ETH zürich

ETH Library

Defect perturbations in Landau-Ginzburg models

Journal Article

Author(s):
Brunner, Ilka; Roggenkamp, Daniel; Rossi, Sebastiano

Publication date:
2010-03

Permanent link:
https://doi.org/10.3929/ethz-b-000017802

Rights / license:
Creative Commons Attribution 4.0 International

Originally published in:
Journal of High Energy Physics 2010(3), https://doi.org/10.1007/JHEP03(2010)015

This page was generated automatically upon download from the ETH Zurich Research Collection.
For more information, please consult the Terms of use.

Published for SISSA by Springer

Received: November 14, 2009
Accepted: February 7, 2010
Published: March 3, 2010

Defect perturbations in Landau-Ginzburg models

Ilka Brunner,[a,b] **Daniel Roggenkamp**[c] **and Sebastiano Rossi**[d]

[a] *Arnold Sommerfeld Center, Ludwig Maximilians Universität, Theresienstr. 37, 80333 München, Germany*

[b] *Excellence Cluster Universe, Technische Universität München, Boltzmannstr. 2, 85748 Garching, Germany*

[c] *Department of Physics and Astronomy, Rutgers University, Piscataway, 08855-0849 NJ, U.S.A.*

[d] *Institut für Theoretische Physik, ETH Zürich, 8093 Zürich, Switzerland*

E-mail: Ilka.Brunner@physik.uni-muenchen.de, roggenka@physics.rutgers.edu, rossise@phys.ethz.ch

Abstract: Perturbations of B-type defects in Landau-Ginzburg models are considered. In particular, the effect of perturbations of defects on their fusion is analyzed in the framework of matrix factorizations. As an application, it is discussed how fusion with perturbed defects induces perturbations on boundary conditions. It is shown that in some classes of models all boundary perturbations can be obtained in this way. Moreover, a universal class of perturbed defects is constructed, whose fusion under certain conditions obey braid relations. The functors obtained by fusing these defects with boundary conditions are twist functors as introduced in the work of Seidel and Thomas.

Keywords: D-branes, Conformal Field Models in String Theory, Topological Field Theories

ArXiv ePrint: 0909.0696

© SISSA 2010

doi:10.1007/JHEP03(2010)015

JHEP03(2010)015

Contents

1 Introduction **1**

2 Brief review of matrix factorizations **4**

3 Perturbed defects and their fusion **8**
- 3.1 Perturbation of defects 8
- 3.2 Fusion of perturbed defects 10
- 3.3 Method to calculate fusion products 11

4 An application: boundary flows from defects **12**
- 4.1 Example: minimal models 12
  - 4.1.1 B-type boundary conditions in minimal models 12
  - 4.1.2 B-type defects in minimal models 13
  - 4.1.3 Inducing boundary flows by defects 14
- 4.2 Example: minimal model orbifolds 16
- 4.3 Example: tensor products of identical LG models 18
- 4.4 Defect induced boundary flows in CFT 19

5 Braid group actions and defects **25**
- 5.1 The defects 27
- 5.2 Realization in terms of matrix factorizations 28
  - 5.2.1 Twist defects 28
  - 5.2.2 Twist defects for spherical matrix factorizations 29
  - 5.2.3 Defect realization of braid groups 30
- 5.3 Examples 31
  - 5.3.1 Degenerate K3 and fibrations 32
  - 5.3.2 A non-geometric Landau-Ginzburg example 34
  - 5.3.3 Hirzebruch-Jung examples 34

A Equivalence of identity matrix factorizations **36**

1 Introduction

A defect in a two-dimensional field theory is a line of inhomogeneity on the surface on which the theory is defined. In general defect lines carry extra degrees of freedom not inherited from the bulk, which determine how excitations are transmitted between the theories on either side.[1]

[1] The theories on the two sides of the defect can either be the same or different. In case the theories are different, defects are sometimes referred to as "interfaces".

JHEP03(2010)015

The theory of defects is closely related to the theory of boundary conditions. Consider for example two conformal field theories on the complex plane separated by a defect located along the real line. Folding the plane along this line results in a surface with boundary for a "doubled theory", which is given by the theory on the upper half plane tensored by the conjugate of the theory on the lower half plane [1, 2].

However, compared to boundary conditions, defects have more structure: They can form junctions, and they can be composed by *fusion*. Fusion is the process in which two parallel defects are brought infinitely close together [3–6]. In the limit, a new defect is created. Of course, taking such a limit is in general a highly singular procedure, which in special situations simplifies however. Most notably, there are so-called topological defects that preserve the full diffeomorphism invariance. They can be moved around freely, in particular without causing any singularities. Hence, they can be fused smoothly.

In the context of supersymmetric $N = (2, 2)$ models one can consider defects preserving A- or B-type supersymmetry. These defects survive the corresponding topological A- or B-twists respectively. On the level of the twisted theory, their fusion is regular and defines a product structure on all such defects [7].

Defects and their fusion have a variety of applications in the context of string theory and conformal field theory. In the string theory context, it was proposed in [8] that defects can be used as spectrum generating symmetries. The main idea is that fusion of a conformal boundary condition describing a D-brane in CFT1 with a topological defect between CFT1 and CFT2, produces a conformal boundary condition in CFT2. Since conformal invariance is equivalent to the classical string equations, fusion with topological defects creates new solutions of classical string theory out of given ones.

Certain special defects arise between UV and IR fixed points of quantum field theories [9]. These defects can be used to describe how boundary conditions behave under the corresponding renormalization group flows. In this way, defects and their fusion can serve as an alternative to the perturbative analysis of this problem. This has been made explicit in the case of $N = (2, 2)$ minimal models in [9].[2]

In the same spirit, defects can be used to describe D-brane monodromies [11]. Namely, there are defects associated to exactly marginal bulk deformations as well. Fusion with a defect associated to a deformation along a closed loop in the bulk moduli space encodes the effect of the corresponding monodromy on boundary conditions.

But defects are not only useful in the analysis of bulk perturbations of theories with boundary. They also relate different boundary renormalization group flows [12–14]. The basic idea is that certain boundary perturbations can be pulled back to the bulk by splitting off defects. In the context of WZW models this has been analyzed in [13]. Turning things around, fusing a perturbed defect with a boundary condition, the defect perturbation descends to a boundary perturbation of the boundary condition emanating from the fusion. Hence, defect perturbations give rise to classes of perturbations of different boundary conditions.

[2]Recently, the paper [10] explored the possibility to use certain topological defects to investigate bulk-boundary flows on the level of the full conformal field theory.

JHEP03(2010)015

Motivated by these observations, in this paper, we will study perturbed defects and their fusion in $N = (2, 2)$ supersymmetric theories, in particular Landau-Ginzburg models (see [10, 15, 16] for recent related work in conformal field theory). Special emphasis is put on the cone construction, which will play a prominent role in the examples we present. In Landau-Ginzburg models, B-type defects have a convenient realization in terms of matrix factorizations [7]. Their fusion is regular and essentially given by a tensor product between Chan-Paton type spaces. Therefore, this framework lends itself easily to the analysis of fusion of perturbed defects. This will be used to discuss how defect perturbations induce boundary perturbations in the way alluded to above.

In the case of the Landau-Ginzburg models with one chiral superfield and superpotential $W = x^d$ we establish that all supersymmetry preserving boundary perturbations arise in this way, *i.e.* all boundary perturbations can be pulled back into the bulk by means of defects. The same applies to $\mathbb{Z}_d$-orbifolds of these models, and to models which are tensor products of two identical models. Although we treat it in the Landau-Ginzburg framework, we expect that the arguments in the latter case generalize to tensor products of arbitrary $N = (2, 2)$ theories with their conjugates.

In the IR, the Landau-Ginzburg orbifolds with superpotential $W = x^d$ are described by $N = 2$-supersymmetric minimal models, which are rational conformal field theories with diagonal[3] modular invariants. Thus, in these models we can relate our considerations to results obtained for diagonal RCFTs. Of course, defect perturbations are much more difficult to deal with on the level of the full conformal field theory, but some special classes of perturbations have been treated in [5].

Apart from the induction of boundary perturbations, we use defect perturbations to construct special classes of defects which show an interesting universal behavior. More precisely, in any theory there are purely reflective defects which impose fixed boundary conditions on the two theories on either side, as well as the "trivial" or identity defect. Between these defects there is a canonical defect changing field, which descends from the identity field on the boundary condition imposed by the reflective defect. It can be used to perturb superpositions of reflective and identity defects. We show that the resulting perturbed defects have the following nice properties.

If the underlying boundary condition is "spherical" in the sense that the BRST-cohomology of boundary fields on it is two-dimensional and the boundary two-point functions are non-degenerate, then the associated defect is group-like as defined in [17], *i.e.* the defect and its dual fuse to the identity defect. Furthermore these defects obey a twisted commutation relation with respect to fusion.

Moreover, if there is a collection $(P_1, \ldots, P_m)$ of spherical boundary conditions, such that there is exactly one BRST-invariant boundary condition changing field between any neighboring P_i, P_j, $|i - j| = 1$ and no one between P_i and P_j, $|i - j| > 1$, *i.e.* the spherical boundary conditions form an A_m-sequence,[4] the associated defects satisfy braid relations with respect to fusion.

[3] with respect to B-type supersymmetry

[4] Examples include the A_m chains on K3 surfaces responsible for the non-abelian gauge symmetries of type II strings.

JHEP03(2010)015

We carry out the construction and discussion in the context of Landau-Ginzburg models, but we expect it to be valid in any $N = (2, 2)$ supersymmetric theory.

In fact, these defects are generalizations of defects describing monodromies around conifold points [11]. Their fusion with boundary conditions provides a world sheet realization of the twist functors introduced in the construction of braid group representations in the group of autoequivalences of certain categories in [18].

This paper is organized as follows: In section 2 we review the matrix factorization formalism used to describe B-type defects and boundary conditions in Landau-Ginzburg models. Section 3 contains a general discussion of perturbed defects and their fusion in this framework. Section 4 is devoted to the fusion of perturbed defects with boundary conditions, and the induced boundary perturbations. Finally in section 5 we construct the universal twist defects and establish that under some conditions their fusion satisfies braid relations. We provide various classes of examples in which these conditions are satisfied.

2 Brief review of matrix factorizations

In Landau-Ginzburg models, B-type supersymmetric D-branes as well as B-type supersymmetry preserving defects have an elegant description in terms of matrix factorizations [7, 19–21], see [22, 23] for reviews.

A matrix factorization P of a polynomial $W \in \mathbb{C}[x_1, \ldots, x_N]$ is given by a pair (P_1, P_0) of free $\mathbb{C}[x_1, \ldots, x_N]$ modules together with homomorphisms $p_s : P_s \to P_{(s+1) \bmod 2}$ between them which compose to W times the identity map, *i.e.* $p_1 p_0 = W \mathrm{id}_{P_0}$ and $p_0 p_1 = W \mathrm{id}_{P_1}$. In the following we will often represent matrix factorizations by

$$P : P_1 \underset{p_0}{\overset{p_1}{\rightleftarrows}} P_0 \,. \tag{2.1}$$

Sometimes it is useful to regard them as two-periodic twisted[5] complexes. Indeed, such matrix factorizations form a category, with morphisms $\mathcal{H}^*(P, Q)$ between two matrix factorizations P and Q given by the cohomology of the Hom-complex of the two twisted complexes associated to P and Q. The latter is a two-periodic untwisted complex.

$$\mathcal{H}^*(P, Q) = H^*(\mathrm{Hom}(P, Q)) \cong H^*(P^* \otimes Q) \tag{2.2}$$

Here, P^* denotes the dual matrix factorization

$$P^* : P_1^* \underset{-p_1^*}{\overset{p_0^*}{\rightleftarrows}} P_0^* \,, \tag{2.3}$$

and the tensor product is the ordinary tensor product of complexes. It will be spelled out explicitly in (2.10) below.

There are always matrix factorizations with modules $P_s = \mathbb{C}[x_1, \ldots, x_N]$ and with maps $p_r = 1$ and $p_{(r+1) \bmod 2} = W$. They are trivial in the sense that they only have zero-morphisms with any other (including themselves) matrix factorization. Two matrix

[5] The differential squares to W instead of zero.

JHEP03(2010)015

factorizations which differ by the addition of such a trivial matrix factorization are equivalent. Indeed, if a matrix representing one of the maps q_i of a matrix factorization Q contains a scalar entry different from zero, such a trivial matrix factorization can always be split off from Q.

More generally, two matrix factorizations Q and Q' are equivalent if one can find maps $u_i : Q_i \to Q'_i$ and $v_i : Q'_i \to Q_i$ such that

$$q'_1 = u_0 q_1 v_1 \,, \quad q'_0 = u_1 q_0 v_0 \,, \quad q_1 = v_0 q'_1 u_1 \,, \quad q_0 = v_1 q'_0 u_0 \tag{2.4}$$

and

$$\begin{aligned} v_0 u_0 &= \mathrm{id}_{Q_0} + \chi_1 q_0 + q_1 \chi_0 \,, \quad & v_1 u_1 &= \mathrm{id}_{Q_1} + q_0 \chi_1 + \chi_0 q_1 \,, \\ u_0 v_0 &= \mathrm{id}_{Q'_0} + \chi'_1 q'_0 + q'_1 \chi'_0 \,, \quad & u_1 v_1 &= \mathrm{id}_{Q'_1} + q'_0 \chi'_1 + \chi'_0 q'_1 \,, \end{aligned} \tag{2.5}$$

for some $\chi_i : Q_i \to Q_{(i+1) \bmod 2}$ and $\chi'_i : Q'_i \to Q'_{(i+1) \bmod 2}$.

Physically, the defects constructed using matrix factorizations can be regarded as composites of a defect-anti-defect pair with a tachyon turned on. The data of the matrix factorization can be summarized in a defect BRST-operator

$$\mathcal{Q} = \begin{pmatrix} 0 & p_1 \\ p_0 & 0 \end{pmatrix} \tag{2.6}$$

containing the tachyon profile. This is an operator in $\mathrm{End}(P_1 \oplus P_0)$, which is odd with respect to the $\mathbb{Z}_2$-grading

$$\sigma = \mathrm{id}_{P_0} - \mathrm{id}_{P_1} \,. \tag{2.7}$$

In this language, $\mathcal{H}(P, Q)$ is just given by the BRST-cohomology on $\mathrm{Hom}(P, Q)$, and the equivalence relation (2.4) becomes

$$\mathcal{Q}' = U \mathcal{Q} V, \quad UV = \mathrm{id}' + \{\mathcal{Q}, O'\}, \quad VU = \mathrm{id} + \{\, \mathcal{Q}, O\} \tag{2.8}$$

for some O and O'.

As was shown in [19, 20, 24], B-type supersymmetric D-branes in Landau-Ginzburg models with chiral superfields $x_1, \ldots, x_N$ and superpotential $W \in \mathbb{C}[x_1, \ldots, x_N]$ can be represented by matrix factorizations of W, where open strings between two such D-branes are described by morphisms between the respective matrix factorizations.

In the same way, it has been argued in [7] that B-type supersymmetry preserving defects between two Landau-Ginzburg models, one with chiral fields $x_1, \ldots, x_N$ and superpotential $W_1 \in \mathbb{C}[x_1, \ldots, x_N]$ and one with chiral superfields $y_1, \ldots, y_M$ and superpotential $W_2 \in \mathbb{C}[y_1, \ldots, y_M]$ can be represented by matrix factorizations of $W_1 - W_2$ over the polynomial ring $\mathbb{C}[x_1, \ldots, x_N, y_1, \ldots, y_N]$.

As mentioned before, one interesting property of $N = 2$-supersymmetric defects is that they can be fused with other such defects or boundary conditions preserving the same supersymmetry. Namely, two such defects can be brought on top of each other to produce a new defect, or a defect can be moved onto a world sheet boundary to change the boundary condition imposed there. This fusion has a very simple realization in terms of the matrix

JHEP03(2010)015

factorization description. For instance, let x_i, y_i, z_i be the chiral superfields of three Landau-Ginzburg models with superpotentials $W_1 \in \mathbb{C}[x_i]$, $W_2 \in \mathbb{C}[y_i]$ and $W_3 \in \mathbb{C}[z_i]$ respectively, which are separated by two defects represented by matrix factorizations P^1 of $W_1 - W_2$ and P^2 of $W_2 - W_3$. Fusing the two defects gives rise to a new defect separating the Landau-Ginzburg model with chiral fields x_i and superpotential W_1 from the one with chiral fields z_i and superpotential W_3. This fused defect is given by the matrix factorization

$$P^1 * P^2 = \left(P^1 \otimes P^2\right)^{\text{red}}_{\mathbb{C}[x_i,z_i]} \,. \tag{2.9}$$

Here, the tensor product of two matrix factorizations is defined by taking the tensor product of the associated twisted complexes. It is also a two-periodic complex which is twisted by the sum of the twists of the tensor factors. More concretely, the tensor product $P \otimes Q$ of matrix factorizations P and Q of W and W' respectively can be written as

$$P \otimes Q : P_1 \otimes Q_0 \oplus P_0 \otimes Q_1 \underset{r_0}{\overset{r_1}{\rightleftarrows}} P_0 \otimes Q_0 \oplus P_1 \otimes Q_1 \tag{2.10}$$

with

$$r_1 = \begin{pmatrix} p_1 \otimes \text{id} & \text{id} \otimes q_1 \\ -\text{id} \otimes q_0 & p_0 \otimes \text{id} \end{pmatrix} , \quad r_0 = \begin{pmatrix} p_0 \otimes \text{id} & -\text{id} \otimes q_1 \\ \text{id} \otimes q_0 & p_1 \otimes \text{id} \end{pmatrix} \tag{2.11}$$

which is a matrix factorization of $W + W'$.

