[5] In spherical polar coordinates this is the statement that we know the complete answer to the heat kernel once we know the value for one of the points at the origin and the other at some $(\chi,0,0)$ for S^3 (cf. (3.15)) and $(y,0,0)$ for H_3^+. The spherical functions also satisfy simple radial Laplacian equations, which ensures that we can also have a simple analytic continuation for them. We refer the reader to section 5.3 of [9] for more properties of these spherical functions.

[5]We can decompose a general $\mathrm{SL}(2,\mathbb{C})$ element g as $g = h_1 t h_2$, where $h_1, h_2 \in \mathrm{SU}(2)$ and t lies in the maximal torus.

For our purposes it is sufficient to make the following observations. In the thermal section, using the spherical coordinates (5.2), we can use a similar reasoning as in section 3.2. We can choose one point to be at the origin and factor out the S^2 angular dependence as in (3.15). Then the other point can be taken to be $(y, 0, 0)$ and we obtain

$$K_a^{(s)}(y, 0; t) = \int_0^\infty d\mu^{(s)}(\lambda)\, \phi_{\lambda,a}^{(s)}(y)\, e^{-t(\lambda^2+s+1)}\,, \tag{5.15}$$

where $\phi_{\lambda,a}^{(s)}(y)$ is the analytic continuation of $K_{a;n}^{(s)}(\chi)$ in (3.17) under $\chi \to -iy$. These functions are expressed in terms of Gegenbauer polynomials in (C.7). In order to perform the analytic continuation explicitly, we can use the definition of the Gegenbauer polynomials in terms of hypergeometric functions

$$C_n^\alpha(\cos\chi) = \frac{\Gamma(2\alpha+n)}{\Gamma(n+1)\Gamma(2\alpha)}\, F\left(2\alpha+n, -n, \alpha+\frac{1}{2}; \sin^2\frac{\chi}{2}\right)\,. \tag{5.16}$$

The right hand side can be defined for complex values of the arguments and in particular under the continuation $n \to -s-1+i\lambda$. Note that the index α takes the values $s+a+1$ in (C.7) and therefore continues to be an integer. Also the sum there continues to be a finite one with an upper limit given by $2a$. It is not easy to perform the integral over λ for general spin and give an explicit form of the heat kernel on AdS_3. However, we can do this integral for a few simple cases and check that the above prescription gives the correct result.

5.3.1 The scalar case

The heat kernel for the case $s = 0$ can be easily evaluated. In this case, we can in fact write the answer slightly more generally, namely directly in terms of the geodesic separation r between the two points. Instead of (3.15) we can start with the expression (3.13). Since the metric $ds^2 \to -ds^2$ in the analytic continuation we continue $\rho \to -ir$. Together with the continuation $n \to -1+i\lambda$, we find that (3.13) becomes

$$K^{(0)}(r; t) = \frac{1}{2\pi^2}\int_0^\infty d\lambda\, \lambda\, e^{-t(\lambda^2+1)}\, \frac{\sin\lambda r}{\sinh r}\,, \tag{5.17}$$

where we have absorbed a sign into the λ measure, see (5.13). After integrating over λ we obtain

$$K^{(0)}(r; t) = \frac{e^{-t}}{(4\pi t)^{3/2}}\frac{r\, e^{-\frac{r^2}{4t}}}{\sinh r}\,. \tag{5.18}$$

The explicit form of the geodesic distance on H_3^+ between the points (y, θ, ϕ) and (y', θ', ϕ') is given by

$$\cosh r = \cosh y' \cosh y - \sinh y' \sinh y \cos\theta' \cos\theta - \sinh y' \sinh y \sin\theta \sin\theta' \cos(\phi' - \phi)\,. \tag{5.19}$$

The expression (5.18) agrees with the heat kernel determined in [3] for the case $m^2 = 0$ — the general case is easily obtained from this since the mass only contributes an additive term to the exponent in (5.18).

5.3.2 The spinor case

For ($s = \frac{1}{2}$) we can again take the answer for the sphere, in this case worked out in (3.20), and perform the above analytic continuation. Instead of writing it in terms of Gegenbauer polynomials we can directly use, for the analytic continuation, the hypergeometric form of the Jacobi polynomial appearing in (3.21)

$$P_n^{(\alpha,\beta)}(\cos\chi) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}\, F\left(n+\alpha+\beta+1, -n, \alpha+1; \sin^2\frac{\chi}{2}\right) . \tag{5.20}$$

After the continuation $n \to -\frac{3}{2} + i\lambda$, (3.20) then becomes

$$K_{ab}^{(\frac{1}{2})} = \delta_{ab}\left[\frac{1}{2\pi^2}\int_0^\infty d\lambda\,\left(\lambda^2+\frac{1}{4}\right)\,\phi_\lambda(y)\, e^{-t(\lambda^2+\frac{3}{2})}\right] , \tag{5.21}$$

with $\phi_\lambda(y) = \cosh\frac{y}{2} F(\frac{3}{2}+i\lambda, \frac{3}{2}-i\lambda, \frac{3}{2}, -\sinh^2\frac{y}{2})$. Here we have again absorbed an overall minus sign into the measure, see (5.13). This agrees with eq. (5.14) of [6] (apart from the same shift in the exponent, see the discussion before (3.20)).

5.3.3 The vector case

For $s = 1$ we can analytically continue the answer for the 3-sphere given in (3.24) and (3.27) using

$$n \to -2 + i\lambda\,, \qquad \chi \to -iy\ . \tag{5.22}$$

For the case where we evaluate the heat kernel between the north pole and the point $(y, 0, 0) = (\chi, 0, 0)$, the geodesic distance r agrees with y. Using the above prescription we then obtain after some straightforward manipulations

$$\begin{aligned}
K_{00}^{(1)}(r,0;t) &= -\sqrt{\frac{\pi}{t}}\frac{e^{-2t}}{2\pi^2}\left(\frac{1}{\sinh^2 r}e^{-\frac{r^2}{4t}} - \frac{\cosh r}{\sinh^3 r}\int_0^r dx e^{-\frac{x^2}{4t}}\right), \qquad (5.23)\\
K_{11}^{(1)}(r,0;t) &= K_{-1-1}^{(1)}(r,0;t),\\
&= \frac{e^{-2t}}{4\pi^2\sin^3 r}\sqrt{\frac{\pi}{t}}\left(\frac{r}{2t}e^{-\frac{r^2}{4t}}\sinh^2 r + e^{-\frac{r^2}{4t}}\sinh r\cosh r - \int_0^r dx e^{-\frac{x^2}{4t}}\right) .
\end{aligned}$$

To check that this result satisfies the heat equation for vectors we recall that the heat equation for a U(1) gauge field is given by (see for instance [3] which we follow by also adding the constant two to the Laplacian)

$$-(\Delta_{(1)}+2)K_{\mu\nu'}(x,x';t) = -\frac{\partial}{\partial t}K_{\mu\nu'}(x,x';t)\,, \tag{5.24}$$

where $x = (y, \theta, \phi)$ and $x' = (y', \theta', \phi')$ are two points on H_3^+. We are interested in the heat kernel satisfying the Lorentz-gauge condition

$$\nabla^\mu K_{\mu\nu'}(x,x';t) = 0\,, \qquad \nabla^{\nu'}K_{\mu\nu'}(x,x';t) = 0\ . \tag{5.25}$$

Thus the initial condition at $t = 0$ is

$$K_{\mu\nu'}(x,x';0) = g_{\mu\nu'}(x)\,\delta^3(x,x') + \nabla_\mu\nabla_{\nu'}\frac{1}{\Delta_{(0)}}\delta^3(x,x')\ . \tag{5.26}$$