In the situation above, P^1 is a matrix factorization of $W_1 - W_2$ and P^2 one of $W_2 - W_3$. Hence, $P^1 \otimes P^2$ is a matrix factorization of $W_1 - W_3 \in \mathbb{C}[x_i, z_i]$, but it is still a matrix factorization over $\mathbb{C}[x_i, y_i, z_i]$. That means that the modules $(P^1 \otimes P^2)_s$ are free $\mathbb{C}[x_i, y_i, z_i]$-modules and also the maps r_s between them depend on the y_i. The notation $\left(P^1 \otimes P^2\right)_{\mathbb{C}[x_i,z_i]}$ means that this matrix factorization has to be regarded as one over $\mathbb{C}[x_i, z_i]$ only.[6] As such, it is of infinite rank, because the modules $(P^1 \otimes P^2)_s$ regarded as modules over $\mathbb{C}[x_i, z_i]$ are free modules of infinite rank. For instance, $\mathbb{C}[x_i, y_i, z_i]$ can be decomposed as

$$\mathbb{C}[x_i, y_i, z_i] = \bigoplus_{(l_1,\ldots,l_N)\in\mathbb{N}_0^N} y_1^{l_1} \ldots y_N^{l_N} \mathbb{C}[x_i, z_i] \tag{2.12}$$

into free $\mathbb{C}[x_i, z_i]$-modules. Physically speaking, the chiral fields y_i of the theory squeezed in between the two defects are promoted to new defect degrees of freedom in the limit where the two defects coincide. However, most of them are trivial. Namely, if both P^1 and P^2 are of finite rank, the matrix factorization $\left(P^1 \otimes P^2\right)_{\mathbb{C}[x_i,z_i]}$ can be reduced to finite rank by splitting off infinitely many trivial matrix factorizations. It is the result of this reduction $\left(P^1 \otimes P^2\right)^{\text{red}}_{\mathbb{C}[x_i,z_i]}$ which describes the fused defect.

In the same way, fusion of B-type defects and B-type boundary conditions in Landau-Ginzburg models can be formulated in the matrix factorization framework. The fusion of a B-type defect separating a Landau-Ginzburg model with chiral fields x_i and superpotential $W_1 \in \mathbb{C}[x_i]$ from one with chiral fields y_i and superpotential $W_2 \in \mathbb{C}[y_i]$ and a B-type

[6] In the following, it will usually be evident which base ring is chosen for matrix factorizations. For ease of notation we will therefore omit subscripts like $\mathbb{C}[x_i, z_i]$ in equation (2.9).

JHEP03(2010)015

boundary condition in the second of these Landau-Ginzburg models can be represented by the matrix factorization

$$P * Q = (P \otimes Q)^{\text{red}}_{\mathbb{C}[x_i]} \,, \tag{2.13}$$

where P is the matrix factorization of $W_1 - W_2$ associated to the defect and Q the matrix factorization of W_2 associated to the boundary condition.

Certain defects are quite universal and exist in any Landau-Ginzburg model, or even any two-dimensional QFT. A special example is the identity defect Id. It is trivial in the sense that inserting it does not change any correlation functions. Nevertheless, it will play an interesting role in this paper. In Landau-Ginzburg models, it is realized by the following matrix factorization. The difference of the same superpotential in different variables can always be factorized as

$$W(x_i) - W(y_i) = \sum_i (x_i - y_i) A_i(x_i, y_i) \,. \tag{2.14}$$

Denoting the rank-one factorizations with factors $p_1^{(i)} = x_i - y_i$ and $p_0^{(i)} = A_i(x_j, y_j)$ by $\mathrm{Id}^{(i)}$, a matrix factorization representing the identity defect in the Landau-Ginzburg model with superpotential W is given by the tensor product

$$\mathrm{Id} = \bigotimes \mathrm{Id}^{(i)} \,. \tag{2.15}$$

Even though the factorization (2.14) is not unique, the equivalence class of Id is unique. Different choices of the A_i lead to equivalent matrix factorizations Id. This is shown in appendix A. Indeed, as expected from the identity defect, fusion with the matrix factorization Id is trivial: $\mathrm{Id} * Q = Q$ [7, 25].

Another universal class of defects are totally reflective defects. Such defects provide boundary conditions for the theories on either side, and do not allow any excitations to be transmitted between the theories. In the context of matrix factorizations, such defects are realized by tensor products of matrix factorizations of the superpotentials on the two sides. Let P be a matrix factorization of $W(x_i)$ and Q of $W(y_i)$. A totally reflective defect imposing boundary condition Q on one side and P on the other side is given by

$$T_{P,Q^*} := P \otimes Q^*, \tag{2.16}$$

where the dual matrix factorization Q^* was defined in (2.3). It arises here because of the different orientations on the two sides of the defect. The fusion of T_{P,Q^*} with matrix factorizations R of $W(y_i)$ have a simple form

$$T_{P,Q^*} * R \equiv P \otimes (Q^* \otimes R) \equiv P \otimes H^*(Q^* \otimes R) \equiv P \otimes \mathcal{H}^*(Q, R) \,. \tag{2.17}$$

Here, the matrix factorization $P \otimes (Q^* \otimes R)$ has to be regarded as a matrix factorization over $\mathbb{C}[x_i]$. The factor $Q^* \otimes R$ is a factorization of $W(y_i) - W(y_i) = 0$. It is a complex with a regular differential δ, which squares to zero and does not depend on the x_i. Thus, the non-zero matrix entries of δ contribute scalars to the matrix factorization $P \otimes (Q^* \otimes R)$ which can be used to reduce it to $P \otimes H^*(Q^* \otimes R)$.

JHEP03(2010)015

In section 5 we will perturb the superposition of the identity and totally reflective defects and show that the resulting defects give rise to some interesting structures.

Everything described above for Landau-Ginzburg models easily carries over to Landau-Ginzburg orbifolds [9]. If Γ is a finite group acting on the ring $\mathbb{C}[x_1, \ldots, x_N]$ of chiral fields of a Landau-Ginzburg model in such a way that the superpotential W is Γ-invariant, one can consider its Γ-orbifold. B-type defects and boundary conditions in such orbifold theories can be represented by Γ-equivariant matrix factorizations [9, 26, 27]. These are matrix factorizations P together with representations ρ_i of Γ on P_i which are compatible with the ring structure, and with respect to which the maps p_i are invariant: $\rho_{(i-1) \bmod 2} p_i = p_i \rho_i$. These conditions ensure that there is an induced representation of Γ on the BRST-cohomology $\mathcal{H}^*(P, Q)$ which can be used to define the BRST-cohomology of the orbifold theory as the Γ-invariant part

$$\mathcal{H}^*_{\text{orb}}(P, Q) = (\mathcal{H}^*(P, Q))^{\Gamma} \tag{2.18}$$

of the respective BRST-cohomology in the underlying unorbifolded model.

Similarly, fusion of two defect matrix factorizations in the orbifold theory is given by the Γ_{squeezed}-invariant part of the fusion of the underlying matrix factorizations

$$P^1 *_{\text{orb}} P^2 = \left(P^1 * P^2\right)^{\Gamma_{\text{squeezed}}} . \tag{2.19}$$

Here, Γ_{squeezed} denotes the orbifold group of the Landau-Ginzburg model squeezed in between the two defects. For more details on B-type defects in Landau-Ginzburg orbifolds see [9].

3 Perturbed defects and their fusion

3.1 Perturbation of defects

The perturbation theory of defects exactly parallels the one of boundary conditions. Interesting new effects arise however, when one considers structures inherent to defects, which are not present in boundary conditions, for instance fusion. Let us start by briefly discussing perturbations of B-type defects in Landau-Ginzburg models, following [27].

Just like for boundaries, there are fields which are confined to defects, and which can be used to perturb the latter. In the context of Landau-Ginzburg models, supersymmetry preserving perturbations of B-type defects P are generated by fields in $\mathcal{H}(P, P)$, and they correspond to deformations of the corresponding matrix factorizations. In the BRST-formulation (2.6) such a deformation is a family

$$\mathcal{Q}(t) = \mathcal{Q}^0 + \sum_{n>1} t^n \psi_n \tag{3.1}$$

of BRST-operators with

$$(\mathcal{Q}(t))^2 = W \tag{3.2}$$

JHEP03(2010)015

for all t. To first order in t this condition means that ψ_1 has to be BRST-closed with respect to the undeformed BRST-operator $\mathcal{Q}^0$

$$\{\mathcal{Q}^0, \psi_1\} = 0\,. \tag{3.3}$$

On the other hand, if ψ_1 is BRST-exact with respect to $\mathcal{Q}^0$

$$\psi_1 = \left[\mathcal{Q}^0, \chi_1\right] \tag{3.4}$$

then the first order deformation can be compensated by the equivalence

$$\mathcal{Q} \mapsto e^{-t\chi_1}\,\mathcal{Q}e^{t\chi_1}\,. \tag{3.5}$$

Thus, to first order, deformations of (equivalence classes of) matrix factorizations P are generated by $\mathcal{H}^1(P,P)$. Of course, not all first order deformations are necessarily integrable. In general, there can be obstructions at higher order, as has been analyzed in [27–30]. For instance at second order, condition (3.2) implies

$$\psi_1^2 + \{\mathcal{Q}^0, \psi_2\} = 0\,. \tag{3.6}$$

If ψ_1 squares to a non-trivial BRST-cohomology class, equation (3.6) cannot be satisfied and the deformation generated by ψ_1 is obstructed. Otherwise, one can find ψ_2 such that (3.6) holds. This can be repeated order by order: given $\psi_1, \ldots, \psi_{n-1}$ such that (3.2) holds to order $n-1$ one has to construct ψ_n such that it is satisfied to order n. If for some n this is not possible, the deformation is obstructed. Otherwise one obtains a family of BRST-operators parametrized by t, or to put it differently a family of matrix factorizations. The corresponding family of equivalence classes is non-constant in $t=0$ if and only if ψ_1 represents a non-trivial class in $\mathcal{H}^1(P,P)$ (see [27] for a more detailed discussion in case of boundaries).

A special case arises, when the undeformed matrix factorization is a direct sum of two matrix factorizations P and P', and the deformation is generated by a "defect changing operator" $T \in \mathcal{H}^1(P,P')$. It describes the bound state formation of the associated defects.

Since in this case $T^2 = 0$, condition (3.2) is automatically satisfied to all orders, and no higher order terms are necessary. In particular, all such deformations are integrable, and the family of matrix factorizations are given by the mapping cones

$$Q(t) = \mathrm{Cone}(tT : P \longrightarrow P') : \; P_1 \oplus P_1' \underset{c_0}{\overset{c_1}{\rightleftarrows}} P_0 \oplus P_0' \tag{3.7}$$

with

$$c_1 = \begin{pmatrix} p_1 & 0 \\ tT|_{P_1} & p_1' \end{pmatrix}, \quad c_0 = \begin{pmatrix} p_0 & 0 \\ tT|_{P_0} & p_0' \end{pmatrix}.$$

Note that all $Q(t)$ for $t \neq 0$ are equivalent.

JHEP03(2010)015

3.2 Fusion of perturbed defects

Our focus in this paper is the fusion of perturbed defects, in particular of those defects which can be obtained as bound states. Of course, the fusion product of a perturbed defect $D(t)$ with another defect D' can be viewed as a perturbation of the fusion product of the unperturbed defect with the other defect

$$D(t) * D' = (D * D')(t)\,. \tag{3.8}$$

Once the obstruction problem is solved for the initial defect D and a given direction $\psi_1 \in \mathcal{H}^1(D, D)$, it is automatically solved for $D*D'$ for an induced direction $\widetilde{\psi}_1 \in \mathcal{H}^1(D*D', D*D')$. Indeed, this is obvious, because fusing a family of defects with another defect one obtains again a family of defects.

The first question that arises is how to determine the induced direction $\widetilde{\psi}_1$. The answer is indeed straight-forward to work out in the Landau-Ginzburg framework.

Let us start with an unperturbed defect corresponding to a matrix factorization of $W_1(x_i) - W_2(y_i)$ with BRST-operator $\mathcal{Q}_1^0$. Adding a perturbation generated by ψ_1 results in a perturbed BRST-operator $\mathcal{Q}_1(t) = \mathcal{Q}_1^0 + \sum_{n>0} t^n \psi_n$. We now take the fusion product with an arbitrary other defect with BRST-operator $\mathcal{Q}_2$, $(\mathcal{Q}_2)^2 = W_2(y_i) - W_3(z_i)$. Fusion creates a new defect with BRST-operator

$$\mathcal{Q}(t) = \mathcal{Q}_1^0 + \sum_{n>0} t^n \psi_n + \mathcal{Q}_2 = \mathcal{Q}^0 + \sum_{n>0} t^n \psi_n \tag{3.9}$$

which correctly squares to $W_1(x_i) - W_3(y_i)$. This equation just reflects the fact that the fusion of a perturbed defect with another defect can be interpreted as a perturbation of the fusion of the unperturbed defects. The BRST-operators $\mathcal{Q}(t)$ appearing above still depend on the chiral fields y_i, which in the fusion process were promoted to new defect degrees of freedom. In terms of matrix factorizations, the result is an infinite dimensional matrix factorization over $\mathbb{C}[x_i, z_i]$. They can be made finite dimensional by equivalence transformations involving stripping off infinitely many trivial matrix factorizations. For the unperturbed fusion there exist polynomial matrices U, V that are inverse to each other up to BRST-equivalence and satisfy

$$U \mathcal{Q}^0 V = [\mathcal{Q}^0]^{\mathrm{red}}\,, \tag{3.10}$$

where $[\mathcal{Q}^0]^{\mathrm{red}}$ is the finite dimensional reduction of $\mathcal{Q}^0$. Once the equivalences U, V are determined for the unperturbed fusion, they can be used to map the perturbing field ψ_1 and the higher order terms ψ_n, $n > 1$ to the induced fields $\widetilde{\psi}_n$ on $[\mathcal{Q}^0]^{\mathrm{red}}$.

As discussed above, in case the perturbed defect $D(t)$ corresponds to a bound state, the same is true for $D(t) * D'$. Hence, also the deformation problem for $D(t) * D'$ is solved at first order and all $\widetilde{\psi}_n = 0$ for $n > 1$. Only $\widetilde{\psi}_1$ needs to be determined.

Consider for instance the bound state

$$P = \mathrm{Cone}(T : P^{(1)} \to P^{(2)}) \tag{3.11}$$

for some $T \in \mathcal{H}^1(P^{(1)}, P^{(2)})$.

JHEP03(2010)015

The fusion with a defect represented by a matrix factorization Q then takes the form

$$\begin{aligned} P * Q &= \mathrm{Cone}(T : P^{(1)} \to P^{(2)}) * Q \\ &= \big[\mathrm{Cone}(T : P^{(1)} \to P^{(2)}) \otimes Q\big]^{\mathrm{red}} \\ &= \big[\mathrm{Cone}(T \otimes \mathrm{id} : P^{(1)} \otimes Q \to P^{(2)} \otimes Q)\big]^{\mathrm{red}} \\ &= \mathrm{Cone}(\widetilde{T} : P^{(1)} * Q \to P^{(2)} * Q) \end{aligned} \tag{3.12}$$

In the last step, the tensor product matrices have been reduced to finite dimension, and the tachyon T was transferred accordingly by means of the equivalences $P^{(i)} \otimes Q \cong P^{(i)} * Q$,.

3.3 Method to calculate fusion products

As alluded to in section 2 the most difficult part in determining the fusion $P*Q = (P\otimes Q)^{\mathrm{red}}$ of two matrix factorizations P of $W_1(x_i)-W_2(y_i)$ and Q of $W_2(y_i)-W_3(z_i)$ is to reduce their tensor product to finite dimension. Indeed, trying to find the corresponding equivalences directly on the level of matrix factorizations can be very intricate. However as put forward in [7] one can make use of the relation between maximal Cohen-Macaulay modules and matrix factorizations [31]. Namely, instead of considering the matrix factorization $Q' = P \otimes Q$ over $\mathbb{C}[x_i, z_i]$, one can equivalently consider an $R = \mathbb{C}[x_i, z_i]/(W_1(x_i) - W_3(z_i))$-module V with a projective resolution

$$\ldots \xrightarrow{v_{n+1}} V_n \xrightarrow{v_n} V_{n-1} \xrightarrow{v_{n-1}} \ldots \xrightarrow{v_1} V_0 = V \to 0\,, \tag{3.13}$$

which after a finite number of steps turns into the two-periodic complex determined by the matrix factorization Q'

$$\begin{aligned} &V_{N+2i} = Q'_0/(W_1 - W_3)Q'_0\,, && V_{N+2i+1} = Q'_1/(W_1 - W_3)Q'_1\,, \\ &v_{N+2i} = q'_0\,, && v_{N+2i+1} = q'_1\,, \end{aligned} \tag{3.14}$$

for all $i \geq 0$. Instead of reducing Q' one can now reduce V to a finite rank R-module $\widetilde{V} \cong V$ and calculate a projective resolution

$$\ldots \xrightarrow{\widetilde{v}_{n+1}} \widetilde{V}_n \xrightarrow{\widetilde{v}_n} \widetilde{V}_{n-1} \xrightarrow{\widetilde{v}_{n-1}} \ldots \xrightarrow{\widetilde{v}_1} \widetilde{V}_0 = \widetilde{V} \to 0\,, \tag{3.15}$$

which also turns two-periodic after a finite number of steps

$$\begin{aligned} &\widetilde{V}_{\widetilde{N}+2i} = S_0/(W_1 - W_3)S_0\,, && \widetilde{V}_{\widetilde{N}+2i+1} = S_1/(W_1 - W_3)S_1\,, \\ &\widetilde{v}_{\widetilde{N}+2i} = s_0\,, && \widetilde{v}_{\widetilde{N}+2i+1} = s_1\,, \end{aligned} \tag{3.16}$$

for all $i \geq 0$. Here the S_i are free $\mathbb{C}[x_i, z_i]$-modules of finite rank, and the two-periodic part of the resolution gives rise to a finite dimensional matrix factorization

$$S : S_1 \underset{s_2}{\overset{s_1}{\rightleftarrows}} S_0 \tag{3.17}$$

of $W_1(x_i) - W_3(z_i)$ over $\mathbb{C}[x_i, z_i]$. Because of projectivity, the isomorphisms $r : V \to \widetilde{V}$ and $r^* : \widetilde{V} \to V$ lift to the resolutions

$$\begin{array}{ccccccc} \ldots \xrightarrow{v_3} & V_2 & \xrightarrow{v_2} & V_1 & \xrightarrow{v_1} & V_0 \cong V & \to 0 \\ & r_2^* \big\updownarrow r_2 & & r_1^* \big\updownarrow r_1 & & r^* \big\updownarrow r & \\ \ldots \xrightarrow{\widetilde{v}_3} & \widetilde{V}_2 & \xrightarrow{\widetilde{v}_2} & \widetilde{V}_1 & \xrightarrow{\widetilde{v}_1} & \widetilde{V}_0 \cong \widetilde{V} & \to 0 \end{array}\,, \tag{3.18}$$

JHEP03(2010)015

and the r_i and r_i^* for $i > N, \widetilde{N}$ provide an equivalence of the matrix factorizations Q' and S. In this way, one can obtain a finite dimensional matrix factorization S equivalent to Q', and also determine the equivalence between the two. The latter can in particular be used to map morphisms of Q' to those of S.

4 An application: boundary flows from defects

A special case of the fusion processes discussed above is the fusion of a perturbed defect with a boundary condition. If the perturbation of the defect is unobstructed, so that it gives rise to a family of supersymmetry preserving defects, fusion with a boundary condition immediately yields a family of boundary conditions. The deformation problem on the boundary does not need to be solved again, it is solved on the level of the defect. Note that one and the same family of defects can be fused with many different boundary conditions. This means that unobstructed directions in the moduli space of different D-branes are in fact related: They are universal flat directions in the sense that they can be pulled back to the bulk using the same defect.

This holds in particular for defects, which can be obtained as bound states. Fusing such a defect with a boundary condition yields a bound state of boundary conditions, where the tachyon is induced by the one on the defect. This again means that certain tachyon condensation processes of D-branes are universal in the above sense and can be pulled back to the bulk, as has been discussed for WZW models in [13]. The advantage of the Landau-Ginzburg language is that the fusion product can easily be calculated, and that it is therefore straight forward to determine the resulting boundary flows. We will illustrate this in some examples.

4.1 Example: minimal models

4.1.1 B-type boundary conditions in minimal models

Consider a Landau-Ginzburg model with superpotential

$$W = x^d \,. \tag{4.1}$$

B-type supersymmetric boundary conditions in these models can be represented by matrix factorizations of W. All matrix factorizations of W can be obtained as cones of the elementary matrix factorizations

$$Q^\ell : Q_1^\ell = \mathbb{C}[x] \underset{x^{d-\ell}}{\overset{x^\ell}{\rightleftarrows}} \mathbb{C}[x] = Q_0^\ell \,. \tag{4.2}$$

As described in section 2, the open string spectrum between two different boundary conditions can be obtained as the BRST-cohomology $\mathcal{H}(Q^{\ell_1}, Q^{\ell_2})$ of the respective matrix factorizations Q^{ℓ_i}

JHEP03(2010)015

$$
\begin{tikzcd}
Q^{\ell_1}: \quad \mathbb{C}[x] \arrow[r, shift left, "x^{\ell_1}"] \arrow[d, "\phi_1"'] \arrow[dr, "t_1"] & \mathbb{C}[x] \arrow[l, shift left, "x^{d-\ell_1}"] \arrow[d, "\phi_0"] \arrow[dl, "t_0"'] \\
Q^{\ell_2}: \quad \mathbb{C}[x] \arrow[r, shift left, "x^{\ell_2}"] & \mathbb{C}[x] \arrow[l, shift left, "x^{d-\ell_2}"]
\end{tikzcd}
$$

A BRST-invariant fermion $t = (t_1, t_0) : Q^{\ell_1} \to Q^{\ell_2}$ has to satisfy

$$x^{d-\ell_2} t_1 + t_0 x^{\ell_1} = 0\,.$$

In the case that $\ell_1 + \ell_2 \leq d$ we can solve for t_0

$$t_0 = -t_1 x^{d-\ell_1-\ell_2}\,, \tag{4.3}$$

otherwise, if $\ell_1 + \ell_2 > d$

$$t_1 = -t_0 x^{\ell_1+\ell_2-d} \tag{4.4}$$

The fermion is BRST-exact if

$$t_1 = x^{\ell_2}\phi_1 + \phi_0 x^{\ell_1}, \quad t_0 = \phi_0 x^{d-\ell_2} + \phi_1 x^{d-\ell_1}\,. \tag{4.5}$$

Hence, the fermionic BRST-cohomology can be described as

$$t = (t_1, t_0) = (t_1, -t_1 x^{d-\ell_1-\ell_2})\,, \;\; t_1 \in x^b \mathbb{C}[x]/\langle x^a \rangle \cong \mathcal{H}^1(Q^{\ell_1}, Q^{\ell_2})\,, \tag{4.6}$$

where $a = \min\{\ell_1, \ell_2\} - 1$ and $b = \max\{d - \ell_1 - \ell_2, 0\}$.

The possible tachyon condensation processes have been described in the Landau-Ginzburg framework in [32]. Deformations of a single Q^ℓ are not integrable, but perturbations with defect changing fields are. As discussed above, they can be represented by cones

$$\mathrm{Cone}(t : Q^{\ell_1} \to Q^{\ell_2}) \tag{4.7}$$

In the following we will demonstrate that all these perturbations are induced by fusion with perturbed defects. The idea is to generate all boundary conditions by fusing defects with the boundary condition corresponding to the elementary matrix factorization Q^1, and to show that the boundary spectra can be induced from the defects. Let us start by introducing the defects which we will use.

4.1.2 B-type defects in minimal models

B-type defects in minimal models can be represented by matrix factorizations of the superpotential

$$W = x^d - y^d\,. \tag{4.8}$$

A nice class of such defects can be easily obtained by grouping the linear factors of

$$W(x) - W(y) = \prod_{l=1}^{d} (x - \eta_l y), \quad \eta_l = e^{\frac{2\pi i l}{d}} \tag{4.9}$$

JHEP03(2010)015

into two sets. One obtains rank-one factorizations

$$P^I : \mathbb{C}[x,y] \underset{p_0^I}{\overset{p_1^I}{\rightleftarrows}} \mathbb{C}[x,y] \tag{4.10}$$

with

$$p_1^I = \prod_{i\in I}(x-\eta_i y), \quad p_0^I = \prod_{i\in D\setminus I}(x-\eta_i y), \tag{4.11}$$

and $D = \{1,\dots,d\}$ and $I \subset D$.

4.1.3 Inducing boundary flows by defects

It is not difficult to see [7] that an elementary matrix factorization Q^ℓ can be obtained by fusing any defect matrix factorization P^I with $|I| = \ell$ with Q^1

$$P^I * Q^1 \cong Q^{|I|}\,. \tag{4.12}$$

Namely, as described in section 3.3 to reduce the infinite dimensional matrix factorization $Q' = P^I \otimes Q^1$ we consider the $R = \mathbb{C}[x]/\langle x^d\rangle$-module $V = \mathrm{coker}(p_1^I \otimes \mathrm{id}_{Q_0}, \mathrm{id}_{P_0} \otimes q_1^1)$ and its R-free resolution

$$\dots \xrightarrow{q_0'} Q_1' \xrightarrow{q_1'} Q_0' \xrightarrow{q_1'} Q_1' \xrightarrow{(p_1^I, q_1^1)} P_0 \otimes Q_0 \to V \to 0\,, \tag{4.13}$$

which turns into the matrix factorization Q' after two steps. But now as an R-module

$$V = \mathrm{coker}(p_1^I, q_1^1) \cong \mathbb{C}[x,y]/\langle \prod_{i\in I}(x-\eta_i y), y\rangle \cong \mathbb{C}[x]/\langle x^{|I|}\rangle =: \widetilde{V}\,, \tag{4.14}$$

which has a two-periodic resolution

$$\dots \xrightarrow{x^{d-|I|}} R \xrightarrow{x^{|I|}} R \xrightarrow{x^{d-|I|}} R \xrightarrow{x^{|I|}} R \to \widetilde{V} \to 0\,, \tag{4.15}$$

corresponding to the matrix factorization $Q^{|I|}$. Hence, $Q' \cong Q^{|I|}$.

Next we will show that also the boundary condition changing spectra between elementary boundary conditions specified by Q^{ℓ_1} and Q^{ℓ_2} can be induced from defect changing spectra of P^{I_1} and P^{I_2}, $|I_i| = \ell_i$ upon fusion with Q^1.

The spectrum between the defects P^{I_1} and P^{I_2} depends very much on the divisibility properties of $p_1^{I_1}, p_1^{I_2}$. In the extreme case $I_1 = I_2$ the spectrum is purely bosonic, whereas in the case that I_1 and I_2 have no common factors it is purely fermionic. For our purposes, we are interested in having many fermionic defect changing fields, and hence we choose I_1 and I_2 such that the cardinality of their intersection is minimized. If $\ell_1 + \ell_2 \le d$, there are non-intersecting I_1 and I_2 with $|I_i| = \ell_i$. Then $p_0^{I_2}$ is always divisible by $p_1^{I_1}$. If on the other hand $\ell_1 + \ell_2 > d$ the intersection $I_1 \cap I_2$ contains at least $\ell_1 + \ell_2 - d$ elements. If I_1 and I_2 are chosen to contain exactly $\ell_1 + \ell_2 - d$ elements, $p_1^{I_1}$ is divisible by $p_0^{I_2}$. This means that the condition for BRST-closedness of a defect changing field $T : P^{I_1} \to P^{I_2}$

$$p_0^{I_2} T_1 + T_0 p_1^{I_1} = 0 \tag{4.16}$$

JHEP03(2010)015

can be solved similarly to the case of the boundary conditions Q^ℓ

$$T_0 = -\frac{p_0^{I_2}}{p_1^{I_1}} T_1, \quad \text{for } \ell_1 + \ell_2 \leq d, \tag{4.17}$$

$$T_1 = -\frac{p_1^{I_1}}{p_0^{I_2}} T_0, \quad \text{for } \ell_1 + \ell_2 > d \tag{4.18}$$

BRST-exact fermions satisfy

$$\begin{aligned} T_1 &= \prod_{m \in I_1} (x - \eta_m y)\phi_1 + \phi_0 \prod_{m \in I_2} (x - \eta_m y) \\ T_0 &= \phi_0 \prod_{m \in D \setminus I_1} (x - \eta_m y) + \prod_{m \in D \setminus I_2} (x - \eta_m y)\phi_1 \end{aligned} \tag{4.19}$$

Thus, the fermionic BRST-cohomology can be described as

$$T = (T_1, T_0) = \left(T_1, -\frac{p_0^{I_2}}{p_1^{I_1}} T_1 \right), \quad T_1 \in \prod_{i \in I_1 \cap I_2} (x - \eta_i y)\mathbb{C}[x,y]/\langle p_1^{I_1}, p_1^{I_2} \rangle \,. \tag{4.20}$$

Next, we will show that upon fusion with Q^1 these defect changing spectra indeed induce the boundary condition changing spectra between the respective Q^{ℓ_i}.

To show that this is the case, we first determine the equivalence of the matrix factorizations $Q' = P^I \otimes Q^1$ and Q^ℓ with $\ell = |I|$. This can be easily done using the method described in section 3.3. Namely, we just have to lift the isomorphism $V \cong \widetilde{V}$ to a map between the resolutions (4.13) and (4.15). Setting $\widehat{R} = \mathbb{C}[x,y]/\langle W(x) \rangle$ we have to construct the R-module homomorphisms r, r^*, r_i, r_i^* in

$$\begin{array}{ccccccccc} \dots & \xrightarrow{q_1'} & \widehat{R}^2 & \xrightarrow{q_0'} & \widehat{R}^2 & \xrightarrow{(p_1^I, y)} & \widehat{R} & \to \operatorname{coker}(p_1^I, y) & \to 0 \\ & & r_2^* \Big\updownarrow r_2 & & r_1^* \Big\updownarrow r_1 & & r^* \Big\updownarrow r & \cong \Big\updownarrow & \\ \dots & \xrightarrow{x^\ell} & R & \xrightarrow{x^{d-\ell}} & R & \xrightarrow{x^\ell} & R & \to \operatorname{coker}(x^\ell) & \to 0 \end{array} \,. \tag{4.21}$$

The isomorphism can be lifted in the following way:

$$\begin{aligned} & r : (1 \mapsto 1\,,\ y^i \mapsto 0)\,,\ r^* : 1 \mapsto 1\,, \\ & r_1 = r \circ (1,\ 0)\,, \qquad r_1^* = \begin{pmatrix} 1 \\ \frac{1}{y}(-p_1^I + x^\ell) \end{pmatrix} \circ r^*\,, \\ & r_2 = r \circ (1,\ 0)\,, \qquad r_2^* = \begin{pmatrix} 1 \\ \frac{1}{y}(-p_0^I + x^{d-\ell}) \end{pmatrix} \circ r^*\,, \\ & \dots\,. \end{aligned} \tag{4.22}$$

Note that $p_1^I = x^\ell + y(\dots)$ and $p_0^I = x^{d-\ell} + y(\dots)$ so that all the morphisms are well defined. As discussed in section 3.3 the morphisms r_1, r_1^*, r_2, r_2^* indeed provide the equivalence of the matrix factorizations $Q' = P^I \otimes Q^1$ and Q^ℓ, and they can be used to transfer defect changing fields $T : P^{I_1} \to P^{I_2}$ to boundary condition changing fields $t : Q^{\ell_1} \to Q^{\ell_2}$. Upon

JHEP03(2010)015

fusion with Q^1, a defect changing field T is transferred to a boundary condition changing field $T \otimes \mathrm{id}_{Q^1}$ on Q'. By means of the equivalence one obtains

$$\begin{aligned} t_1 &= r_2^{(2)} \circ (T \otimes \mathrm{id}_{Q^1})_1 \circ r_1^{(1)*} = r \circ T_1 \circ r^* \,, \\ t_0 &= r_1^{(2)} \circ (T \otimes \mathrm{id}_{Q^1})_0 \circ r_2^{(1)*} = r \circ T_0 \circ r^* \,, \end{aligned} \tag{4.23}$$

where $r_i^{(i)}, r_i^{(i)*}$ denote the equivalences of $P^{I_i} \otimes Q^1 \cong Q^{\ell_i}$ respectively. Thus, the t_i are obtained from the T_i by setting $y = 0$. Comparing the fermionic spectra (4.20) of defect changing fields $P^{I_1} \to P^{I_2}$ and the ones (4.6) of boundary condition changing fields $Q^{\ell_1} \to Q^{\ell_2}$, one finds that the entire fermionic boundary spectra can be induced by defect changing fields upon fusion with Q^1. Thus, for A-type minimal models, all boundary RG flows can be pulled back to the bulk using defects.