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Since H_3^+ is a maximally symmetric space, we can write the heat kernel, following [3], as

$$K_{\mu\nu'}(x,x';t) = F(t,u)\partial_\mu\partial_{\nu'}u + \partial_\mu\partial_{\nu'}S(t,u)\,, \tag{5.27}$$

where $1+u = \cosh r$, and r is the geodesic distance between the points x and x' given by (5.19). The heat equation (5.24) then reduces to

$$\begin{aligned}(\Delta_{(1)}+1)F(t,u) &= \partial_t F(t,u)\,,\\ \Delta_{(1)}S(t,u) - 2\int_u^\infty F(t,v)dv &= \partial_t S(t,u)\,,\end{aligned} \tag{5.28}$$

while the Lorentz gauge condition (5.25) becomes

$$\frac{\partial F}{\partial u}(1+u) + F + \partial_t\partial_u S = 0\,, \tag{5.29}$$

and the initial conditions on F and S are

$$F(0,u) = -\delta^3(x,x') \qquad S(0,u) = \frac{1}{\Delta_{(0)}}\delta^3(x,x') = -\frac{1}{4\pi}\coth r\,. \tag{5.30}$$

The correct solution is then

$$\begin{aligned}F(r,t) &= -\frac{e^{-\frac{r^2}{4t}}}{(4\pi t)^{3/2}}\frac{r}{\sinh r}\,,\\ S(r,t) &= -\frac{2}{(4\pi)^{3/2}\sqrt{t}}\frac{\cosh r}{\sinh r}\int_0^r e^{-\frac{x^2}{4t}}\,.\end{aligned} \tag{5.31}$$

Note that this solution differs form that found in [3], for which the Lorentz gauge condition was not implemented and which therefore satisfied the boundary condtion $K_{\mu\nu'}(x,x';0) = g_{\mu\nu'}(x)\delta^3(x,x')$, which is different from (5.26). In fact, [3] had to subtract out a scalar degree of freedom from the trace of their heat kernel to obtain the physical one loop determinant for vectors. This is unnecessary for the solution given in (5.31) since the Lorentz gauge condition guarantees that only the physical degrees of freedom contribute.

In order to compare (5.27) to (5.23) we need to convert the coordinate basis implicit in (5.27) to the tangent space indices of (5.23). For the case where x is the north pole and $x' = (r,0,0)$ the relations turn out to be

$$\begin{aligned}K^{(1)}_{00}(r,0;t) &= -F(r,t)\cosh r - \frac{\partial^2}{\partial r^2}S(r,t)\,,\\ K^{(1)}_{11}(r,0;t) &= K^{(1)}_{-1-1}(r,0;t) = -F(r,t) - \frac{1}{\sinh r}\frac{\partial}{\partial r}S(r,t)\,,\end{aligned} \tag{5.32}$$

where we have used (2.45). Substituting (5.31) we then reproduce indeed (5.23) up to an overall factor of e^{-2t}. The origin of this factor is that in (5.24), following [3], we have analyzed the heat equation for $(\Delta_{(1)}+2)$, rather than for the Laplacian $\Delta_{(1)}$ itself.

5.4 The coincident heat kernel

It is difficult to do the integrals over λ for the heat kernel in general. However it is easy to obtain the expression for the coincident heat kernel for arbitrary spin s. One need only consider the integrand of (5.14) to notice that the coincident traced heat kernel $K^{(s)}_{aa}(x,x;t)$ is given by

$$\begin{aligned}
K^{(s)}_{aa}(x,x;t) &= (2s+1)\int_0^\infty d\mu^{(s)}(\lambda)\, e^{E^{(s)}_\lambda t} \\
&= (2-\delta_{s,0})\,\frac{1}{2\pi^2}\int_0^\infty d\lambda\,(\lambda^2+s^2)\,e^{-t(\lambda^2+s+1)} \\
&= \frac{1}{(4\pi t)^{\frac{3}{2}}}\,(2-\delta_{s,0})\,(1+2s^2 t)\,e^{-t(s+1)}\ .
\end{aligned} \tag{5.33}$$

For $s = 1, 2$ this agrees precisely with the answers of Giombi *et.al.* [3] (up to shifts in the exponent which come from mass terms), as well as with the general expression for the zeta function in [9].

6 Heat kernel on thermal H_3^+

6.1 The thermal identification

We are actually interested in determining the heat kernel for thermal AdS_3. Thermal AdS_3 is obtained from Euclidean AdS_3 (i.e. H_3^+) described above by identifying points under a $\mathbb{Z}$ action. To identify the relevant $\mathbb{Z}$ action it is useful to write H_3^+ in double polar coordinates (5.4), which were obtained from the corresponding coordinates on S^3 by the continuation

$$i\psi = \rho\,, \qquad i\eta = -t\ . \tag{6.1}$$

Translating the thermal identifications (4.1) of S^3 into the analytically continued variables then corresponds to

$$t \sim t - i\beta\,, \qquad \phi \sim \phi + \vartheta\ . \tag{6.2}$$

Thus β has the interpretation of the inverse temperature. In addition, the analytically continued variables, τ and $\bar{\tau}$ of (4.3) are now

$$\tau = \vartheta + i\beta\,, \qquad \bar{\tau} = \vartheta - i\beta\,, \tag{6.3}$$

and are indeed complex conjugates of one another.

6.2 The heat kernel

As discussed in section 5, we could analytically continue the harmonic analysis on S^3 to that on H_3^+. We are now considering quotients of these two spaces. The identifications being made in the quotienting are also analytic continuations of each other, as seen in the previous subsection. We therefore expect that the expressions for the heat kernel on the thermal quotient of S^3 described in section 4 should be analytically continued as well. However, it should be pointed out that the group $\Gamma \cong \mathbb{Z}_N$ generated by γ in the S^3 case is finite in order for the identifications to make global sense. There is no such constraint in the case of the identifications on H_3^+, and therefore the group is just $\mathbb{Z}$. This difference however only plays a role when taking into account the sum over the images to obtain the full heat kernel: in the thermal S^3 case (4.6) is a finite sum, while the corresponding sum for H_3^+ (see below) will involve an infinite sum over m.

However, this is a global aspect of the quotienting which we expect to be irrelevant to the analytic continuation of a particular image point to the heat kernel. Indeed, the analysis of section 4.1 was essentially algebraic, and thus can be equally applied for the case of H_3^+. There we had written the expressions in terms of group integrals and as traces over the appropriate SU(2) representations. These group theoretic operations carry over into the noncompact case though care should be taken in the group integrals and definitions of the trace. This is normally accomplished through the various ingredients of the harmonic analysis on the noncompact groups that we have mentioned so far. The additional feature we need to use in our analytic continuation of the results of section 4.1 is the trace. For a noncompact group one can define what is called the the Harish-Chandra (or global) character which is defined as a distributional analogue of the usual trace. In the case of $\mathrm{SL}(2,\mathbb{C})$ this has been worked out and will be explained more explicitly below.

Using these ingredients we will assume the analysis of section 4.1 can be carried through in an identical fashion for $\mathrm{SL}(2,\mathbb{C})$; in the following we shall consider, for ease of notation, the case $s>0$ — the calculation for $s=0$ is almost identical. Instead of the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ character given in (4.20) we now end up with a character of the $\mathrm{SL}(2,\mathbb{C})$ element $M = \mathrm{diag}(e^{\frac{i\tau}{2}}, e^{\frac{-i\tau}{2}})$. The $\mathrm{SL}(2,\mathbb{C})$ character for an element with diagonal entries (α,α^{-1}) is given by (see e.g. [19, p. 100] or [20, p. 117] — note that there is a typo in [20])

$$\chi_{(j_1,j_2)}(\alpha) = \frac{\alpha^{2j_1+1}\bar{\alpha}^{2j_2+1} + \alpha^{-2j_1-1}\bar{\alpha}^{-2j_2-1}}{|\alpha-\alpha^{-1}|^2} . \tag{6.4}$$