4.2 Example: minimal model orbifolds

As a next example, we consider the $\mathbb{Z}_d$-orbifold of the Landau-Ginzburg model with superpotential $W = x^d$, where the orbifold group acts on the chiral superfield x by phase multiplication. In fact, these orbifold theories are mirror to the original unorbifolded Landau-Ginzburg models, and B-type defects and boundary conditions in the orbifolds correspond to A-type defects and boundary conditions in the unorbifolded theories. As mentioned in section 2, B-type boundary conditions and matrix factorizations in Landau-Ginzburg orbifolds are represented by equivariant matrix factorizations of the respective superpotentials. For instance the $\mathbb{Z}_d$-equivariant rank one factorizations of W are given by

$$Q_m^\ell : \mathbb{C}[x][m+\ell] \underset{x^{d-\ell}}{\overset{x^\ell}{\rightleftarrows}} \mathbb{C}[x][m] \,, \tag{4.24}$$

where the $\mathbb{Z}_d$-representation on $\mathbb{C}[x][m]$ is specified by the action of the generator on $1 \in \mathbb{C}[x]$: $1 \mapsto e^{\frac{2\pi i m}{d}} 1$. Also in the orbifold models all matrix factorizations can be obtained by cones of such rank-one factorizations.

B-type defects in the orbifold models are represented by $\Gamma = \mathbb{Z}_d \times \mathbb{Z}_d$-equivariant matrix factorizations of $W(x) - W(y)$, where the first $\mathbb{Z}_d$ acts on x only, and the second one on y. Indeed, by means of the orbifold construction, one can obtain such factorizations out of the non-equivariant P^I defined in equation (4.10). Roughly speaking, one chooses a representation of the stabilizer subgroup $\Gamma_{\mathrm{stab}} \subset \Gamma$ under which P^I is invariant and then takes its $\Gamma/\Gamma_{\mathrm{stab}}$-orbit. In this case Γ_{stab} is the diagonal $\mathbb{Z}_d$-subgroup, so one obtains a sum

$$P_m^I = \bigoplus_{\gamma \in \mathbb{Z}_d} P^{\gamma(I)}[m] \,, \tag{4.25}$$

where the representation of the diagonal $\mathbb{Z}_d$-subgroup is indicated by $(\cdot)[m]$ and

$$\gamma(\{i_1, \ldots, i_r\}) = \{i_1 + \gamma \bmod d, \ldots, i_r + \gamma \bmod d\}$$

is the cyclic shift of I. This is a diagonal d-dimensional matrix factorization, on which the action of Γ is non-diagonal however. But it can be diagonalized. Denoting the basis in

JHEP03(2010)015

which the matrix factorization is diagonal by e_i, we can diagonalize the Γ-action by the change of basis

$$\hat{e}_n = \sum_{i=1}^{d} \eta^{\text{in}} e_i \tag{4.26}$$

which is inverted by

$$e_m = \frac{1}{d} \sum_i \eta^{mi} \hat{e}_i . \tag{4.27}$$

In the basis $\hat{e}_n$, P^I_M takes the form

$$P^I_m : \mathbb{C}[x,y]^d \begin{pmatrix} {\scriptstyle [m+|I|,0]} \\ {\scriptstyle [m+1+|I|,-1]} \\ \vdots \\ {\scriptstyle [m+(d-1)+|I|,-(d-1)]} \end{pmatrix} \underset{p^I_0(X,Y)}{\overset{p^I_1(X,Y)}{\rightleftarrows}} \mathbb{C}[x,y]^d \begin{pmatrix} {\scriptstyle [m,0]} \\ {\scriptstyle [m+1,-1]} \\ \vdots \\ {\scriptstyle [m+(d-1),-(d-1)]} \end{pmatrix} , \tag{4.28}$$

where X and Y are the $d \times d$-matrices $X = x1_d$ and $Y = y\Xi$, with Ξ the d-dimensional shift matrix

$$\Xi_{ab} = \delta^{(d)}_{a,b+1} . \tag{4.29}$$

Note that because of the orbit formation in the orbifold construction not all of the P^I_m are inequivalent. In fact, $P^I_m \cong P^J_m$ if J is a (cyclic) shift of I, *i.e.* $J = I + n \bmod d$.

The fusion of P^I_m with Q^1_0 can be easily calculated. Using the method already employed in section 4.1.2, one obtains

$$P^I_m * Q^1_0 \cong \bigoplus_{i \in \mathbb{Z}_d} Q^{|I|}_{m+i}[-i] , \tag{4.30}$$

where here $\cdot[-i]$ denotes the representation of the second $\mathbb{Z}_d$, the orbifold group of the model squeezed in between defect and boundary. The fusion in the orbifold model is the Γ_{squeezed}-invariant part of (4.30)

$$P^I_m *_{\text{orb}} Q^1_0 \cong Q^{|I|}_m . \tag{4.31}$$

Hence, also in the orbifold theory, one can generate all elementary matrix factorizations Q^ℓ_m by fusing defect factorizations P^I_m with Q^1_0.

Indeed, there is another way to obtain the fusion (4.30). Namely, we one can use the diagonal form (4.25) P^I_m, which is a direct sum of the ordinary rank-one factorizations P^I. In this way, one reduces the problem to the problem in the non-orbifolded situation discussed in section 4.1.2. Since the result of the fusion of P^I with Q^1 only depends on the cardinality of I, the fusion $P^I_m * Q^1_0$ just gives a direct sum of d equal summands $Q^{|I|}$. To bring this in the basis in which the Γ-action is diagonal, we have to do the change of basis (4.26), (4.27). Being diagonal, this does not change the result however, and we arrive at (4.30).

Having established that one can generate all Q^ℓ_m by fusing defects P^I_m with Q^1_0, we would like to show next that also the entire spectra of boundary operators $Q^{\ell_1}_{m_1} \to Q^{\ell_2}_{m_2}$ can be induced upon fusion with Q^1_0 by defect changing operators $P^{I_1}_{m_1} \to P^{I_2}_{m_2}$ with $|I_i| = \ell_i$.

JHEP03(2010)015

As in the unorbifolded theory we choose I_1, I_2 in such a way that $|I_1 \cap I_2|$ is minimized. Indeed, it is easy to see that for a given $T(x,y) \in \mathcal{H}^1(P^{I_1}, P^{I_2})$ in the unorbifolded theory of degree $\ell_1 + m_1 - m_2 \bmod d$ for T_1 and degree $\ell_2 + m_2 - m_1$ for T_0, $T(X,Y) \in \mathcal{H}^1_{\text{orb}}(P^{I_1}_{m_1}, P^{I_2}_{m_2})$. To transfer this boundary condition changing field to the fused boundary we make use of the diagonal form (4.25). This allows us to reduce the problem to the unorbifolded problem. Using the equivalences in the unorbifolded case (4.23) and then projecting to the invariant part of the fusion product, we obtain

$$t = r \circ T \circ r^* = T(x, 0) \tag{4.32}$$

for the boundary changing field. As in the unorbifolded case, comparing defect and boundary BRST-cohomology we find that before the orbifold projection we obtain the entire boundary spectra this way. But of course also the projections agree. Hence, we arrive at the conclusion that in the orbifold models as well the entire boundary changing spectra can be induced from defect changing fields by fusion of defects P^I_m with Q^1_0.

4.3 Example: tensor products of identical LG models

Another simple example is the tensor product of two identical minimal models. It turns out that this is not any simpler than the more general case of a product of two arbitrary identical Landau-Ginzburg models. In fact, for ease of notation we will consider tensor products of Landau-Ginzburg models with their conjugates in the following. That means the models have chiral superfields $x_1, \ldots, x_N, \widehat{x}_1, \ldots, \widehat{x}_N$ and superpotential $W(x_i) - W(\widehat{x}_i)$ instead of $W(x_i) + W(\widehat{x}_i)$. The construction below easily carries over to the tensor product of identical Landau-Ginzburg models.

Now, given any matrix factorization $Q(x_i, \widehat{x}_i)$ of $W(x_i) - W(\widehat{x}_i)$, we define the defect matrix factorization of $W(x_i) - W(\widehat{x}_i) - W(y_i) + W(\widehat{y}_i)$ as the tensor product

$$P_Q := Q(x_i, y_i) \otimes \mathrm{Id}(\widehat{x}_i, \widehat{y}_i)\,, \tag{4.33}$$

where $\mathrm{Id}(\widehat{x}_i, \widehat{y}_i)$ is the matrix factorization representing the identity defect in the second tensor factor.

Of course, fusing P_Q with the identity matrix factorization

$$E = \mathrm{Id}(y_i, \widehat{y}_i) \tag{4.34}$$

between the two tensor factors gives back the matrix factorization Q:

$$P_Q * E \cong Q\,. \tag{4.35}$$

Here one only needs to make repeated use of the fact that $\mathrm{Id} * P \cong P$ for any matrix factorization P. Hence, in these models every matrix factorization Q can be obtained by fusing a defect matrix factorization P_Q with a fixed matrix factorization E. This is true in particular for families $Q(t)$ of matrix factorizations. Therefore, all perturbations of boundary conditions can be pulled back into the bulk by means of the defects P_Q in these models.

JHEP03(2010)015

Of course it is also clear from the above that the defect changing fields

$$\mathcal{H}^1(Q^1, Q^2) \otimes \mathrm{id}_{\mathrm{Id}} \subset \mathcal{H}^1(P_{Q^1}, P_{Q^2}) \tag{4.36}$$

induce the respective boundary condition changing fields between the fused boundaries $P_{Q^i} * E \cong Q^i$.

In fact, the arguments used to arrive at this conclusion do not depend on the matrix factorization formalism, so we expect that the result carries over to tensor products of arbitrary $N = (2,2)$-theories with their conjugates.

4.4 Defect induced boundary flows in CFT

The fusion of perturbed defects has been considered on the level of the full conformal field theory for rational models with diagonal modular invariants in [5]. There, only defects which preserve both, the holomorphic and antiholomorphic W-algebras on the full complex plane were considered. These defects in particular preserve both copies of the Virasoro algebra and are therefore topological, which implies that their fusion is non-singular.

To ensure that also the perturbed defects (taken here to extend parallel to the real line) can be moved smoothly along the imaginary axis, the perturbations are restricted to chiral defect fields $\phi(z)$, $\frac{\partial}{\partial \bar{z}}\phi(z) = 0$. Defects perturbed by chiral fields still commute with the Hamiltonian generating translations along the imaginary axis, and hence can be fused smoothly with parallel defects. Moreover, the perturbations are further restricted in [5] by demanding that only fields in a single fixed representation occur.

The result of the fusion of two defects perturbed in this way is obtained as a bunch of defects resulting from the fusion of the unperturbed defects perturbed again by defect changing fields in the fixed representation.

For rational CFTs with charge conjugate modular invariant, defect operators corresponding to topological defects can immediately be written down [3]

$$D_J = \sum_j \frac{S_{Jj}}{S_{0j}} \mathrm{P}_{j\bar{j}} \ . \tag{4.37}$$

Here, J, j specify irreducible representations of the chiral symmetry algebra and take values in some index set $\mathcal{I}$. $\mathrm{P}_{j\bar{j}}$ are projection operators on the representation spaces $\mathcal{V}_j \otimes \mathcal{V}_{\bar{j}}$, and S denotes the modular S-matrix of the respective characters χ_j. Fusion of these defects can be obtained by composing the respective operators, and using the Verlinde formula, one easily obtains

$$D_J * D_{J'} = \sum_{J''} \mathcal{N}_{JJ'}^{J''} D_{J''}, \tag{4.38}$$

where $\mathcal{N}$ denote the fusion rule coefficients.

Likewise, boundary states for symmetry preserving boundary conditions are given by Cardy's formula

$$\|J\rangle\rangle = \sum_j \frac{S_{Jj}}{\sqrt{S_{0j}}} |j\rangle\rangle \ , \tag{4.39}$$

where $|j\rangle\rangle$ denote the Ishibashi states in the sector $\mathcal{V}_j \otimes \overline{\mathcal{V}}_{\bar{j}}$.

JHEP03(2010)015

Fusion of topological defects with boundary conditions can be calculated by applying the respective defect operators to the boundary states. For the defects and boundary states above this yields

$$D_J \|J'\rangle\rangle = \sum_{J''} \mathcal{N}_{JJ'}^{J''} \|J''\rangle\rangle \ . \tag{4.40}$$

This implies in particular that all Cardy boundary conditions $\|J\rangle\rangle$ can be obtained by fusing the topological defects D_J with the boundary condition $\|0\rangle\rangle$ associated to the vacuum representation. This is very much like in the case of B-type boundary conditions in Landau-Ginzburg models discussed in sections 4.1 and 4.2 above, where all boundary conditions can be produced by fusing defects with a fixed linear matrix factorization.

The spectra of defect changing fields between the defects D_J and the spectrum of boundary condition changing fields between the Cardy boundary conditions $\|J\rangle\rangle$ can be easily determined in the RCFT setting

$$\mathcal{H}_{JJ'}^{D} = \bigoplus_{j,j',j''} \mathcal{N}_{jj'}^{j''} \mathcal{N}_{j''J'}^{J} \mathcal{V}_j \otimes \overline{\mathcal{V}}_{j'} \ , \tag{4.41}$$

$$\mathcal{H}_{JJ'}^{B} = \bigoplus_{j} \mathcal{N}_{J'j}^{J} \mathcal{V}_j \ . \tag{4.42}$$

The chiral defect changing fields are obtained by setting $j' = 0$ in (4.41). One immediately finds that the space of chiral defect changing fields between defects D_J and $D_{J'}$ is isomorphic to the space of boundary condition changing fields between $\|J\rangle\rangle$ and $\|J'\rangle\rangle$.

This implies that indeed all boundary perturbations can be obtained by fusing the boundary condition $\|0\rangle\rangle$ with chirally perturbed defects.

For instance a boundary flow of a sum of boundary conditions $\|J\rangle\rangle \oplus \|J'\rangle\rangle$ generated by a boundary condition changing field in representation j can be obtained by fusing boundary condition $\|0\rangle\rangle$ with a defect $D_J \oplus D_{J'}$ perturbed by the chiral defect changing field in that same representation.[7]

On a formal level this is quite similar to what we have seen for matrix factorizations: All boundary conditions can be generated by fusing defects with a particular boundary condition, and all boundary condition changing fields can be induced by choosing particular defect changing fields.

To make the relation completely precise, one can consider the example of the supersymmetric minimal model with A-type boundary conditions and defects, or equivalently B-type boundary conditions and defects in the orbifold of the minimal model. Here both a description in terms of matrix factorizations and in terms of rational conformal field theory is available. On the level of matrix factorizations this is the Landau-Ginzburg orbifold discussed in section 4.2.

The $N = (2,2)$-superconformal minimal models $\mathcal{M}_k$ are rational with respect to the $N = 2$ super Virasoro algebra at central charge $c_k = \frac{3k}{k+2}$. In fact, the bosonic part of this algebra can be realized as the coset W-algebra

$$(\mathrm{SVir}_{c_k})_{\mathrm{bos}} = \frac{\widehat{\mathfrak{su}}(2)_k \oplus \widehat{\mathfrak{u}}(1)_4}{\widehat{\mathfrak{u}}(1)_{2k+4}} \ , \tag{4.43}$$

[7] Representations of fields are not changed when they are transferred in the fusion process.

JHEP03(2010)015

and the respective coset CFT can be obtained from $\mathcal{M}_k$ by a non-chiral GSO projection.

The Hilbert space $\mathcal{H}^k$ of $\mathcal{M}_k$ decomposes into irreducible highest weight representations of holomorphic and antiholomorphic super Virasoro algebras, but it is convenient to decompose it further into irreducible highest weight representations $\mathcal{V}_{[l,m,s]}$ of the bosonic subalgebra (4.43). These representations are labelled by

$$[l,m,s] \in \mathcal{I}_k := \{(l,m,s)\,|\,0 \le l \le k,\, m \in \mathbb{Z}_{2k+4},\, s \in \mathbb{Z}_4,\, l+m+s \in 2\mathbb{Z}\}/\sim\,, \tag{4.44}$$

where $[l,m,s] \sim [k-l, m+k+2, s+2]$ is the field identification. The highest weight representation of the full super Virasoro algebra are given by

$$\mathcal{V}_{[l,m]} := \mathcal{V}_{[l,m,(l+m)\bmod 2]} \oplus \mathcal{V}_{[l,m,(l+m)\bmod 2+2]}\,. \tag{4.45}$$

For $(l+m)$ even $\mathcal{V}_{[l,m]}$ is in the NS-, for $(l+m)$ odd in the R-sector. Here $[l,m] \in \mathcal{J}_k := \{(l,m)\,|\,0 \le l \le k,\, m \in \mathbb{Z}_{2k+4}\}/\sim$, $[l,m] \sim [k-l, m+k+2]$. The Hilbert spaces of $\mathcal{M}_k$ in the NSNS- and RR-sectors then read

$$\mathcal{H}^k_{NSNS} \cong \bigoplus_{\substack{[l,m]\in\mathcal{J}_k \\ l+m\,\text{even}}} \mathcal{V}_{[l,m]} \otimes \overline{\mathcal{V}}_{[l,m]}\,, \qquad \mathcal{H}^k_{RR} \cong \bigoplus_{\substack{[l,m]\in\mathcal{J}_k \\ l+m\,\text{odd}}} \mathcal{V}_{[l,m]} \otimes \overline{\mathcal{V}}_{[l,m]}\,. \tag{4.46}$$

To obtain a CFT with a modular invariant partition function from this fully supersymmetric theory, one needs to perform a GSO projection. In the case at hand, there are two possibilities, a type 0A and a type 0B projection, distinguished by the action of $(-1)^F$. We will consider the type 0B case, where states in the sector $\mathcal{V}_{[l,m,s]} \otimes \mathcal{V}_{[l,m,-s]}$ are invariant under the projection.