Thus the final answer for the integrated heat kernel for the case of thermal AdS_3 takes the form (cf. (4.20))

$$K^{(s)}(\tau,\bar{\tau};t) = 2\cdot\frac{\tau_2}{2\pi}\sum_{m\in\mathbb{Z}}\int_0^\infty d\lambda\,\chi_{\lambda,s}(e^{\frac{im\tau}{2}})\,e^{-t(\lambda^2+s+1)} \tag{6.5}$$

with

$$\chi_{\lambda,s}(e^{\frac{im\tau}{2}}) = \frac{1}{2}\frac{\cos(ms\tau_1 - m\lambda\tau_2)}{|\sin\frac{m\tau}{2}|^2} , \tag{6.6}$$

which is just the character of M evaluated for $j_1 = \frac{1}{2}(s-1+i\lambda)$ and $j_2 = \frac{1}{2}(-s-1+i\lambda)$. Since $s>0$ we also have to consider the contribution where the roles of j_1 and j_2 are interchanged, and this is responsible for the overall factor of 2 in (6.5). For fixed m the integral over λ of

$$\frac{\tau_2}{2\pi\,|\sin\frac{m\tau}{2}|^2}\int_0^\infty d\lambda\,\cos(ms\tau_1 - m\lambda\tau_2)e^{-t(\lambda^2+s+1)} \tag{6.7}$$

can be peformed by Gaussian integration, and we obtain

$$\frac{\tau_2}{4\sqrt{\pi t}|\sin\frac{m\tau}{2}|^2}\cos(ms\tau_1)e^{-\frac{m^2\tau_2^2}{4t}}\,e^{-(s+1)t} . \tag{6.8}$$

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The term with $m = 0$ diverges; it describes the integrated heat kernel on H_3^+ since for $m = 0$ the two points $y = \gamma^m(x) = x$ and x coincide. The divergence is then simply a consequence of the infinite volume of H_3^+. In any case, the contribution with $m = 0$ is independent of τ, and therefore not of primary interest to us. Subtracting it out, the final result is then

$$K^{(s)}(\tau,\bar{\tau};t) = \sum_{m=1}^{\infty} \frac{\tau_2}{\sqrt{4\pi t}|\sin\frac{m\tau}{2}|^2} \cos(sm\tau_1) e^{-\frac{m^2\tau_2^2}{4t}} e^{-(s+1)t} \,. \tag{6.9}$$

This is the central result of the paper which we shall use extensively below.

For the case $s = 1$, (6.9) gives exactly the answer of [3] for the transverse components as given in their eqs. (4.16) and (4.17). (Note that their $2\pi\tau$ is our τ; furthermore the relative factor e^{2t} comes from the curvature contribution in their eq. (2.15).) For the case of $s = 2$, while the contribution from the transverse components is not separately considered in [3], it can be inferred from their result eq. (4.25) (together with eq. (4.22)). In fact the first term in their eq. (4.25) is exactly equal to (6.9) with $s = 2$ (again up to a relative factor of e^{2t} coming from the curvature contribution). In the next section we also check that the correct one loop graviton determinant is reproduced by this result.

The expression (6.9) for the case of $s = 0$ and $s = 1$ is of the form given by the Selberg trace formula for scalars and transverse vectors. In fact, the heat kernel for these cases were written down in [21] using the Selberg trace formula — see their eqs. (B.1) and (B.2). (A general reference for the trace formula in this context is [22], section 3.4, see also [23, 24]). The trace formula essentially gives a path integral like interpretation to the heat kernel answer. To summarize the salient points we note that the sum over m is a sum over closed paths of non-zero winding number m and of length $m\tau_2$ weighted with a classical action $\frac{m^2\tau_2^2}{4t}$. The denominator in (6.9) is proportional to $|1 - q^m|^2$ (with $q = e^{i\tau}$). This is the semiclassical (or van-Vleck) determinant. Finally, from the explicit form of the $s = 1$ case quoted in eq. (B.1) of [21], one interprets the $\cos m\tau_1$ piece of (6.9) as a monodromy term. This suggests that the general spin s answer given by us here can be understood in terms of a general Selberg trace formula for symmetric traceless tensors of rank s. We should like to mention though that the Selberg trace formula is generally applied to quotients of H_3^+ of finite volume. In such cases there is an additional finite piece coming from the $m = 0$ (or 'direct') term. As mentioned earlier, for the thermal quotient this is a trivial (q independent) volume divergence.

7 Partition function of $\mathcal{N} = 1$ supergravity

As an interesting application of the formalism we have developed in the previous sections we can now evaluate the one loop partition function of $\mathcal{N} = 1$ supergravity in thermal H_3^+ and explicitly check the argument of Maloney and Witten [1]. We will, in the process, also derive the expressions for the one loop determinant in the bosonic (pure gravity) sector reproducing the results of the check of [3].

The field content of $\mathcal{N} = 1$ supergravity consists of the graviton of spin $s = 2$, and the Majorana gravitino of spin $s = 3/2$. The complete one loop partition function of $\mathcal{N} = 1$

supergravity is therefore the product of the graviton and gravitino contribution

$$Z_{1-\text{loop}} = Z_{1-\text{loop}}^{\text{graviton}} \cdot Z_{1-\text{loop}}^{\text{gravitino}} . \tag{7.1}$$

The calculation of the two contributions will be described in detail below, first for the graviton (section 7.1), and then for the gravitino (section 7.2). In each case we can reduce the calculation of the one loop partition function to determinants of the form $\det(-\Delta_{(s)} + m_s^2)$, where $\Delta_{(s)}$ denotes an appropriate spin s Laplacian, while m_s is a mass shift. In turn these determinants can be easily deduced from the heat kernel since we have

$$-\log\det(-\Delta_{(s)} + m_s^2) = \int_0^\infty \frac{dt}{t} K^{(s)}(\tau,\bar{\tau};t)\, e^{-m_s^2 t} , \tag{7.2}$$

where $K^{(s)}$ is the spin s heat kernel that was determined above (6.9). Thus the knowledge of the heat kernel allows us to calculate the one loop partition functions fairly directly.

7.1 The one loop determinant for the graviton

The one loop contribution of the graviton to the effective action has been evaluated by several authors [25–27]. Including the gauge fixing terms and the ghosts, the one loop partition function for the graviton in D spacetime dimensions is given by [27]

$$Z_{1-\text{loop}}^{\text{graviton}} = \det{}^{-1/2}(\Delta_{(2)}^{\text{LL}} - 2R/D) \cdot \det{}^{1/2}(\Delta_{(1)}^{\text{LL}} - 2R/D) , \tag{7.3}$$

where $\Delta_{(2)}^{\text{LL}}$ and $\Delta_{(1)}^{\text{LL}}$ denote the Lichnerowicz Laplacians on rank 2 symmetric traceless and vectors, respectively, while R is the scalar curvature. For H_3^+ the curvature tensors, in units of the radius of AdS$_3$, are

$$R_{\mu\rho\nu\sigma} = \frac{R}{6}\,(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\sigma}g_{\nu\rho}) , \qquad R_{\mu\nu} = \frac{R}{3}\, g_{\mu\nu} , \qquad R = -6 . \tag{7.4}$$

Note that the convention for the scalar curvature used in [27] differs by a sign from the above (conventional) definition.

To convert the Lichnerowicz Laplacian to the ordinary Laplacian we use the relations [25]

$$\begin{aligned} \Delta_{(2)}^{\text{LL}}\, T_{\mu\nu} &= -\Delta_{(2)} T_{\mu\nu} - 2R_{\mu\rho\nu\sigma}T^{\rho\sigma} + R_{\mu\rho}T^{\rho}{}_{\nu} + R_{\nu\rho}T_{\mu}{}^{\rho} \\ \Delta_{(1)}^{\text{LL}}\, T_{\mu} &= -\Delta_{(1)} T_{\mu} + R_{\mu\rho}T^{\rho} , \end{aligned} \tag{7.5}$$

where $T_{\mu\nu}$ and T_μ are arbitrary symmetric traceless tensors and vectors, respectively. Using (7.4) we then find

$$\begin{aligned} (\Delta_{(2)}^{\text{LL}} - 2R/D)\, T_{\mu\nu} &= (-\Delta_{(2)} - 2)\, T_{\mu\nu} \\ (\Delta_{(1)}^{\text{LL}} - 2R/D)\, T_{\mu} &= (-\Delta_{(1)} + 2)\, T_{\mu} . \end{aligned} \tag{7.6}$$

Thus the one loop partition function of the graviton is given by

$$Z_{1-\text{loop}}^{\text{graviton}} = \det{}^{-1/2}(-\Delta_{(2)} - 2) \cdot \det{}^{1/2}(-\Delta_{(1)} + 2) , \tag{7.7}$$

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which can be directly evaluated in terms of the heat kernel. In fact, using (7.2) we simply have

$$\begin{aligned}\log Z^{\text{graviton}}_{1-\text{loop}} &= -\frac{1}{2}\log(\det(-\Delta_{(2)}-2)) + \frac{1}{2}\log(\det(-\Delta_{(1)}+2)) \\ &= \frac{1}{2}\int_0^\infty \frac{dt}{t}\left(K^{(2)}(\tau,\bar{\tau};t)\,e^{2t} - K^{(1)}(\tau,\bar{\tau};t)\,e^{-2t}\right) .\end{aligned} \tag{7.8}$$

Using the expression (6.9) for the heat kernel, and performing the t-integral with the help of

$$\frac{1}{4\pi^{1/2}}\int_0^\infty \frac{dt}{t^{3/2}} e^{-\frac{\alpha^2}{4t}-\beta^2 t} = \frac{1}{2\alpha}e^{-\alpha\beta}, \tag{7.9}$$

we then obtain

$$\begin{aligned}\log Z^{\text{graviton}}_{1-\text{loop}} &= \frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m|\sin\frac{m\tau}{2}|^2}\left(\cos(2m\tau_1)e^{-m\tau_2} - \cos(m\tau_1)e^{-2m\tau_2}\right) \\ &= \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{q^{2m}}{1-q^m}+\frac{\bar{q}^{2m}}{1-\bar{q}^m}\right) = -\sum_{n=2}^{\infty}\log|1-q^n|^2 ,\end{aligned} \tag{7.10}$$

where $q = \exp(i\tau)$, and in the last line we have expanded out the geometric series. Thus the one loop gravity partition function is given by

$$Z^{\text{graviton}}_{1-\text{loop}} = \prod_{n=2}^{\infty}\frac{1}{|1-q^n|^2} . \tag{7.11}$$

This was argued to be the result for pure gravity in [1] by a quantum extension of the argument of Brown and Henneaux [2]. It also reproduces precisely the calculation of [3]. Including the tree level contribution $|q|^{-2k}$, the total one loop gravity partition function is just the product of a left- and a right-moving Virasoro vacuum representation at $c = \bar{c} = 24k$ [1]. Since there are no bulk propagating states in $3d$ gravity, the perturbative partition function simply counts the contributions of the so-called boundary Brown-Henneaux states which are obtained by acting on the $\mathrm{SL}(2,\mathbb{C})$ invariant vacuum by the Virasoro generators L_{-n} (with $n \geq 2$).

7.2 One loop determinant for the gravitino

The calculation for the one loop gravitino partition function is slightly more complicated. The gravitino that is of relevance to us is a Majorana gravitino, but it is actually easier to study first the case of a Dirac gravitino. Its action is given by [28]

$$S = -\int d^3z\sqrt{g}\,\bar{\psi}_\mu(\Gamma^{\mu\nu\rho}D_\nu\psi_\rho + \hat{m}\,\Gamma^{\mu\nu})\psi_\nu . \tag{7.12}$$

Here Γ^μ are defined as $\Gamma^\mu = \gamma^a e^\mu_a$ with e^μ_a being the vielbeins, and

$$\gamma_0 = \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} . \tag{7.13}$$

The Γ-matrices satisfy the usual Clifford algebra, $\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu}$, and we define

$$\begin{aligned}\Gamma^{\mu\nu} &= \frac{1}{2}(\Gamma^\mu\Gamma^\nu - \Gamma^\nu\Gamma^\mu) \\ \Gamma^{\mu\nu\rho} &= \frac{1}{3!}(\Gamma^\mu\Gamma^\nu\Gamma^\rho - \Gamma^\nu\Gamma^\mu\Gamma^\rho + \text{ cyclic}) \ .\end{aligned} \tag{7.14}$$

Furthermore the covariant derivative is given by

$$D_\mu \psi_\nu = \partial_\mu \psi_\nu + \frac{1}{8}\omega_\mu^{ab}[\gamma_a, \gamma_b]\psi_\nu - \tilde{\Gamma}^\rho_{\mu\nu}\psi_\rho \, , \tag{7.15}$$

where $\tilde{\Gamma}^\rho_{\mu\nu}$ are the Christoffel symbols, while ω_μ^{ab} refers to the spin connection. For a massless gravitino $\hat{m}$ is related to the radius of AdS$_3$ by

$$\hat{m}^2 = \frac{1}{4} \ . \tag{7.16}$$

The gravitino Lagrangian has the gauge symmetry

$$\delta\psi_\mu = D_\mu \epsilon - \hat{m}\Gamma_\mu \epsilon \, , \tag{7.17}$$

and thus we need to worry about isolating the gauge invariant degrees of freedom. To do so we shall fix a gauge and use the Fadeev-Popov method, following [29]. To start with we remove from ψ_μ the gauge trivial part

$$\psi_\mu = \varphi_\mu + \frac{\Gamma_\mu}{3}\psi \, , \tag{7.18}$$

where $\Gamma^\mu \varphi_\mu = 0$ and $\psi = \Gamma^\mu \psi_\mu$. The remaining field φ_μ we then further decompose as

$$\varphi_\mu = \varphi_\mu^\perp + \left(D_\mu - \frac{1}{3}\Gamma_\mu \hat{D}\right)\xi \, , \qquad \text{where} \qquad D^\mu \varphi^\perp_\mu = \Gamma^\mu \varphi^\perp_\mu = 0 \ . \tag{7.19}$$

Here $\hat{D} = \Gamma^\mu D_\mu$, and $D_\mu \psi$ is defined by

$$D_\mu \psi = \partial_\mu \psi + \frac{1}{8}\omega_\mu^{ab}[\gamma_a, \gamma_b]\psi \ . \tag{7.20}$$

With respect to this decomposition the gravitino Lagrangian (7.12) then becomes (the details are described in appendix D)

$$\begin{aligned} S = -\int d^3 z \sqrt{g} \Big(& \bar{\varphi}^{\perp\mu}(\hat{D} - \hat{m})\varphi^\perp_\mu - \frac{2}{9}\bar{\xi}(\hat{D} - 3\hat{m})[\Delta_{(1/2)} - 3/4]\xi \\ & + \frac{2}{9}\bar{\xi}\,[\Delta_{(1/2)} - 3/4]\,\psi - \frac{2}{9}\bar{\psi}\,[\Delta_{(1/2)} - 3/4]\,\xi + \frac{2}{9}\bar{\psi}\,(\hat{D} - 3\hat{m})\,\psi \Big) \ . \end{aligned} \tag{7.21}$$

Furthermore, the change in the measure is equal to [29]

$$\mathcal{D}\phi_\mu = \mathcal{D}\varphi^\perp_\mu \, \mathcal{D}\xi \, \mathcal{D}\psi \, \det{}^{-2}\left[\Delta_{(1/2)} - 3/4\right] \, , \tag{7.22}$$

where the power of -2 comes from the fact that we are dealing with a two-component Dirac fermion. It follows from (7.17) that the components transform under a gauge transformation as

$$\delta\varphi^\perp_\mu = 0 \, , \quad \delta\xi = \epsilon \, , \quad \delta\psi = (\hat{D} - 3\hat{m})\epsilon \ . \tag{7.23}$$