The defects of the theory with either GSO projection have been given in [7]. The general form of the defect operators in the Cardy case is

$$\mathcal{D} = \sum_{\substack{[l,m,s],\bar{s} \\ s-\bar{s}\,\text{even}}} \mathcal{D}^{[l,m,s,\bar{s}]} \mathrm{P}_{[l,m,s,\bar{s}]}\,, \tag{4.47}$$

where $\mathrm{P}_{[l,m,s,\bar{s}]}$ is a projector on the modules $\mathcal{V}_{[l,m,s]} \otimes \overline{\mathcal{V}}_{[l,m,\bar{s}]}$ of the bosonic subalgebra. The solutions for the coefficients are given by

$$\mathcal{D}^{[l,m,s,\bar{s}]}_{[L,M,S,\bar{S}]} = e^{-i\pi\frac{\bar{S}(s+\bar{s})}{2}} \frac{S_{[L,M,S-\bar{S}][l,m,s]}}{S_{[0,0,0],[l,m,s]}}\,, \tag{4.48}$$

where the different defects are specified by $[L,M,S,\bar{S}]$ with $[L,M,S-\bar{S}] \in \mathcal{I}_k$, and

$$S_{[L,M,S][l,m,s]} = \frac{1}{k+2} e^{-i\pi\frac{Ss}{2}} e^{i\pi\frac{Mm}{k+2}} \sin\left(\pi\frac{(L+1)(l+1)}{k+2}\right) \tag{4.49}$$

is the modular S-matrix for the coset representations $\mathcal{V}_{[l,m,s]}$.

In the orbifold theory, modding out the $\mathbb{Z}_{k+2}$ phase symmetry acting on the $\mathfrak{u}_{2k+4}$ labels projects the Hilbert space of $\mathcal{M}_k$ on the subsector with $m=0$. Together with the twisted sectors, the new Hilbert space takes the form

$$\mathcal{H}^k_{NSNS} \cong \bigoplus_{\substack{[l,m]\in\mathcal{J}_k \\ l+m\,\text{even}}} \mathcal{V}_{[l,m]} \otimes \overline{\mathcal{V}}_{[l,-m]}\,, \qquad \mathcal{H}^k_{RR} \cong \bigoplus_{\substack{[l,m]\in\mathcal{J}_k \\ l+m\,\text{odd}}} \mathcal{V}_{[l,m]} \otimes \overline{\mathcal{V}}_{[l,-m]}\,. \tag{4.50}$$

JHEP03(2010)015

The defect operators of the orbifold theory look very similar to that of the original theory:

$$\mathcal{D}^{\text{orb}} = \sum_{\substack{[l,m,s],\bar{s} \\ s-\bar{s}\,\text{even}}} \mathcal{D}^{[l,m,s,\bar{s}]} \mathrm{P}^{\text{orb}}_{[l,m,s,\bar{s}]}\,, \tag{4.51}$$

where now $\mathrm{P}^{\text{orb}}_{[l,m,s,\bar{s}]}$ is a projector on $\mathcal{V}_{[l,m,s]} \otimes \mathcal{V}_{[l,-m,\bar{s}]}$. The coefficients are given by (4.48) just like in the unorbifolded case.

For this reason, also the fusion algebra between defects in the orbifold theory is the same as the one in the original unorbifolded model [7]

$$\mathcal{D}^{\text{orb}}_{[L_1,M_1,S_1,\bar{S}_1]} \mathcal{D}^{\text{orb}}_{[L_2,M_2,S_2,\bar{S}_2]} = \sum_L \mathcal{N}^L_{L_1 L_2} \mathcal{D}^{\text{orb}}_{[L,M_1+M_2,S_1+S_2,\bar{S}_1+\bar{S}_2]}\,. \tag{4.52}$$

For $L = 0$ these defects are group like. For the original theory, the defects $\mathcal{D}_{[0,M,0,0]}$ realize the action of the orbifold group, whereas in the orbifold theory, the $\mathcal{D}^{\text{orb}}_{[0,M,0,0]}$ realize the corresponding quantum symmetry.

For later use, we calculate the defect changing spectrum. For this, we use the folding trick and map the defects to permutation boundary states in the doubled theory. These boundary states have been analyzed in detail for the unorbifolded case [33, 34]. To summarize, in the original, unorbifolded theory, the permutation B type boundary states are given by

$$\begin{aligned} &\|[L,M,S_1,S_2]\rangle\rangle \\ &= \frac{1}{2\sqrt{2}} \sum_{l,m,s_1,s_2} \frac{S_{Ll}}{S_{0l}}\, e^{i\pi Mm/(k+2)}\, e^{-i\pi(S_1 s_1 - S_2 s_2)/2}\, |[l,m,s_1] \otimes [l,-m,-s_2]\rangle\rangle^{\sigma}\,, \end{aligned} \tag{4.53}$$

where the sum runs over all l, m, s_1 and s_2 for which

$$l + m + s_1 \quad \text{and} \quad s_1 - s_2 \quad \text{are even.} \tag{4.54}$$

Here, $|[l,m,s_1] \otimes [l,-m,-s_2]\rangle\rangle^{\sigma}$ are B-type permutation Ishibashi states in the sectors

$$\left(\mathcal{V}_{[l,m,s_1]} \otimes \mathcal{V}_{[l,-m,-s_2]}\right) \otimes \left(\bar{\mathcal{V}}_{[l,m,s_2]} \otimes \bar{\mathcal{V}}_{[l,-m,-s_1]}\right), \tag{4.55}$$

which means that they intertwine the respective supersymmetry algebras of the two tensor factors. In the orbifold theory, the boundary states are similarly given by

$$\begin{aligned} &\|[L,M,S_1,S_2]\rangle\rangle_{\text{orb}} \\ &= \frac{1}{2\sqrt{2}} \sum_{l,m,s_1,s_2} \frac{S_{Ll}}{S_{0l}}\, e^{i\pi Mm/(k+2)}\, e^{-i\pi(S_1 s_1 - S_2 s_2)/2}\, |[l,m,s_1] \otimes [l,m,-s_2]\rangle\rangle^{\sigma}_{\text{orb}}\,, \end{aligned} \tag{4.56}$$

where now the permutation Ishibashi states with $m \neq 0$ come from the twisted sectors

$$\left(\mathcal{V}_{[l,m,s_1]} \otimes \mathcal{V}_{[l,m,-s_2]}\right) \otimes \left(\bar{\mathcal{V}}_{[l,m,s_2]} \otimes \bar{\mathcal{V}}_{[l,m,-s_1]}\right). \tag{4.57}$$

These boundary states can be obtained directly from the defect (4.51) by means of the folding trick. It can also be obtained from the un-orbifolded boundary states (4.53) using

JHEP03(2010)015

the orbifold construction. To see this, note that the boundary states are invariant under the diagonal $\mathbb{Z}_{k+2} \subset \mathbb{Z}_{k+2} \times \mathbb{Z}_{k+2}$, leading to resolved boundary states labelled by an additional $\mathbb{Z}_{k+2}$-representation label M', which specifies the representation in the boundary sectors. Orbifolding by the second $\mathbb{Z}_{k+2}$-factor introduces an orbit of boundary states of different M. The result is (4.56). In our notation we do not distinguish between labels M and M' although from the orbifold point of view these labels play different roles.

Note that the permutation boundary states of the two theories differs only in a minus sign in front of one of the m labels in the Ishibashi states. As a consequence, also the one-loop amplitudes are almost identical to the ones

$$\begin{aligned}
\langle\!\langle [L,M,S_1,S_2] \| q^{\frac{1}{2}(L_0+\bar{L}_0)-\frac{c}{12}} \| [\hat{L},\hat{M},\hat{S}_1,\hat{S}_2] \rangle\!\rangle = & \sum_{[l_i',m_i',s_i']} \chi_{[l_1',m_1',s_1']}(\tilde{q})\, \chi_{[l_2',m_2',s_2']}(\tilde{q}) \\
& \sum_{\hat{l}} \Big[\mathcal{N}_{\hat{l}\hat{L}}{}^{L}\, \mathcal{N}_{l_1'l_2'}{}^{\hat{l}}\, \delta^{(2k+4)}(\Delta M + m_1' - m_2') \\
& \times \Big(\delta^{(4)}(\Delta S_1 + s_1')\, \delta^{(4)}(\Delta S_2 + s_2') + \delta^{(4)}(\Delta S_1 + 2 + s_1')\, \delta^{(4)}(\Delta S_2 + 2 + s_2') \Big) \\
& + \mathcal{N}_{\hat{l}\, k-\hat{L}}{}^{L}\, \mathcal{N}_{l_1'l_2'}{}^{\hat{l}}\, \delta^{(2k+4)}(\Delta M + k + 2 + m_1' - m_2') \\
& \times \Big(\delta^{(4)}(\Delta S_1 + 2 + s_1')\, \delta^{(4)}(\Delta S_2 + s_2') + \delta^{(4)}(\Delta S_1 + s_1')\, \delta^{(4)}(\Delta S_2 + 2 + s_2') \Big) \Big] ,
\end{aligned}$$

of the original unorbifolded theory. Here $\Delta M = \hat{M} - M$ and $\Delta S_i = \hat{S}_i - S_i$. In particular, we find that the boundary spectrum of $\| [0,0,0,0] \rangle\!\rangle$ is isomorphic to the bulk spectrum, which is expected, because it is isomorphic to the spectrum of defect fields on the trivial defect.

In the orbifold theory one obtains

$$\begin{aligned}
\langle\!\langle [L,M,S_1,S_2] \| q^{\frac{1}{2}(L_0+\bar{L}_0)-\frac{c}{12}} \| [\hat{L},\hat{M},\hat{S}_1,\hat{S}_2] \rangle\!\rangle_{\text{orb}} = & \sum_{[l_i',m_i',s_i']} \chi_{[l_1',m_1',s_1']}(\tilde{q})\, \chi_{[l_2',m_2',s_2']}(\tilde{q}) \\
& \sum_{\hat{l}} \Big[\mathcal{N}_{\hat{l}\hat{L}}{}^{L}\, \mathcal{N}_{l_1'l_2'}{}^{\hat{l}}\, \delta^{(2k+4)}(\Delta M + m_1' + m_2') \\
& \times \Big(\delta^{(4)}(\Delta S_1 + s_1')\, \delta^{(4)}(\Delta S_2 + s_2') + \delta^{(4)}(\Delta S_1 + 2 + s_1')\, \delta^{(4)}(\Delta S_2 + 2 + s_2') \Big) \\
& + \mathcal{N}_{\hat{l}\, k-\hat{L}}{}^{L}\, \mathcal{N}_{l_1'l_2'}{}^{\hat{l}}\, \delta^{(2k+4)}(\Delta M + k + 2 + m_1' + m_2') \\
& \times \Big(\delta^{(4)}(\Delta S_1 + 2 + s_1')\, \delta^{(4)}(\Delta S_2 + s_2') + \delta^{(4)}(\Delta S_1 + s_1')\, \delta^{(4)}(\Delta S_2 + 2 + s_2') \Big) \Big] ,
\end{aligned}$$

where as before $\Delta M = \hat{M} - M$ and $\Delta S_i = \hat{S}_i - S_i$. As alluded to above, the only difference between the open string spectra for the orbifold and the original theory is the sign with which m_2 enters. The reason for this is of course that in the bulk of the orbifold theory $\mathcal{V}_{[l,m,s]}$ is paired with $\bar{\mathcal{V}}_{[l,-m,s]}$ instead of $\bar{\mathcal{V}}_{[l,m,s]}$ so that the B-type permutation Ishibashi states are from a conjugate sector compared to the original theory. A modular transformation to the open string sector then leads to a sign flip for m_2', which is the only difference.

Let us now discuss the B-type boundary states in minimal models and their spectra.

JHEP03(2010)015

For the original unorbifolded model, the boundary states are given by

$$\|L,S\rangle\rangle = \sqrt{k+2}\sum_{l+s\in 2\mathbb{Z}} \frac{S_{[L,0,S],[l,0,s]}}{\sqrt{S_{[l,0,s],[0,0,0]}}}\,|[l,0,s]\rangle\rangle\,. \tag{4.58}$$

These boundary states are not of Cardy type, but require an additional projection on Ishibashi states that satisfy B-type boundary conditions.

This is different in the orbifold theory, where indeed the standard Cardy construction can be applied to B-type boundary conditions. The Ishibashi states $|[l,m,s]\rangle\rangle_{\rm orb}$ are from the sectors $\mathcal{V}_{[l,m,s]}\otimes\bar{\mathcal{V}}_{[l,-m,-s]}$ and the boundary states are explicitly given by

$$\|[L,M,S]\rangle\rangle_{\rm orb} = \sum_{[l,m,s]} \frac{S_{[L,M,S][l,m,s]}}{\sqrt{S_{[0,0,0][l,m,s]}}}|[l,m,s]\rangle\rangle_{\rm orb}\,. \tag{4.59}$$

These boundary states can of course also be obtained from the states (4.58) by the orbifold construction. The spectrum of boundary condition changing fields between two such orbifold boundary conditions is given by the partition function

$$\begin{aligned}&\langle\langle[L,M,S]|q^{\frac{1}{2}(L_0+\bar{L}_0)-\frac{c}{24}}\|[L',M',S']\rangle\rangle_{\rm orb}\\ &\quad=\sum_{[l,m,s]}\big(N^l_{LL'}\delta^{(4)}(S'-S+s)\delta^{(2k+4)}(M'-M+m)\\ &\qquad+N^{k-l}_{LL'}\delta^{(4)}(S'-S+s+2)\delta^{(2k+4)}(M'-M+m+k+2)\big)\chi_{[l,m,s]}(q)\end{aligned} \tag{4.60}$$

Being a special case of an RCFT with diagonal modular invariant, one expects from the general discussion in section 4.4 that all supersymmetry preserving boundary flows between B-type boundary conditions in the orbifold theory should be generated by fusion of chirally perturbed B-type topological defects with the boundary condition $\|[0,0,0]\rangle\rangle_{\rm orb}$. Indeed, fusion of the defects $\mathcal{D}^{\rm orb}_{[L_1,M_1,S_1,\bar{S}_1]}$ and boundary conditions $\|[L_2,M_2,S_2]\rangle\rangle$ is given by

$$\mathcal{D}^{\rm orb}_{[L_1,M_1,S_1,\bar{S}_1]}\|[L_2,M_2,S_2]\rangle\rangle_{\rm orb} = \sum_L \mathcal{N}^L_{L_1L_2}\|L,M_1+M_2,S_1-\bar{S}_1+S_2\rangle\rangle_{\rm orb}\,. \tag{4.61}$$

In particular $\mathcal{D}^{\rm orb}_{[L,M,S,0]}\|[0,0,0]\rangle\rangle_{\rm orb} = \|[L,M,S]\rangle\rangle_{\rm orb}$. Hence, all boundary conditions can be obtained by fusing boundary condition $\|[0,0,0]\rangle\rangle_{\rm orb}$ with defects $\mathcal{D}^{\rm orb}_{[L,M,S,0]}$. Moreover, the chiral defect changing spectrum between defects $\mathcal{D}^{\rm orb}_{[L_1,M_1,S_1,0]}$ and $\mathcal{D}^{\rm orb}_{[L_2,M_2,S_2,0]}$, which can be obtained from the spectrum (4.58) of boundary condition changing operators of permutation boundary conditions in the folded model by setting $[l_2',m_2',s_2'] = [0,0,0]$ is isomorphic to the spectrum of boundary condition changing operators between boundary conditions $\|[L_1,M_1,S_1]\rangle\rangle_{\rm orb}$ and $\|[L_2,M_2,S_2]\rangle\rangle_{\rm orb}$.

Thus, in orbifolds of minimal models, we have explicitly seen that perturbations of B-type supersymmetric boundary condition can be pulled back into the bulk by chirally perturbed topological defects.