In particular, we can therefore fix the gauge $\psi = 0$, for which the corresponding Fadeev-Popov determinant is

$$\Delta_{\rm FP} = \det{}^{-2}(\hat{D} - 3\hat{m}) \ . \tag{7.24}$$

To perform the one loop integration we also need to add a gauge fixing term in the action in (7.21). This is done by treating $\hat{m}$ as an independent variable not given by the relation (7.16) in the intermediate steps of the one loop integration; this amounts to adding an explicit gauge fixing term [29]. After performing the integration over $\varphi^{\perp}_{\mu}$, ξ, and ψ we then obtain the one loop determinant

$$\begin{aligned} Z^{\rm Dirac}_{1-{\rm loop}} &= \det{}^{-2}\left[\Delta_{(1/2)} - 3/4\right] \det{}^{-2}(\hat{D} - 3\hat{m}) \\ &\times \det{}^{2}(\hat{D} - \hat{m})_{\varphi^\perp} \det{}^{2}(\hat{D} - 3\hat{m})_{\xi} \det{}^{2}\left[\Delta_{(1/2)} - 3/4\right]_{\xi} \det{}^{-2}(\hat{D} - 3\hat{m})_{\psi} \,, \end{aligned} \tag{7.25}$$

where the first line arise from the change in the measure and the Fadeev-Popov determinant, while the terms in the second line come from integrating out $\varphi^{\perp}$, ξ and ψ, as indicated by the suffices. Simplifying and taking the square of the operators in the determinants then leads to (see eq. (D.19) and (D.20))

$$Z^{\rm Dirac}_{1-{\rm loop}} = \frac{\det{}^{2}(\hat{D} - \hat{m})_{(3/2)}}{\det{}^{2}(\hat{D} - 3\hat{m})_{(1/2)}} = \frac{\det(-\Delta_{(3/2)} - \frac{9}{4})}{\det(-\Delta_{(1/2)} + \frac{3}{4})} \ . \tag{7.26}$$

The actual one loop determinant for the Majorana gravitino that appears in $\mathcal{N} = 1$ supergravity is the square root of (7.26), i.e.

$$Z^{\rm gravitino}_{1-{\rm loop}} = \left(\frac{\det(-\Delta_{(3/2)} - \frac{9}{4})}{\det(-\Delta_{(1/2)} + \frac{3}{4})}\right)^{1/2} \,, \tag{7.27}$$

and its logarithm is hence given by

$$\begin{aligned} \log Z^{\rm gravitino}_{1-{\rm loop}} &= \frac{1}{2}\log(\det(-\Delta_{(3/2)} - 9/4)) - \frac{1}{2}\log(\det(-\Delta_{(1/2)} + 3/4)) \\ &= -\frac{1}{2}\int_0^\infty \frac{dt}{t}\left(\hat{K}^{(3/2)}(\tau,\bar{\tau};t)\, e^{\frac{9}{4}t} - K^{(1/2)}(\tau,\bar{\tau};t)\, e^{-\frac{3}{4}t}\right) \ . \end{aligned} \tag{7.28}$$

Since we are dealing with fermions of spin $s = \frac{1}{2}$ and $s = \frac{3}{2}$, the heat kernels $K^{(1/2)}(\tau,\bar{\tau};t)$ and $K^{(3/2)}(\tau,\bar{\tau};t)$ that appear here differ slightly from (6.9). Indeed, for the thermal partition function one has to impose antiperiodic boundary conditions for the fermions along the thermal circle. In our heat kernel calculation we have summed over the images (labelled by m) that describe the contribution from wrapping the thermal circle m times. Thus for fermions we need to introduce an additional factor of $(-1)^m$. With this modification, and after performing the t-integral with the help of (7.9) we then obtain

$$\begin{aligned} \log Z^{\rm gravitino}_{1-{\rm loop}} &= -\frac{1}{2}\sum_{m=1}^{\infty}\frac{(-1)^m}{m|\sin\frac{m\tau}{2}|^2}\left[\cos\left(\tfrac{3}{2}m\tau_1\right)e^{-\frac{m\tau_2}{2}} - \cos\left(\tfrac{m}{2}\tau_1\right)e^{-\frac{3m\tau_2}{2}}\right] \\ &= -\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\left[\frac{q^{\frac{3m}{2}}}{1-q^m} + \frac{\bar{q}^{\frac{3m}{2}}}{1-\bar{q}^m}\right] = \sum_{n=1}^{\infty}\log|1 + q^{n+\frac{1}{2}}|^2 \,, \end{aligned} \tag{7.29}$$

where the sum over n comes again from the geometric series. Thus the partition function of the $\mathcal{N}=1$ gravitino is given by

$$Z_{1-\text{loop}}^{\text{gravitino}} = \prod_{n=1}^{\infty} |1+q^{n+\frac{1}{2}}|^2 \,. \tag{7.30}$$

Together with (7.11) and the tree level contribution this then gives

$$Z_{\text{combined}} = |q|^{-2k} \prod_{n=2}^{\infty} \frac{|1+q^{n-\frac{1}{2}}|^2}{|1-q^n|^2} \,, \tag{7.31}$$

where the factor $|q|^{-2k}$ is the contribution of the tree level partition function. This partition function has indeed the form of a trace

$$Z = \text{Tr}\Big(q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\Big) \tag{7.32}$$

over the irreducible vacuum representation of the $\mathcal{N}=1$ super Virasoro algebra at $c=\bar{c}=24k$, as argued on the basis of a quantum Brown-Henneaux reasoning in [1].

Incidentally, if we impose instead periodic boundary conditions for the fermions along the thermal circle, we would obtain (7.29) without the factor of $(-1)^m$. Performing the same steps as above this would then lead to

$$Z'_{\text{combined}} = |q|^{-2k} \prod_{n=2}^{\infty} \frac{|1-q^{n-\frac{1}{2}}|^2}{|1-q^n|^2} = \text{Tr}\Big((-1)^F q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\Big) \,, \tag{7.33}$$

which corresponds, as expected, to the introduction of a $(-1)^F$ factor in the dual conformal field theory partition function.

8 Final remarks

We have seen how the heat kernel (and therefore the one loop determinants) for arbitrary spin s fields on (thermal) AdS_3 can be obtained in a group theoretic way. The simplicity of the final answer (6.9), expressed in terms of characters of $\text{SL}(2,\mathbb{C})$ (see (6.5)), is a reflection of the underlying symmetry of the spacetime. It is interesting to observe that the computation of the one loop (super)gravity answers of section 7 essentially assembles these $\text{SL}(2,\mathbb{C})$ characters into a (super) Virasoro character, where the $\text{SL}(2,\mathbb{C})$ is the global part of the asymptotic isometry group given by the two copies of the Virasoro algebra. We therefore believe there is useful insight to be gained by viewing the one loop heat kernel answers in this group theoretic way.

Amongst the potential applications of the results given here are checks of the conjectures made in [30] for the one loop behaviour of chiral or log gravity. An explicit calculation of the one loop fluctuations of the chiral (log) gravity action should be amenable to a similar analysis.

Moving further onto more nontrivial theories of gravity, the heat kernel can be expected to play a useful role in a better understanding of one loop string theory on AdS_3 [31]. This was, in fact, one of the prime motivations for this work. One expects the one loop string computation to be assembled as a sum of heat kernel contributions of different spin (and mass). The exact answer of [31] does actually reflect this property. These and related matters are currently under investigation [32], and we hope to report on them soon.

Finally, the considerations of this paper can be generalized, using a similar group theoretic approach, to higher dimensional AdS spacetimes (and their quotients). Once again, this is likely to be useful in the investigation of the one loop quantum string/M dynamics on these spacetimes. Another case of interest is AdS_2 where the methods of this paper could be useful in evaluating Sen's quantum entropy function (see, for instance, [33–35]).