This can be compared to our discussion of boundary flows in Landau-Ginzburg orbifolds in section 4.2. Namely, the minimal model $\mathcal{M}_{d-2}$ is the IR fixed point of a Landau-Ginzburg model with superpotential $W = x^d$, and the same is true for the $\mathbb{Z}_d$-orbifolds of the respective models.

JHEP03(2010)015

Thus, B-type boundary conditions and defects in the (orbifold of the) minimal model $\mathcal{M}_{d-2}$ can nicely be represented by (equivariant) matrix factorizations. For the minimal model one obtains[8]

$$\begin{array}{rcl} Q^\ell & \leftrightarrow & \|\ell-1,0\rangle\rangle \\ P^{\{m,m+1,\ldots,m+\ell\}} & \leftrightarrow & \mathcal{D}_{[\ell,\ell+2m,0,0]} \end{array} . \tag{4.62}$$

Note that there are more defect matrix factorizations than there are topological defects in the CFT. Namely, only those matrix factorizations P^I have an interpretation as topological defects in the CFT for which I is a set of consecutive integers (mod d) [33, 34].

In the orbifold theory, the relation is

$$\begin{array}{rcl} Q_m^\ell & \leftrightarrow & \|[\ell-1,\ell-1-2m,0]\rangle\rangle \\ P_M^{\{m,m+1,\ldots,m+\ell\}} & \leftrightarrow & \mathcal{D}_{[\ell,\ell-2M,0,0]} \end{array} . \tag{4.63}$$

Note that in the orbifold model $P_M^{\{m,m+1,\ldots,m+l\}} \cong P_M^{\{m',m'+1,\ldots,m'+l\}}$.

Comparing the discussion of B-type boundary flows in the matrix factorization approach and the full CFT we find complete agreement. The boundary condition $\|[0,0,0]\rangle\rangle_{\rm orb}$ out of which all boundary conditions can be generated by means of fusion with topological defects corresponds to the matrix factorization Q_0^1, which was used in the same way in the matrix factorization approach. Of course, also the defects which are used for this purpose coincide, when we restrict to P_M^I with index sets consisting of consecutive (mod d) integers. Finally, inspection of the defect changing spectra in the CFT and the matrix factorization approach shows that we have indeed chosen the same defect changing fields to induce boundary condition changing fields in the fusion with $\|[0,0,0]\rangle\rangle_{\rm orb}$ and Q_0^1 respectively.

5 Braid group actions and defects

In string theory, actions of braid groups on D-brane categories arise in various contexts. For example one finds braid group actions on A-type branes whenever the target space manifold contains an A_m chain of Lagrangian spheres $\mathcal{L}_i$, which have intersections

$$(\mathcal{L}_i \cap \mathcal{L}_j) = \begin{cases} 1 & |i-j| = 1 \\ 0 & |i-j| > 1 \end{cases} \tag{5.1}$$

In particular, such configurations arise when the compactification manifold degenerates into a singular space with singularity of type A_m. On the level of the homology it is well-known that probe cycles undergo a Picard-Lefschetz monodromy transformation when encircling a locus where one of the spheres shrinks to zero size. This transformation acts as

$$L_{\mathcal{L}}(x) = x - \langle[\mathcal{L}]|x\rangle[\mathcal{L}] , \tag{5.2}$$

where the bracket $\langle\ldots\rangle$ denotes the intersection number between the two cycles. Such transformations satisfy the braid group relations on n strands

$$\begin{aligned} L_{\mathcal{L}_i}L_{\mathcal{L}_j}L_{\mathcal{L}_i} &= L_{\mathcal{L}_j}L_{\mathcal{L}_i}L_{\mathcal{L}_j} , \quad \text{for } |i-j| = 1 \\ L_{\mathcal{L}_i}L_{\mathcal{L}_j} &= L_{\mathcal{L}_j}L_{\mathcal{L}_i} , \quad \text{for } |i-j| > 1 . \end{aligned} \tag{5.3}$$

[8] Note that the matrix factorizations only describe B-type boundary states and defects with the same spin structure, which we chose by setting all the S-labels to zero.

JHEP03(2010)015

Picard Lefschetz transformations play a role in the context of BPS solitons (described by A-type D-branes) in Landau-Ginzburg models [35]. Here, the intersection numbers are realized as soliton numbers, and the Picard Lefschetz monodromy captures their changes under deformations of the superpotential.

A natural question is whether this braid group action extends to the level of the full topological D-brane category rather than just the charges. This question has been answered in the work of Seidel [36], who constructed braid group actions on categories of A-branes, which is mathematically described by the Fukaya category. Via mirror symmetry, this action should carry over to an action on the mirror B-brane category. Indeed, this was constructed by Seidel and Thomas in [18]. The authors introduce the notion of spherical objects E in the derived category of coherent sheaves on the target space manifold X which satisfy the condition

$$\mathrm{Ext}^i(E,E) = \begin{cases} \mathbb{C} & i = 0, n \\ 0 & i \neq 0, n \end{cases}, \tag{5.4}$$

where n is the complex dimension of X. To any such object they associate a Fourier-Mukai transformation which describes an autoequivalence of the derived category of coherent sheaves on X. The kernel of the Fourier-Mukai transformation

$$\mathcal{K}^{\mathrm{C}}_{\mathcal{Q}} = \mathrm{Cone}(r : \mathcal{Q} \boxtimes \mathcal{Q}^\vee \to \mathcal{O}_{\Delta X}), \tag{5.5}$$

is determined by the large volume complex $\mathcal{Q}$ associated to the B-brane Q [18, 37]. Here $\mathcal{Q}^\vee$ denotes the dual of $\mathcal{Q}$, and $\mathcal{Q} \boxtimes \mathcal{Q}^\vee = \pi_1^*(\mathcal{Q}) \otimes \pi_2^*(\mathcal{Q}^\vee)$ is obtained by tensoring the pullbacks on $X \times X$ of $\mathcal{Q}$ and $\mathcal{Q}^\vee$ from its first and second factor by means of the respective projections $\pi_i : X \times X \to X$. Moreover, $\mathcal{O}_{\Delta X}$ is the structure sheaf of the diagonal $\Delta X \subset X \times X$, and the map r is the restriction map to ΔX. If for instance $\mathcal{Q} = \mathcal{O}_X$, then the map r restricts $\mathcal{O}_{X \times X} = \mathcal{O}_X \boxtimes \mathcal{O}_X$ to $\mathcal{O}_{\Delta X}$. Transformations of this type in for example describe the effect on B-branes of monodromies around conifold points in Kähler moduli spaces. These are points where B-branes $\mathcal{Q}$ become massless. The action on the B-brane charges is encoded in the periods near the conifold point and can be represented by equation (5.2).

Braid group actions can be obtained provided that there is an A_m-chain $(E_1, \ldots, E_m)$ of spherical objects. Mimicking condition (5.1) on the A-side, this means

$$\sum \dim(\mathrm{Ext}^k(E_i, E_j)) = \begin{cases} 1 & |i-j| = 1 \\ 0 & |i-j| > 1 \end{cases} \tag{5.6}$$

In a different but related context, the braid group has appeared in the context of 4-dimensional gauge theories with surface operators [38, 39].

If the non-linear sigma model with target space X has a Landau-Ginzburg phase, the derived category of coherent sheaves on X is equivalent to the category of B-branes in the corresponding Landau-Ginzburg orbifold, *i.e.* the associated category of equivariant matrix factorizations.[9] In these cases representations of braid groups in the autoequivalences of the derived category of coherent sheaves on X carry over to the respective category of matrix factorizations.

[9] The equivalence can be realized for instance in terms of gauged linear sigma models [40].

JHEP03(2010)015

In the following we will present a world sheet realization of these braid group representations. Namely, we will construct defects which satisfy braid relations on the level of their fusion. Since they can be fused with boundary conditions they give rise to functors (in this case autoequivalences) on the respective D-brane categories. One should point out however, that defects have a richer structure than the associated functors on D-brane categories. First of all they are objects in the full conformal field theory, not just in the topologically twisted theories. Moreover, defects can form junctions etc.

The construction we use is rather general and should apply to any $N = (2,2)$-supersymmetric theory. It is certainly not limited to theories which have a non-linear sigma model phase. However, we will restrict our considerations to Landau-Ginzburg models, in which everything can be spelled out relatively explicitly.

5.1 The defects

The defects that are relevant for us are conifold type defects considered in [11]. Following terminology from the work of Seidel and Thomas [18] we will call them twist defects. Such defects exist in any theory, and in case the theory has $N = (2,2)$ supersymmetry one can lift them to the respective B-twisted model. Namely, in any theory we have the trivial identity defect and totally reflective defects. The identity defect Id is a topological defect and maps via fusion any boundary condition back to itself. On the other hand, a totally reflective defect is a defect that provides boundary conditions for each of the two adjacent theories. In a theory with $N = (2,2)$ supersymmetry we can choose the boundary conditions to be of B-type, the defect then preserves B-type supersymmetry and can be fused on the level of the B-twisted theory. Choosing for example the boundary condition P on the one side of the defect and its world sheet parity dual P^* on the other side, we obtain the defect $T_{P,P^*} = P \otimes P^*$. Fusing it with a boundary condition Q one obtains a copy of P for every boundary condition changing field between P and Q

$$T_{P,P^*} * Q = \mathcal{H}^*(P,Q) \otimes Q\,. \tag{5.7}$$

Consider now a defect that is a superposition of the identity defect Id and the totally reflective defect $T_{P,P^*}\{1\}$, *i.e.* T_{P,P^*} shifted by one. There is a universal defect changing field between $T_{P,P^*}\{1\}$ and Id, which can be used to perturb this configuration. To see this, note that there is always a bosonic defect changing field between T_P and Id which has its origin in the fact that there is an identity field on the boundary condition P. Accordingly, there is a canonical fermion between $T_{P,P^*}\{1\}$ and Id.

This construction mimics the form of the Fourier-Mukai kernel (5.5). Here, $\mathcal{O}_{\Delta X}$ and $\mathcal{Q} \boxtimes \mathcal{Q}^\vee$ correspond to the identity and the purely reflective defects respectively.

Physically, fusion of boundary conditions with this defect mimics how D-branes behave when one moves along a closed path in Kähler moduli space which encloses a locus Δ_P on which a D-brane P becomes massless. Since copies of P and its anti-brane can be produced at no cost in energy, a probe D-brane Q which is carried around Δ_P forms bound states with P provided there is a suitable tachyon. As we will see explicitly in the next sections this is exactly how fusion with the bound state of T_{P,P^*} and Id acts on boundary conditions.

JHEP03(2010)015

The Id defect preserves a copy of Q, whereas the totally reflective defect creates as many copies of P as there are tachyons between P and Q. Finally the universal defect changing field between T_{P,P^*} and Id induces a bound state formation between all the copies of P and the copy of Q.

5.2 Realization in terms of matrix factorizations

5.2.1 Twist defects

The matrix factorizations corresponding to the identity and the totally reflective defects in Landau-Ginzburg models have been explicitly described in section 2.

Since the reflective defect is really a product of two boundary conditions the space of defect changing fields between the totally reflective defect T_{P,P^*} and the identity defect Id is isomorphic to the space of boundary conditions changing fields between the two boundary conditions

$$\mathcal{H}(T_{P,P^*}, \mathrm{Id}) \equiv \mathcal{H}(P \otimes P^*, \mathrm{Id}) \equiv \mathcal{H}(P, P)\,. \tag{5.8}$$

In particular, via this isomorphism the identity field id_P on P gives rise to a canonical defect changing field ψ_P

$$\mathcal{H}^0(T_{P,P^*}, \mathrm{Id}) \ni \psi_P \mapsto \mathrm{id}_P \in \mathcal{H}^0(P, P)\,, \tag{5.9}$$

which can be used to perturb the superposition of the (shifted) tensor product and the identity defect. The outcome

$$D_P = \mathrm{Cone}(\psi_P : T_{P,P^*} \to \mathrm{Id}) \tag{5.10}$$

of this perturbation obeys some nice universal relations. Indeed, the D_P are the twist defects alluded to above. To see this, let us first describe how these defects act on matrix factorizations

$$D_P * Q \cong D_P \otimes Q \cong \mathrm{Cone}(\psi_P \otimes \mathrm{id}_Q : T_{P,P^*} \otimes Q \to \mathrm{Id} \otimes Q). \tag{5.11}$$

As discussed in section 2, $T_{P,P^*} \otimes Q \cong P \otimes \mathcal{H}(P, Q)$ and $\mathrm{Id} \otimes Q \cong Q$. Indeed, via these isomorphisms, the morphism $\psi_P \otimes \mathrm{id}_Q$ is mapped to the evaluation map $\mathrm{ev} : P \otimes \mathcal{H}(P, Q) \to Q$. This can be seen as follows. By definition, the morphism ψ_P is mapped to $\psi_P \mapsto \mathrm{id}_P$ under the isomorphism $\mathcal{H}(T_{P,P^*}, \mathrm{Id}) \cong \mathcal{H}(P, P)$. Indeed, it also maps to $\psi_P \mapsto \mathrm{id}_{P^*}$ under the isomorphism $\mathcal{H}(T_{P,P^*}, \mathrm{Id}) \cong \mathcal{H}(P^*, P^*)$, and hence $\psi_P \otimes \mathrm{id}_Q \mapsto \mathrm{id}_{P^*} \otimes \mathrm{id}_Q$ under $\mathcal{H}(T_{P,P^*} \otimes Q, \mathrm{Id} \otimes Q) \cong \mathcal{H}(P^* \otimes Q, P^* \otimes Q)$. Here $P^* \otimes Q$ are matrix factorizations of 0 and therefore ordinary complexes. But now $\mathcal{H}(P^* \otimes Q, P^* \otimes Q) \cong \mathcal{H}(P \otimes P^* \otimes Q, Q) \cong \mathcal{H}(P \otimes \mathcal{H}(P, Q), Q)$, and $\mathrm{id}_{P^*} \otimes \mathrm{id}_Q \mapsto \mathrm{ev}$ under this isomorphism, because id_{V^*} is mapped to the evaluation map under the canonical isomorphism $\mathrm{Hom}(V^*, V^*) \to (V \otimes V^*)^*$.

Thus, we obtain

$$D_P * Q \cong \mathrm{Cone}(\mathrm{ev} : P \otimes \mathcal{H}(P, Q) \to Q)\,. \tag{5.12}$$

JHEP03(2010)015

A similar form can be found for the fusion with the dual of a twist defect which is represented by the dual matrix factorization D_P^*. Namely

$$D_P^* * Q \cong (Q^* * D_P)^* \cong (\mathrm{Cone}(\mathrm{id}_{Q^*} \otimes \psi_P : Q^* \otimes T_{P,P^*} \to Q^* \otimes \mathrm{Id}))^* \,. \tag{5.13}$$

With arguments similar to the ones above one arrives at

$$\begin{aligned} D_P^* * Q &\cong (\mathrm{Cone}(\mathrm{ev} : \mathcal{H}(Q,P) \otimes P^* \to Q^*))^* \\ &\cong \mathrm{Cone}(\mathrm{ev}^* : Q \to (\mathcal{H}(Q,P))^* \otimes P) \,. \end{aligned} \tag{5.14}$$

The action (5.12) of the defects D_P on matrix factorizations is realized by twist functors as introduced in [18]. There Seidel and Thomas show that under certain assumptions such twist functors generate representations of braid groups in the groups of autoequivalences of certain derived categories. In the next subsections, we follow their arguments to establish that under similar conditions, twist defects satisfy braid relations with respect to fusion.

5.2.2 Twist defects for spherical matrix factorizations

A matrix factorization P is called **n-spherical** if

$$\mathcal{H}^i(P,P) = \begin{cases} \mathbb{C}\,, & i = 0, n \\ 0\,, & \text{otherwise} \end{cases} \tag{5.15}$$

and for every matrix factorization Q, the operator product

$$\mathcal{H}^i(P,Q) \otimes \mathcal{H}^{n-i}(Q,P) \to \mathcal{H}^n(P,P) \tag{5.16}$$

is non-degenerate. Note that we have extended the grading of $\mathcal{H}^*$ from $\mathbb{Z}_2$ to $\mathbb{Z}$ by means of the R-charge, and n is chosen such that $-n$ is the R-charge anomaly of the disk amplitudes in the models under consideration.[10] Condition (5.16) is nothing but the non-degeneracy of the boundary two-point function, which holds for unitary theories. We restrict our considerations to such theories and will assume condition (5.16) in the following.

One nice property of spherical matrix factorizations P is that the associated twist defects D_P are indeed group-like *i.e.*

$$D_P * D_P^* = \mathrm{id} = D_P^* * D_P \,. \tag{5.17}$$

In particular, they act on matrix factorization categories as equivalences. To see this, we calculate the fusion $D_P * (D_P^* * Q)$ for P a spherical and Q any matrix factorization. Using equations (5.14) and (5.12) one arrives at

$$D_P * (D_P^* * Q) \cong \mathrm{Cone}\begin{pmatrix} P \otimes \mathcal{H}(P,Q) & \xrightarrow{f} & \mathcal{H}(P,P) \otimes (\mathcal{H}(Q,P))^* \otimes P \\ \downarrow \mathrm{ev} & & \downarrow \mathrm{ev} \\ Q & \xrightarrow{\mathrm{ev}^*} & (\mathcal{H}(Q,P))^* \otimes P \end{pmatrix}, \tag{5.18}$$

[10] For models with a realization in terms of non-linear sigma models, n is the complex dimension of the target space.

JHEP03(2010)015

where the induced map f acts as a dualization $\mathcal{H}(P,Q) \longrightarrow \mathcal{H}(P,P) \otimes (\mathcal{H}(Q,P))^*$ of the operator product $\mathcal{H}(P,Q) \otimes \mathcal{H}(Q,P) \to \mathcal{H}(P,P)$. Now, since by assumption P is spherical, the non-degeneracy (5.16) of the operator product $\mathcal{H}^{n-i}(P,Q) \otimes \mathcal{H}^i(Q,P) \to \mathcal{H}^n(P,P)$ implies that we have an isomorphism

$$P \otimes \mathcal{H}(P,Q) \xrightarrow{\cong} \mathcal{H}^n(P,P) \otimes (\mathcal{H}(Q,P))^* \otimes P \tag{5.19}$$

in the upper row of (5.18). Furthermore, in the right column, we have an isomorphism

$$\mathcal{H}^0(P,P) \otimes (\mathcal{H}(Q,P))^* \otimes P \xrightarrow{\cong} (\mathcal{H}(Q,P))^* \otimes P\,. \tag{5.20}$$

These isomorphisms can be used to reduce the matrix factorization (5.18) to Q. Therefore,

$$D_P * (D_P^* * Q) \cong Q\,. \tag{5.21}$$

In a similar way one obtains $D_P^* * (D_P * Q) \cong Q$.

The fact that D_P is group-like for spherical P can be used to show the following relation between twist defects associated to spherical matrix factorizations P_1, P_2:

$$D_{P_2} * D_{P_1} \cong D_{D_{P_2}P_1} * D_{P_2}\,. \tag{5.22}$$

For this, we again fuse the equation with a matrix factorization Q

$$\begin{aligned} D_{P_2} * (D_{P_1} * Q) \cong & \operatorname{Cone}\left(\begin{array}{ccc} P_2 \otimes \mathcal{H}(P_1,Q) \otimes \mathcal{H}(P_2,P_1) & \xrightarrow{\text{ev}} & P_2 \otimes \mathcal{H}(P_2,Q) \\ \downarrow \text{ev} & & \downarrow \text{ev} \\ P_1 \otimes \mathcal{H}(P_1,Q) & \xrightarrow{\text{ev}} & Q \end{array}\right) \\ \cong & \operatorname{Cone}(\mathcal{H}(P_1,Q) \otimes D_{P_2}P_1 \xrightarrow{g} D_{P_2}Q)\,. \end{aligned} \tag{5.23}$$

Since D_{P_2} is group-like $\mathcal{H}(P_1,Q) \cong \mathcal{H}(D_{P_2}P_1, D_{P_2}Q)$ and the map g factors through the evaluation map

$$\mathcal{H}(P_1,Q) \otimes D_{P_2}P_1 \xrightarrow{\cong} \mathcal{H}(D_{P_2}P_1, D_{P_2}Q) \otimes D_{P_2}P_1 \xrightarrow{\text{ev}} D_{P_2}Q\,. \tag{5.24}$$

Thus,

$$\begin{aligned} D_{P_2} * (D_{P_1} * Q) &\cong \operatorname{Cone}(\mathcal{H}(D_{P_2}P_1, D_{P_2}Q) \otimes D_{P_2}P_1 \xrightarrow{\text{ev}} D_{P_2}Q) \\ &\cong D_{D_{P_2}P_1} * (D_{P_2} * Q)\,. \end{aligned} \tag{5.25}$$

5.2.3 Defect realization of braid groups

Relation (5.22) can be used to construct defect realizations of braid groups in the following way. An A_m**-sequence of spherical matrix factorizations** is a collection $(P_1, \ldots, P_m)$ of spherical matrix factorizations P_i such that

$$\dim \mathcal{H}(P_i, P_j) = \begin{cases} 1\,, & |i-j| = 1 \\ 0 & |i-j| > 1 \end{cases}\,. \tag{5.26}$$

JHEP03(2010)015

Given such a collection, with the preparations of the last sections, it is easy to see that the associated twist defects D_{P_i} satisfy braid relations (5.3). Namely for $|i-j|>1$ $\mathcal{H}(P_i,P_j)=0$, so that from (5.12) one reads off that $D_{P_i}P_j \cong P_j$. Therefore, relation (5.22) implies

$$D_{P_i} * D_{P_j} \cong D_{P_j} * D_{P_i} \quad \text{for } |i-j|>1\,. \tag{5.27}$$

Moreover, using $\dim \mathcal{H}(P_{i+1},P_i)=1$ one obtains

$$D_{P_{i+1}} * P_i \cong \mathrm{Cone}(P_{i+1}[-r_i] \xrightarrow{f_i\neq 0} P_i) \tag{5.28}$$

$$D^*_{P_i} * P_{i+1} \cong \mathrm{Cone}(P_{i+1} \xrightarrow{g_i\neq 0} P_i[r_i]) \tag{5.29}$$

for some shifts r_i. But since $\dim \mathcal{H}(P_{i+1},P_i)=1$, f_i and g_i are multiples of each other. In particular,

$$D_{P_2} * P_1[r_i] \cong D^*_{P_1} * P_2\,. \tag{5.30}$$

Now we can conclude the other braid relations:

$$\begin{aligned} D_{P_i} * D_{P_{i+1}} * D_{P_i} &\cong D_{P_i} * D_{D_{P_{i+1}}*P_i} * D_{P_{i+1}} \\ &\cong D_{P_i} * D_{D^*_{P_i}*P_{i+1}} * D_{P_{i+1}} \\ &\cong D_{D_{P_i}*D^*_{P_i}*P_{i+1}} * D_{P_i} * D_{P_{i+1}} \\ &\cong D_{P_{i+1}} * D_{P_i} * D_{P_{i+1}}\,,. \end{aligned} \tag{5.31}$$

Here, the first equation is obtained by means of (5.22). In the second equation use was made of (5.30) and the fact that shifts do not change twist defects. The third equation is again obtained by means of relation (5.22), where one has to note that because twist defects of spherical matrix factorizations are group-like, their fusion products with spherical matrix factorizations are still spherical. Finally, in the last equation one again employs (5.17).

Summarizing, we have established the following. Given a boundary conditions P, the superpositions of the shifted purely reflective defects T_{P,P^*} and the trivial defect Id exhibit a universal defect changing field. The corresponding perturbations lead to twist defects D_P, which have some universal properties. For spherical boundary conditions P the associated twist defects D_P are group-like, and with respect to fusion, satisfy the commutation relation (5.22). Moreover, A_m-sequences of boundary conditions give rise to a collection of twist defects, which satisfy braid relations. We have discussed this explicitly in the context of B-type defects in Landau-Ginzburg models, but we expect that the constructions should apply to general $N=(2,2)$-supersymmetric theories.

5.3 Examples

In this section we will present some examples of Landau-Ginzburg models, which exhibit A_m-sequences of B-type boundary conditions. By means of the construction above they give rise to B-type defects satisfying braid relations.

JHEP03(2010)015

5.3.1 Degenerate K3 and fibrations

Geometrically, the appearance of A_m-chains of homology cycles has played a prominent role in the discussion of heterotic-IIA duality [41, 42]. In particular, local singularities of K3-surfaces are responsible for the non-abelian gauge symmetry enhancement of the IIA string compactified on K3. Via het-IIA duality this is dual to the non-Abelian gauge symmetries appearing at special points in the moduli space of the toroidally compactified weakly coupled heterotic string. Here, we are particularly interested in the case that the enhanced symmetry is A_m, meaning that the K3 should locally exhibit an orbifold singularity $\mathbb{C}^2/\mathbb{Z}_{m+1}$. Standard examples arise as suitable hypersurfaces in weighted projective spaces. Consider for example a degree 12 hypersurface in $\mathbb{P}_{(1,3,4,4)}[12]$. Projective equivalence acts on the affine coordinates as

$$(x_1, x_2, x_3, x_4) \mapsto (\lambda x_1, \lambda^3 x_2, \lambda^4 x_3, \lambda^4 x_4) \tag{5.32}$$

For $\lambda^4 = 1$ this transformation leaves x_3 and x_4 invariant, leading to a local $\mathbb{C}^2/\mathbb{Z}_4$-singularity in $x_1 = 0 = x_2$. This singularity is resolved by three spheres intersecting in an A_3 pattern. The hypersurface intersects the singular fixed point set in three points. Hence, there are three A_3-chains on this K3 surface. Note also that this means that the Picard lattices of K3's embedded in this weighted projective space generically have rank 10 (1 canonical holomorphic $(1,1)$ coming from the hyperplane bundle plus 3×3 spheres coming from the resolution of the singularities) so that the embedding requires a restriction to a particular part of the K3 moduli space.

This model has a Landau-Ginzburg orbifold phase with superpotential

$$W = x_1^{12} + x_2^4 + x_3^3 - x_4^3 \tag{5.33}$$

and orbifold group $\Gamma = \mathbb{Z}_{12}$. Therefore, one can realize the B-branes supported on the exceptional divisors by $\mathbb{Z}_{12}$-equivariant matrix factorizations of W. They have been obtained in [43]. All the building blocks have already appeared in sections 4.1 and 4.2. The relevant matrix factorizations are

$$E_m^n = Q_m^1(x_1) \otimes Q_0^1(x_2) \otimes P^{\{n\}}(x_3, x_4)_0 \,. \tag{5.34}$$

This is a tensor product of three linear matrix factorizations: the ordinary one-variable factorizations Q^1 in the first two factors, and a permutation matrix factorization $P^{\{n\}}$ in the last two factors. Here, m specifies the $\mathbb{Z}_{12}$-representation of the matrix factorization, and $n \in \mathbb{Z}_3$ denotes which of the three A_3-sequences E_m^n belongs to. To motivate that these are good candidates for the matrix factorizations realizing the A_3-sequences of B-type boundary conditions, one can use a simplified version of the arguments in [40, 44]. The latter suggests that the B-type boundary conditions associated to E_m^n are localized at the zero locus of the factorization

$$x_1 = x_2 = x_3 - \eta_n x_4 = 0 \,, \tag{5.35}$$

which is the $\mathbb{Z}_4$-singularity. This is blown up by the exceptional divisors, and it is known *e.g.* from orbifold theories, that the fractional branes of the corresponding $\mathbb{Z}_4$ represent B-branes wrapping the exceptional divisors at large volume.

JHEP03(2010)015

Of coures, one can just check that the E^n_{3m} are spherical and form A_3-sequences. The Witten-index between the matrix factorizations E^n_M and E^n_N is easily calculated to be

$$\begin{aligned} I_{M,N} &= \big((1-g^{-1})(1-g^3)(1+g^4)\big)_{M,N} \\ &= \big(2-g^{-1}-g^{-3}+g^{-4}+g^4-g^3-g\big)_{M,N}\,, \end{aligned} \tag{5.36}$$

where

$$g_{M,N} = \delta^{(12)}_{M-N,1} \tag{5.37}$$

is the $\mathbb{Z}_{12}$-shift matrix. Since these matrix factorizations are tensor products of matrix factorizations which do not have bosons and fermions at the same degree, and since furthermore no cancellation occurred in the expansion of (5.36) one can indeed even read off the dimensions of the corresponding open string Hilbert spaces $\mathcal{H}^*(E^n_M, E^n_N)$. One obtains that the E^n_m are spherical and that (E^n_0, E^n_3, E^n_6) constitute A_3-sequences of spherical objects.

Let us turn to models with three dimensional target spaces. Examples that gained particular importance in the context of string dualities are K3 fibrations. Here, one expects an enhanced gauge symmetry at points in moduli space where the fiber exhibits an ADE degeneration. For instance, a hypersurface in weighted projective space $\mathbb{P}_{1,1,6,8,8}[24]$ is a K3-fibration over $\mathbb{P}^1$ with fiber a hypersurface in $\mathbb{P}_{(1,3,4,4)}[12]$. To see this, we intersect the hypersurface with a linear equation in the coordinates x_0, x_1 of weight 1. A special case is $x_0 = 0$ for which the hypersurface equation reduces to

$$x_1^{24} + x_2^4 + x_3^3 - x_4^3 = 0 \tag{5.38}$$

from which we recover the K3 hypersurface equation (5.33) by substituting $y_2 = x_2^2$ which is single valued because of quasi-projective equivalence. In complete analogy to the above discussion, the fibers degenerate at the points (5.35). Hence, one expects an A_3-intersection pattern for the matrix factorizations

$$\hat{E}^n_m = Q^1_m(x_0) \otimes Q^1_0(x_1) \otimes Q^1_0(x_2) \otimes P^{\{n\}}_0(x_3, x_4)\,. \tag{5.39}$$

Indeed, the intersection matrix between the $\hat{E}^n_m$ is given by

$$\begin{aligned} I &= (1-g^{-1})^2(1-g^{-6})(1+g^8) \\ &= -2g^{-1} + g^{-2} - g^{-6} + 2g^{-7} - g^{-8} + g^8 - 2g^7 + g^6 - g^2 + 2g\,, \end{aligned} \tag{5.40}$$

where now g denotes the $\mathbb{Z}_{24}$-shift matrix. Again, no cancellation occured in the expansion of (5.40), and one can read off that the matrix factorizations $\hat{E}^n_m$ are spherical, and that $(\hat{E}^n_0, \hat{E}^n_6, \hat{E}^n_{12})$ constitute A_3-sequences. Many more examples can be constructed in a similar manner, making use of the divisibility patterns of the weights.

A_n-sequences can also be obtained in these examples using tensor products of the one variable matrix factorizations Q^1 only. These factorizations have the advantage that they are universally available in any theory with an R-charge, since any quasi-homogeneous superpotential can be factorized as $W = \sum_i x_i A_i$. Geometrically they come from the embedding quasi-projective space.

JHEP03(2010)015

Moreover, one easily sees that tensor products of linear matrix factorizations Q^1 are always spherical: The spectrum for the ith factor of the tensor product consists of the identity and one fermion (see *e.g.* section 4.1). The part of the spectrum of the tensor product factorizations which survives the orbifold projection consists of the identity and the product of the fermionic fields in each factor. Hence condition (5.15) is always satisfied. This means in particular that the corresponding twist defects are always group-like and the induced functors on the B-brane categories are autoequivalences. Geometrically, these universal autoequivalences correspond to the monodromy in Kähler moduli space around the locus where the highest dimensional D-brane becomes massless.

Making additional assumptions on the divisibility of the weights, one can construct further examples which exhibit A_n-sequences of spherical tensor product boundary conditions.

5.3.2 A non-geometric Landau-Ginzburg example

A very simple class of Landau-Ginzburg models exhibiting A_m-sequences of spherical B-type boundary conditions are the LG-orbifolds with two chiral superfields x_1, x_2, superpotential

$$W = x_1^d + x_2^d \tag{5.41}$$

and orbifold group $\Gamma = \mathbb{Z}_d$ whose generator acts on the x_i by

$$G : (x_1, x_2) \mapsto (\omega x_1, \omega^{-1} x_2), \quad \omega = e^{\frac{2\pi i}{d}} \tag{5.42}$$

The intersection matrix for the linear tensor product factorizations $F_m = Q_m^1 \otimes Q_0^1$ in this model is given by

$$I = (1 - g^{-1})(1 - g) = 2 - g^{-1} - g\,, \tag{5.43}$$

where g denotes the $\mathbb{Z}_d$-shift matrix. As in the previous examples, also here one can read off the dimensions of the BRST-cohomologies from I. Any collection of $d-1$ of the F_m forms an A_{d-1}-sequence of spherical matrix factorizations.

This model is non geometric in the sense that it has (generically) non-integer central charge and thus no direct geometrical interpretation. Note however its close relationship to the corresponding noncompact models $\mathbb{C}^2/\mathbb{Z}_d$ which can be obtained by setting the superpotential to zero.

5.3.3 Hirzebruch-Jung examples

The examples discussed in the last section can be generalized to the Landau-Ginzburg models

$$\left(W = x_1^d + x_2^d\right) / \mathbb{Z}_{d(k)}\,. \tag{5.44}$$

As before, one considers $\mathbb{Z}_d$-orbifolds of the Landau-Ginzburg models with two chiral superfiels x_1, x_2 and superpotential $W = x_1^d + x_2^d$, where now the orbifold generators act as

$$G : (x_1, x_2) \mapsto (\omega x_1, \omega^k x_2), \quad \omega = e^{\frac{2\pi i}{d}}\,. \tag{5.45}$$

Here, d and k are assumed to be coprime, and the previous examples are obtained by setting $k = d - 1$.