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A Change of sections as change of basis

In this section we show that the tensor harmonics are completely independent of the choice of the section. A different choice of section just results in a different choice of the basis in which the tensor harmonics are expressed. We demonstrate this by evaluating the tensor harmonics given in (2.19) for the section $\hat{\sigma}$, where $\hat{\sigma}$ is defined via (2.9). Instead of (2.19) we obtain

$$\begin{aligned}
\hat{\Psi}_a^{(n;m_1,m_2)}(g) &= \sum_{p_1,p_2} \langle s,a|\frac{n}{2}+s,p_1;\frac{n}{2},p_2\rangle\, D^{(\frac{n}{2}+s)}_{p_1,m_1}(h^{-1}\cdot g_L^{-1})\, D^{(\frac{n}{2})}_{p_2,m_2}(h^{-1}\cdot g_R^{-1}) \\
&= \sum_{p_1,p_2} \langle s,a|\frac{n}{2}+s,p_1;\frac{n}{2},p_2\rangle \\
&\quad \times \sum_{q_1,q_2} D^{(\frac{n}{2}+s)}_{p_1,q_1}(h^{-1})\, D^{(\frac{n}{2})}_{p_2,q_2}(h^{-1})\, D^{(\frac{n}{2}+s)}_{q_1,m_1}(g_L^{-1})\, D^{(\frac{n}{2})}_{q_2,m_2}(g_R^{-1})\ .
\end{aligned}$$

Next we observe that

$$\sum_{p_1,p_2} \langle s,a|\frac{n}{2}+s,p_1;\frac{n}{2},p_2\rangle\, D^{(\frac{n}{2}+s)}_{p_1,q_1}(h^{-1})\, D^{(\frac{n}{2})}_{p_2,q_2}(h^{-1}) = \sum_b D^{(s)}_{ab}(h^{-1})\langle s,b|\frac{n}{2}+s,q_1;\frac{n}{2},q_2\rangle \tag{A.1}$$

since the Clebsch-Gordon coefficients describe the decomposition of the tensor product into the spin s representation. Thus we obtain

$$\begin{aligned}\hat{\Psi}_a^{(n;m_1,m_2)}(g) &= \sum_b D_{ab}^{(s)}(h^{-1}) \sum_{q_1,q_2} \langle s,b|\frac{n}{2}+s,q_1;\frac{n}{2},q_2\rangle\, D_{q_1,m_1}^{(\frac{n}{2})}(g_L^{-1})\, D_{q_2,m_2}^{(\frac{n}{2})}(g_R^{-1}) \\ &= \sum_b D_{ab}^{(s)}(h^{-1})\, \Psi_b^{(n;m_1,m_2)}(g)\ . \end{aligned} \tag{A.2}$$

On the other hand, the basis (2.10) with respect to which this tensor harmonic is defined also changes as we change the section. In fact, it follows directly from (2.10) that

$$\hat{\theta}_a(x) = \sum_b \sigma(x) D_{ab}^{(s)}(h)\, \mathbf{v}_b = \sum_b D_{ab}^{(s)}(h)\, \theta_b(x)\ . \tag{A.3}$$

This basis thus transforms precisely in the opposite way to the tensor harmonics, so that

$$\sum_a \hat{\Psi}_a\, \hat{\theta}_a = \sum_a \Psi_a\, \theta_a\ . \tag{A.4}$$

Thus the actual tensor harmonic is completely independent of the choice of the section, as had to be the case.

B Vielbeins for the thermal section

In this section, we will obtain the vielbein for the thermal section using the two different coordinates (2.5) and (2.7). For the case of $G = \mathrm{SU}(2)$, a natural basis for the tangent space at the identity of $\mathrm{SU}(2)\times\mathrm{SU}(2)/\mathrm{SU}(2)$ is given by $\mathbf{T}_a = (T_a, -T_a)$, $a = 1,2,3$, where

$$T_1 = i\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\ , \qquad T_2 = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ , \qquad T_3 = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ . \tag{B.1}$$

In the coordinates (2.5) the thermal section is given by (2.26) and (2.27). The tangent vector $\sigma(g)(T_a, -T_a)$ describes the variation

$$g_L \mapsto \tilde{g}_L = g_L + \epsilon g_L\, T_a \tag{B.2}$$

$$g_R \mapsto \tilde{g}_R = g_R - \epsilon g_R\, T_a \tag{B.3}$$

and this leads to

$$\begin{aligned}\tilde{g}_L\, \tilde{g}_R^{-1} &= (g_L + \epsilon g_L\, T_a)\cdot(g_R^{-1} + \epsilon T_a g_R^{-1}) \\ &= g_L\cdot g_R^{-1} + \epsilon g_L\, T_a\, g_R^{-1} + \epsilon g_L\, T_a\, g_R^{-1} + \mathcal{O}(\epsilon^2)\ . \end{aligned} \tag{B.4}$$

Hence the corresponding tangent vector for $(G\times G)/G$ is simply

$$\delta g = g_L\, T_a\, g_R^{-1}\ . \tag{B.5}$$

For the above section one then finds

$$\begin{aligned} g_L(\chi,\theta,\phi)\, T_3\, g_R(\chi,\theta,\phi)^{-1} &= i\begin{pmatrix} e^{i\chi}\cos^2\frac{\theta}{2} - e^{-i\chi}\sin^2\frac{\theta}{2} & \cos\chi\,\sin\theta\, e^{i\phi} \\ \cos\chi\,\sin\theta\, e^{-i\phi} & e^{i\chi}\sin^2\frac{\theta}{2} - e^{-i\chi}\cos^2\frac{\theta}{2} \end{pmatrix} \\ &= \partial_\chi\, g \end{aligned} \tag{B.6}$$

as well as

$$g_L(\chi,\theta,\phi)\, T_2\, g_R(\chi,\theta,\phi)^{-1} = i \begin{pmatrix} -\sin\theta & \cos\theta\, e^{i\phi} \\ \cos\theta\, e^{-i\phi} & \sin\theta \end{pmatrix} = \frac{1}{\sin\chi}\,\partial_\theta\, g\,, \tag{B.7}$$

and

$$g_L(\chi,\theta,\phi)\, T_1\, g_R(\chi,\theta,\phi)^{-1} = \begin{pmatrix} 0 & -e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix} = \frac{1}{\sin\chi\,\sin\theta}\,\partial_\phi\, g\ . \tag{B.8}$$

Thus the corresponding vielbein is the standard vielbein defined by

$$\mathbf{e}_3 = \partial_\chi\,, \qquad \mathbf{e}_2 = \frac{1}{\sin\chi}\partial_\theta\,, \qquad \mathbf{e}_1 = \frac{1}{\sin\chi\,\sin\theta}\partial_\phi\ . \tag{B.9}$$

In the double polar coordinates (2.7) the thermal section is given by (2.30) and (2.31). The same arguments as above then imply that the corresponding vielbein is

$$g_L(\psi,\eta,\varphi)\, T_1\, g_R(\psi,\eta,\varphi)^{-1} = \begin{pmatrix} 0 & -e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix} = \frac{1}{\sin\psi}\,\partial_\varphi\, g\,, \tag{B.10}$$

$$g_L(\psi,\eta,\varphi)\, T_2\, g_R(\psi,\eta,\varphi)^{-1} = \begin{pmatrix} -e^{-i\eta}\sin\psi & ie^{i\varphi}\cos\psi \\ ie^{-i\varphi}\cos\psi & -e^{i\eta}\sin\psi \end{pmatrix} = \partial_\psi\, g\,, \tag{B.11}$$

and

$$g(\psi,\eta,\varphi)_L\, T_3\, g_R(\psi,\eta,\varphi)^{-1} = i \begin{pmatrix} e^{-i\eta} & 0 \\ 0 & e^{i\eta} \end{pmatrix} = -\frac{1}{\cos\psi}\,\partial_\eta\, g\,, \tag{B.12}$$

leading to

$$\mathbf{e}_1 = \frac{1}{\sin\psi}\,\partial_\varphi\,, \qquad \mathbf{e}_2 = \partial_\psi\,, \qquad \mathbf{e}_3 = -\frac{1}{\cos\psi}\,\partial_\eta\ . \tag{B.13}$$