JHEP03(2010)015

The corresponding non-compact models have appeared in the context of non-supersymmetric orbifolds $\mathbb{C}^2/\mathbb{Z}_{d(k)}$ [45–48]. The orbifold singularity can be resolved using the Hirzebruch-Jung resolution, which replaces the singular point by a chain of $\mathbb{P}_1$'s whose intersection pattern is determined by the continuous fraction expansion

$$\frac{d}{k} = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \frac{1}{\ldots 1/a_f}}} = [a_1, \ldots, a_f]. \tag{5.46}$$

The a_i are the self-intersection numbers of the f exceptional $\mathbb{P}_1$'s blown up in the resolution. The intersection number between subsequent spheres is 1. Hence, for those (d, k) such that the continuous fraction expansion of $\frac{d}{k}$ contains a string $a_s = a_{s+1} = \ldots = a_{s+r-1} = 2$ the corresponding orbifold model contains an A_r-sequence of spherical B-branes wrapping the associated exceptional $\mathbb{P}_1$'s.

Because of the close relationship between non-compact orbifold and Landau-Ginzburg models such A_r-chains must also be present in the LG models (5.44). As an example, let us consider the case that the string of 2's is located at the beginning of the continuous fraction expansion of $\frac{d}{k}$, followed by one further integer $b > 2$:

$$\frac{d}{k} = [2, \ldots, 2, b] = \frac{(r+1)b - r}{rb - (r-1)}\,. \tag{5.47}$$

As candidates for the A_r-sequence of spherical matrix factorizations we again choose tensor products

$$G_m = Q_m^{\Delta} \otimes Q_0^1, \quad \Delta = d - k\,. \tag{5.48}$$

Quite generally, if the continued fraction expansion of $\frac{d}{k}$ starts with a 2, we can conclude that $\Delta \leq \frac{d}{2}$. In the specific case (5.47) $\Delta = b - 1$. The intersection matrix of the G_m is given by

$$\begin{aligned} I &= \left(\sum_{j=-\Delta}^{\Delta-1} sgn(j) g^j \right) \left(1 - g^{-k}\right) \\ &= 2 \sum_{j=0}^{\Delta-1} g^j - \sum_{j=\Delta}^{2\Delta-1} g^j - \sum_{j=-\Delta}^{-1} g^j\,, \end{aligned} \tag{5.49}$$

where g is the d-dimensional shift matrix. Note that no cancellation occurs in the expansion (5.49). Since the G_m are tensor products of matrix factorizations which do not have bosons and fermions at the same degree, one can therefore read off the dimensions of the BRST cohomology groups directly from I. One finds that the G_m are spherical and that $\dim \mathcal{H}^*(G_{a\Delta}, G_{b\Delta}) = 1$ for $|a - b| = 1$. Indeed, the $G_{a\Delta}$ with $a \in \{0, \ldots, r-1\}$ constitute an A_r-sequence of spherical matrix factorizations. Namely, since $(r-1)\Delta < d$, no summand $g^{n\Delta}$ with $r > |n| \geq 2$ appears in (5.49), and hence $\dim \mathcal{H}^*(G_{a\Delta}, G_{b\Delta}) = 0$ for all $a, b \in \{0, \ldots, r-1\}$ with $|a - b| > 1$.

In exactly the same way one obtains A_r-sequences of spherical matrix factorizations for models in which the single integer b in the continuous fraction expansion (5.47) is replaced by an arbitrary string

$$\frac{d}{k} = [2, \ldots, 2, b_{r+1}, \ldots, b_f]\,. \tag{5.50}$$

JHEP03(2010)015

The discussion of the general case

$$\frac{d}{k} = [a_1, \ldots, a_i, 2, \ldots, 2, b_{r+i+1}, \ldots, b_f] \tag{5.51}$$

is slightly more involved and can be found in [49].

Acknowledgments

D. R. is supported by a DFG research fellowship and partially by DOE-grant DE-FG02-96ER40949. I. B. is supported by a EURYI award of the European Science Foundation. We would like to thank Matthias Gaberdiel and Ingo Runkel for stimulating discussions.

A Equivalence of identity matrix factorizations

In this appendix it is shown that the equivalence class of the identity matrix factorization (2.15) does not depend on the choices made in its definition.

For any polynomial $W \in \mathbb{C}[x_1, \ldots, x_N]$, the difference $W(x_i) - W(y_i)$ can be written in factorized form

$$W(x_i) - W(y_i) = \sum_i (x_i - y_i) A_i(x_i, y_i)\,, \tag{A.1}$$

for some polynomials $A_i \in \mathbb{C}[x_1, \ldots, x_N, y_1, \ldots, y_N]$, which are however not completely determined by (A.1). For a choice of A_i define the identity matrix factorization Id as the tensor product

$$\mathrm{Id} = \bigotimes \mathrm{Id}^{(i)} : \mathrm{Id}_1 \underset{\mathrm{id}_0}{\overset{\mathrm{id}_1}{\rightleftarrows}} \mathrm{Id}_0 \tag{A.2}$$

of the rank one matrix factorizations $\mathrm{Id}^{(i)}$ with maps $p_1^{(i)} = (x_i - y_i)$ and $p_0^{(i)} = A_i(x_j, y_j)$. This matrix factorization of course depends on the choice of A_i. In order to show that its equivalence class is independent of this choice, we use a convenient Koszul-type representation of tensor product matrix factorizations (*c.f.* [11, 34]):

$$\mathrm{Id} : \mathrm{Id}_1 = \Lambda^{\mathrm{odd}} V^* \underset{\delta}{\overset{\delta}{\rightleftarrows}} \Lambda^{\mathrm{even}} V^* = \mathrm{Id}_0\,, \tag{A.3}$$

where $V = \mathbb{C}[x_1, \ldots, x_N, y_1, \ldots, y_N]^N$ and

$$\delta = \imath_\Delta + A \wedge \cdot\,, \qquad \Delta = \sum_i (x_i - y_i) e_i\,, \quad A = \sum_i A_i e_i^*\,. \tag{A.4}$$

Here (e_i) is a basis of V and (e_i^*) its dual basis of V^*; $\imath_\Delta$ and $A \wedge \cdot$ denote insertion of the vector Δ and multiplication with the form A respectively. Obviously, $\delta^2 = \imath_\Delta(A) = W(x_i) - W(y_i)$.

Now, for any other choice of factorization

$$W(x_i) - W(y_i) = \sum_i (x_i - y_i) A_i'(x_i, y_i)\,, \qquad A_i' = A_i + a_i\,, \tag{A.5}$$

JHEP03(2010)015

one obtains in the same way a matrix factorization (A.3), where now the map δ is replaced by

$$\delta' = \imath_\Delta + A' \wedge \cdot = \delta + a \wedge \cdot \,. \tag{A.6}$$

Of course

$$W(x_i) - W(y_i) = (\delta')^2 = \delta^2 + \{\delta, a \wedge \cdot\} = \delta^2 + \imath_\Delta(a)\,, \tag{A.7}$$

from which it follows that $a \wedge \cdot$ is δ-closed, and more explicitly $\imath_\Delta(a) = 0$. This implies however that $a = \imath_\Delta(\omega)$ for some $\omega \in \Lambda^2 V^*$. Since ω is a two-form multiplication with it commutes with multiplication of any other form, *i.e.* $\omega \wedge \pi \wedge \cdot = \pi \wedge \omega \wedge \cdot$ for all $\pi \in \Lambda^* V^*$, and therefore

$$\begin{aligned} e^{-\omega\wedge\cdot}\delta e^{\omega\wedge\cdot} &= e^{-\omega\wedge\cdot}\imath_\Delta e^{\omega\wedge\cdot} + A \wedge \cdot = e^{-\omega\wedge\cdot}\left[\imath_\Delta, e^{\omega\wedge\cdot}\right] + \delta \\ &= \imath_\Delta(\omega) \wedge \cdot + \delta = \delta + a \wedge \cdot = \delta'\,. \end{aligned} \tag{A.8}$$

Thus $e^{-\omega} : \Lambda^* V^* \longrightarrow \Lambda^* V^*$ defines an isomorphism from Id to Id$'$, and hence the equivalence class of Id is independent of the choice of A_i.

References

[1] E. Wong and I. Affleck, *Tunneling in quantum wires: a boundary conformal field theory approach*, *Nucl. Phys.* **B 417** (1994) 403 [SPIRES].

[2] C. Bachas, J. de Boer, R. Dijkgraaf and H. Ooguri, *Permeable conformal walls and holography*, *JHEP* **06** (2002) 027 [hep-th/0111210] [SPIRES].

[3] V.B. Petkova and J.B. Zuber, *Generalised twisted partition functions*, *Phys. Lett.* **B 504** (2001) 157 [hep-th/0011021] [SPIRES].

[4] J. Fröhlich, J. Fuchs, I. Runkel and C. Schweigert, *Duality and defects in rational conformal field theory*, *Nucl. Phys.* **B 763** (2007) 354 [hep-th/0607247] [SPIRES].

[5] I. Runkel, *Perturbed defects and T-systems in conformal field theory*, *J. Phys.* **A 41** (2008) 105401 [arXiv:0711.0102] [SPIRES].

[6] C. Bachas and I. Brunner, *Fusion of conformal interfaces*, *JHEP* **02** (2008) 085 [arXiv:0712.0076] [SPIRES].

[7] I. Brunner and D. Roggenkamp, *B-type defects in Landau-Ginzburg models*, *JHEP* **08** (2007) 093 [arXiv:0707.0922] [SPIRES].

[8] C. Bachas, *On the symmetries of classical string theory*, arXiv:0808.2777 [SPIRES].

[9] I. Brunner and D. Roggenkamp, *Defects and bulk perturbations of boundary Landau-Ginzburg orbifolds*, *JHEP* **04** (2008) 001 [arXiv:0712.0188] [SPIRES].

[10] S. Fredenhagen, M.R. Gaberdiel and C. Schmidt-Colinet, *Bulk flows in Virasoro minimal models with boundaries*, *J. Phys.* **A 42** (2009) 495403 [arXiv:0907.2560] [SPIRES].

[11] I. Brunner, H. Jockers and D. Roggenkamp, *Defects and D-brane monodromies*, arXiv:0806.4734 [SPIRES].

[12] K. Graham and G.M.T. Watts, *Defect lines and boundary flows*, *JHEP* **04** (2004) 019 [hep-th/0306167] [SPIRES].

JHEP03(2010)015

[13] C. Bachas and M. Gaberdiel, *Loop operators and the Kondo problem*, *JHEP* **11** (2004) 065 [hep-th/0411067] [SPIRES].

[14] A. Alekseev and S. Monnier, *Quantization of Wilson loops in Wess-Zumino-Witten models*, *JHEP* **08** (2007) 039 [hep-th/0702174] [SPIRES].

[15] D. Manolopoulos and I. Runkel, *A monoidal category for perturbed defects in conformal field theory*, arXiv:0904.1122 [SPIRES].

[16] M. Kormos, I. Runkel and G.M.T. Watts, *Defect flows in minimal models*, *JHEP* **11** (2009) 057 [arXiv:0907.1497] [SPIRES].

[17] J. Fröhlich, J. Fuchs, I. Runkel and C. Schweigert, *Kramers-Wannier duality from conformal defects*, *Phys. Rev. Lett.* **93** (2004) 070601 [cond-mat/0404051] [SPIRES].

[18] P. Seidel and R.P. Thomas, *Braid group actions on derived categories*, *Duke Math Jour.* **108** (2001) 37.

[19] A. Kapustin and Y. Li, *D-Branes in Landau-Ginzburg models and algebraic geometry*, *JHEP* **12** (2003) 005 [hep-th/0210296] [SPIRES].

[20] I. Brunner, M. Herbst, W. Lerche and B. Scheuner, *Landau-Ginzburg realization of open string TFT*, *JHEP* **11** (2006) 043 [hep-th/0305133] [SPIRES].

[21] M. Khovanov and L. Rozansky, *Topological Landau-Ginzburg models on a world-sheet foam*, *Adv. Theor. Math. Phys.* **11** (2007) 233 [hep-th/0404189] [SPIRES].

[22] H. Jockers and W. Lerche, *Matrix factorizations, D-branes and their deformations*, *Nucl. Phys. Proc. Suppl.* **171** (2007) 196 [arXiv:0708.0157] [SPIRES].

[23] J. Knapp, *D-Branes in topological string theory*, arXiv:0709.2045 [SPIRES].

[24] C.I. Lazaroiu, *On the boundary coupling of topological Landau-Ginzburg models*, *JHEP* **05** (2005) 037 [hep-th/0312286] [SPIRES].

[25] A. Kapustin and L. Rozansky, *On the relation between open and closed topological strings*, *Commun. Math. Phys.* **252** (2004) 393 [hep-th/0405232] [SPIRES].

[26] S.K. Ashok, E. Dell'Aquila and D.-E. Diaconescu, *Fractional branes in Landau-Ginzburg orbifolds*, *Adv. Theor. Math. Phys.* **8** (2004) 461 [hep-th/0401135] [SPIRES].

[27] K. Hori and J. Walcher, *F-term equations near Gepner points*, *JHEP* **01** (2005) 008 [hep-th/0404196] [SPIRES].

[28] S.K. Ashok, E. Dell'Aquila, D.-E. Diaconescu and B. Florea, *Obstructed D-branes in Landau-Ginzburg orbifolds*, *Adv. Theor. Math. Phys.* **8** (2004) 427 [hep-th/0404167] [SPIRES].

[29] J. Knapp and H. Omer, *Matrix factorizations, minimal models and Massey products*, *JHEP* **05** (2006) 064 [hep-th/0604189] [SPIRES].

[30] N. Carqueville, *Matrix factorisations and open topological string theory*, *JHEP* **07** (2009) 005 [arXiv:0904.0862] [SPIRES].

[31] D. Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, *Trans. Amer. Math. Soc.* **260** (1980) 35.

[32] K. Hori, *Boundary RG flows of $N = 2$ minimal models*, hep-th/0401139 [SPIRES].

[33] I. Brunner and M.R. Gaberdiel, *Matrix factorisations and permutation branes*, *JHEP* **07** (2005) 012 [hep-th/0503207] [SPIRES].

JHEP03(2010)015

[34] H. Enger, A. Recknagel and D. Roggenkamp, *Permutation branes and linear matrix factorisations*, *JHEP* **01** (2006) 087 [hep-th/0508053] [SPIRES].

[35] K. Hori, A. Iqbal and C. Vafa, *D-branes and mirror symmetry*, hep-th/0005247 [SPIRES].

[36] P. Seidel, *Lagrangian two-spheres can be symplectically knotted*, *J. Diff. Geom.* **52** (1999) 145.

[37] P.S. Aspinwall and M.R. Douglas, *D-brane stability and monodromy*, *JHEP* **05** (2002) 031 [hep-th/0110071] [SPIRES].

[38] S. Gukov and E. Witten, *Gauge theory, ramification and the geometric Langlands program*, hep-th/0612073 [SPIRES].

[39] S. Gukov, *Surface operators and Knot homologies*, arXiv:0706.2369 [SPIRES].

[40] M. Herbst, K. Hori and D. Page, *Phases of $N = 2$ theories in 1+1 dimensions with boundary*, arXiv:0803.2045 [SPIRES].

[41] E. Witten, *String theory dynamics in various dimensions*, *Nucl. Phys.* **B 443** (1995) 85 [hep-th/9503124] [SPIRES].

[42] P.S. Aspinwall, *Enhanced gauge symmetries and $K3$ surfaces*, *Phys. Lett.* **B 357** (1995) 329 [hep-th/9507012] [SPIRES].

[43] I. Brunner, M.R. Gaberdiel and C.A. Keller, *Matrix factorisations and D-branes on $K3$*, *JHEP* **06** (2006) 015 [hep-th/0603196] [SPIRES].

[44] P.S. Aspinwall, *The Landau-Ginzburg to Calabi-Yau dictionary for D-branes*, *J. Math. Phys.* **48** (2007) 082304 [hep-th/0610209] [SPIRES].

[45] A. Adams, J. Polchinski and E. Silverstein, *Don't panic! Closed string tachyons in ALE space-times*, *JHEP* **10** (2001) 029 [hep-th/0108075] [SPIRES].

[46] J.A. Harvey, D. Kutasov, E.J. Martinec and G.W. Moore, *Localized tachyons and RG flows*, hep-th/0111154 [SPIRES].

[47] E.J. Martinec and G.W. Moore, *On decay of k-theory*, hep-th/0212059 [SPIRES].

[48] G.W. Moore and A. Parnachev, *Localized tachyons and the quantum McKay correspondence*, *JHEP* **11** (2004) 086 [hep-th/0403016] [SPIRES].

[49] S. Rossi, *Boundaries and defects in topological Landau-Ginzburg models*, Ph. D. Thesis (2009).

JHEP03(2010)015