C Evaluation of the radial heat kernel on S^3

To evaluate (3.16) it is convenient to write $l_1 = \frac{\hat{n}}{2} \pm \frac{s}{2}$, $l_2 = \frac{\hat{n}}{2} \mp \frac{s}{2}$, where $\hat{n} = n+s$. Then the Racah formula for the Clebsch-Gordan coefficent appearing in (3.16) is particularly simple

$$\begin{aligned} |\langle \frac{\hat{n}-s}{2}, k; \frac{\hat{n}+s}{2}, -k+a | s, a\rangle|^2 &= |\langle \frac{\hat{n}+s}{2}, -k+a; \frac{\hat{n}-s}{2}, k | s, a\rangle|^2 \\ &= \left[\frac{(\hat{n}-s)!(2s+1)!}{(\hat{n}+s+1)!}\right] \times \frac{(\frac{\hat{n}+s}{2}-k+a)!(\frac{\hat{n}+s}{2}+k-a)!}{(\frac{\hat{n}-s}{2}-k)!(\frac{\hat{n}-s}{2}+k)!(s+a)!(s-a)!}\ . \end{aligned} \tag{C.1}$$

The sum we need to carry out — we are suppressing for the moment the k-independent bracket $[\cdot]$ in (C.1), as well as $a_{\hat{n}}^{(s)} e^{E_{\hat{n}}^{(s)} t}$ — is

$$\begin{aligned} K_{a;n}^{(s)}(\chi) = \frac{1}{(s+a)!(s-a)!} &\sum_{k=-\frac{\hat{n}-s}{2}}^{\frac{\hat{n}-s}{2}} \frac{(\frac{\hat{n}+s}{2}-k+a)!(\frac{\hat{n}+s}{2}+k-a)!}{(\frac{\hat{n}-s}{2}-k)!(\frac{\hat{n}-s}{2}+k)!} \\ &\times \left(e^{i(2k-a)\chi} + e^{-i(2k-a)\chi}\right)\,, \end{aligned} \tag{C.2}$$

where the two terms in the last line come from the two different choices $l_1 = \frac{\hat{n}}{2} \pm \frac{s}{2}$ and $l_2 = \frac{\hat{n}}{2} \mp \frac{s}{2}$.(We are assuming here that $s > 0$ — for $s = 0$ the second term is not present.) Note that this expression is symmetric under $a \mapsto -a$, since this can be absorbed into relabelling $k \mapsto -k$. We may therefore, without loss of generality, restrict ourselves to $a \geq 0$.

Putting $p = k + \frac{\hat{n}-s}{2}$, the first exponential in (C.2) becomes

$$z^{-a} \sum_{p=0}^{\hat{n}-s} \frac{(p+s-a)!}{p!(s-a)!} \frac{(\hat{n}-p+a)!}{(\hat{n}-s-p)!(s+a)!} z^{(2p-\hat{n}+s)} , \tag{C.3}$$

where we have written $z = e^{i\chi}$. To evaluate this sum let us define the generating function

$$F_{s,a}(w,z) = \left[\sum_{p=0}^{\infty} \frac{(p+s-a)!}{p!(s-a)!} (wz)^p \right] \times \left[\sum_{q=0}^{\infty} \frac{(q+s+a)!}{q!(s+a)!} (wz^{-1})^q \right] , \tag{C.4}$$

whose $w^{\hat{n}-s}$ coefficient is precisely the sum in (C.3) (without the prefactor of z^{-a}). The sums in (C.4) can be worked out straightforwardly, and we obtain

$$\begin{aligned} F_{s,a}(w,z) &= \frac{1}{(1-wz)^{s-a+1}} \frac{1}{(1-wz^{-1})^{s+a+1}} = \frac{1}{[(1-wz)(1-wz^{-1})]^{s+a+1}} (1-wz)^{2a} \\ &= \frac{1}{(1-2w\cos\chi + w^2)^{s+a+1}} (1-wz)^{2a} . \end{aligned} \tag{C.5}$$

The first term in $F_{s,a}(w,z)$ is precisely the generating function for the Gegenbauer polynomials

$$\frac{1}{(1-2w\cos\chi+w^2)^{\lambda}} = \sum_{p=0}^{\infty} C_p^{\lambda}(\cos\chi)\, w^p \tag{C.6}$$

(see 8.930 of [16]), and thus we find for (C.2)

$$K_{a;n}^{(s)}(\chi) = (2-\delta_{s,0}) \sum_{r=0}^{min(2a,n)} (-1)^r \frac{(2a)!}{r!(2a-r)!} \cos[(r-a)\chi]\, C_{n-r}^{s+a+1}(\cos\chi) , \tag{C.7}$$

where we have now restored the z^{-a} term from (C.3) and included the second exponential in (C.2), i.e. added in the term with $\chi \mapsto -\chi$. (For prefactor $(2-\delta_{s,0})$ guarantees that the result is also correct for $s = 0$.) In addition we have used that $\hat{n}-s = n$. We note in passing that for $a = 0$ this simplifies to $K_{0;n}^{(s)}(\chi) = (2-\delta_{s,0})\, C_n^{s+1}(\cos\chi)$. We also remind the reader that this expression is only valid for $a \geq 0$, and that $K_{a;n}^{(s)}(\chi)$ is invariant under $a \mapsto -a$.

Including the prefactors that were left out in going to (C.2) we then obtain for (3.16)

$$K_a^{(s)}(\chi,t) = \frac{1}{2\pi^2} \sum_{n=0}^{\infty} \frac{(n+1)!(2s)!}{(n+2s)!} K_{a;n}^{(s)}(\chi)\, e^{-((n+s)(n+s+2)-s)t} . \tag{C.8}$$

In the scalar case, $s = 0$, we have $a = 0$, and the formula agrees with (3.13) since the first Gegenbauer polynomial simply equals

$$C_n^1(\cos\chi) = \frac{\sin(n+1)\chi}{\sin\chi} . \tag{C.9}$$

D Gravitino action

In this appendix we provide the details for the derivation of the action (7.21). We start with the gravitino Lagrangian (7.12), and express ψ in terms of $\varphi^\perp$, ξ, and ψ, using (7.18) and (7.19). The resulting terms are all quadratic in these fields, and we shall analyze them in turn.

The quadratic term in $\varphi^\perp$ is given by

$$-\int d^3z\sqrt{g}\,\bar{\varphi}^\perp_\mu(\Gamma^{\mu\nu\rho}D_\nu+\hat{m}\Gamma^{\mu\rho})\varphi^\perp_\rho\,, \tag{D.1}$$

where in Euclidean space $\bar{\varphi}^\perp=(\varphi^\perp)^\dagger$. Using that $\Gamma^\mu\varphi^\perp_\mu=0$ as well as $\{\Gamma^\mu,\Gamma^\nu\}=2g^{\mu\nu}$ and the definition (7.14), we find

$$-\int d^3z\sqrt{g}\,\bar{\varphi}^{\perp\mu}(\Gamma^\nu D_\nu-\hat{m})\varphi^\perp_\mu\,. \tag{D.2}$$

The cross term between $\varphi^\perp$ *and* ξ is of the form

$$-\int d^3z\sqrt{g}\,\left(\bar{\varphi}^{\perp\rho}\Gamma^\mu D_\mu D_\rho\,\xi+D^\rho\xi^\dagger\Gamma^\mu D_\mu\varphi^\perp_\rho\right)\,, \tag{D.3}$$

where we have used that $\Gamma^\mu\varphi^\perp_\mu=D^\mu\varphi^\perp_\mu=0$. Both terms actually vanish. For the first term we use

$$(D_\mu D_\rho-D_\rho D_\mu)\xi=\frac{1}{8}R_{\mu\rho\sigma\delta}[\Gamma^\sigma,\Gamma^\delta]\xi \tag{D.4}$$

to move D_ρ to the left of D_μ, where it vanishes (up to a total derivative) since $D_\rho\varphi^{\perp\rho}=0$. Thus the first term equals

$$-\frac{1}{8}\int d^3z\sqrt{g}\,\bar{\varphi}^{\perp\rho}\Gamma^\mu R_{\mu\rho\sigma\delta}[\Gamma^\sigma,\Gamma^\delta]\xi\,, \tag{D.5}$$

which is seen to vanish upon using (7.4) and $\Gamma^\mu\varphi^\perp_\mu=0$. Similar manipulations can be used to show that the second term in (D.3) also vanishes.

The cross term between $\varphi^\perp$ *and* ψ vanishes directly upon using $D^\mu\varphi^\perp_\mu=\Gamma^\mu\varphi^\perp_\mu=0$.

The quadratic term involving the spinor component ξ arises from

$$-\int d^3z\sqrt{g}\,(\overline{\tilde{D}_\mu\xi})\,(\Gamma^{\mu\nu\rho}D_\nu+\hat{m}\Gamma^{\mu\rho})\tilde{D}_\rho\xi\,, \tag{D.6}$$

where $\tilde{D}_\rho=D_\rho-\frac{\Gamma_\rho}{3}\hat{D}$ is the differential operator that appeared in the defining equation for ξ, (7.19). Using $\Gamma^\mu\tilde{D}_\mu\xi=0$, and performing the same steps as in the analysis leading to (D.2), we can rewrite (D.6) as

$$-\int d^3z\sqrt{g}\,(\overline{\tilde{D}^\mu\xi})\,(\Gamma^\rho D_\rho-\hat{m})\tilde{D}_\mu\xi\,. \tag{D.7}$$

Next we integrate by parts to move the operator $\tilde{D}^\mu$ to the right. Using $\Gamma^\mu\tilde{D}_\mu\xi=0$ the term proportional to $\hat{m}$ reduces to

$$-\hat{m}\int d^3z\sqrt{g}\left(\bar{\xi}\,D_\mu\left(D^\mu-\frac{\Gamma^\mu}{3}\hat{D}\right)\xi\right)\,, \tag{D.8}$$

where we have written out $\tilde{D}^\mu$ in terms of the covariant derivative D^μ and $\hat{D}$. For the first term in (D.7) integration by parts leads to

$$\underbrace{\int d^3z\sqrt{g}\,\bar{\xi}\,D_\mu(\Gamma^\sigma D_\sigma)\tilde{D}^\mu\xi}_{A}\,\underbrace{-\frac{1}{3}\int d^3z\sqrt{g}\,\bar{\xi}(\Gamma^\rho D_\rho)\Gamma_\mu(\Gamma^\sigma D_\sigma)\tilde{D}^\mu\xi}_{B}\ . \tag{D.9}$$

For B we use $\{\Gamma_\mu, \Gamma^\sigma\} = 2\delta^\sigma_\mu$ as well as $\Gamma_\mu\tilde{D}^\mu\xi = 0$ to obtain

$$B = -\frac{2}{3}\int d^3z\sqrt{g}\,\bar{\xi}(\Gamma^\rho D_\rho)D_\mu\left(D^\mu - \frac{1}{3}\Gamma^\mu\hat{D}\right)\xi\ . \tag{D.10}$$

For A we use the commutation relation

$$(D_\mu D_\sigma - D_\sigma D_\mu)\tilde{D}^\mu\xi = R_{\mu\sigma}\tilde{D}^\mu\xi + \frac{1}{8}R_{\mu\sigma\nu\delta}[\Gamma^\nu, \Gamma^\delta]\tilde{D}^\mu\xi\ . \tag{D.11}$$

to rewrite it as

$$A = \int d^3z\sqrt{g}\,\left(\bar{\xi}(\Gamma^\sigma D_\sigma)D_\mu\tilde{D}^\mu\xi + \bar{\xi}\,\Gamma^\sigma R_{\mu\sigma}\tilde{D}^\mu\xi + \bar{\xi}\,\Gamma^\sigma\frac{1}{8}R_{\mu\sigma\nu\delta}[\Gamma^\nu, \Gamma^\delta]\tilde{D}^\mu\xi\right)\ . \tag{D.12}$$

Substituting the explicit expressions (7.4) for the the curvature tensor and Ricci tensor of H_3^+, the last two terms of (D.12) become

$$\frac{R}{3}\int d^3z\sqrt{g}\,\left(\bar{\xi}\,\Gamma^\mu\tilde{D}_\mu\xi - \frac{1}{2}\bar{\xi}\,\Gamma^\mu\tilde{D}_\mu\xi\right) = 0\,, \tag{D.13}$$

which vanish because of $\Gamma^\mu\tilde{D}_\mu\xi = 0$. The first term of A in (D.12) has the same form as B in (D.10), and thus the total contribution quadratic in ξ equals

$$\int d^3z\sqrt{g}\,\left[\frac{1}{3}\bar{\xi}\,(\Gamma^\sigma D_\sigma)D_\mu\left(D^\mu - \frac{\Gamma^\mu}{3}\hat{D}\right)\xi - \hat{m}\,\bar{\xi}D_\mu\left(D^\mu - \frac{\Gamma^\mu}{3}\hat{D}\right)\xi\right]\ . \tag{D.14}$$

Using (D.4) we can simplify

$$D_\mu\left(D^\mu - \frac{\Gamma^\mu}{3}\hat{D}\right)\xi = D_\mu\left(D^\mu - \frac{1}{3}\Gamma^\mu\Gamma^\sigma D_\sigma\right)\xi = \frac{2}{3}\left(\Delta_{(1/2)} + \frac{R}{8}\right)\xi\ . \tag{D.15}$$

Thus the final answer for the quadratic ξ term takes the form

$$\frac{2}{9}\int d^3z\sqrt{g}\,\bar{\xi}(\Gamma^\sigma D_\sigma - 3\hat{m})(\Delta_{(1/2)} + R/8)\xi\ . \tag{D.16}$$

The cross term between ξ and ψ can be analyzed similarly, and it leads to

$$\frac{2}{9}\int d^3z\sqrt{g}\,\left[\bar{\psi}(\Delta_{(1/2)} + R/8)\xi - \bar{\xi}(\Delta_{(1/2)} + R/8)\psi\right]\ . \tag{D.17}$$

The quadratic term in ψ reduces with similar manipulations to

$$-\frac{2}{9}\int d^3z\sqrt{g}\,\bar{\psi}(\hat{D} + 3\hat{m})\psi\ . \tag{D.18}$$

Combing (D.2), (D.16), (D.17) and (D.18), and setting $R = -6$ then finally leads to eq. (7.21).

For the derivation of (7.26) we also need the identities

$$-(\Gamma^\mu D_\mu + \hat{m})(\Gamma^\rho D_\rho - \hat{m})\,\varphi_\sigma^\perp = \left(-D^\mu D_\mu + \frac{5R}{12} + \hat{m}^2\right)\varphi_\sigma^\perp = \left(-\Delta_{(3/2)} - \frac{9}{4}\right)\varphi_\sigma^\perp \quad \text{(D.19)}$$

and

$$-(\Gamma^\sigma D_\sigma + 3\hat{m})(\Gamma^\rho D_\rho - 3\hat{m})\,\xi = \left(-D^\mu D_\mu + \frac{R}{4} + 9\,\hat{m}^2\right)\xi = \left(-\Delta_{(1/2)} + \frac{3}{4}\right)\xi\,. \quad \text{(D.20)}$$

They follow upon using (D.4) and the analogue for spin 3/2

$$(D_\mu D_\rho - D_\rho D_\mu)\varphi_\nu^\perp = R^\sigma{}_{\nu\rho\mu}\varphi_\sigma^\perp + \frac{1}{8}R_{\mu\rho\sigma\delta}[\Gamma^\sigma, \Gamma^\delta]\varphi_\nu^\perp\,, \quad \text{(D.21)}$$

as well as (7.4). We have also substituted the value of $\hat{m}^2$ from (7.16).

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