Invariant curve theorem for quasiperiodic twist mappings and stability of motion in Fermi-Ulam problem

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Abstract

In this paper the monotone twist theorem is extended to the quasiperiodic case and applied to establish boundedness of the velocity in a system of a particle bouncing elastically between two quasiperiodically moving walls. It is shown that the velocity of the particle is uniformly bounded in time if the frequencies satisfy a Diophantine inequality. This answers a question recently asked in [8].

1 Introduction

Quasiperiodic maps arise naturally in one-degree-of-freedom oscillatory Hamiltonian systems depending quasiperiodically on time. Indeed, if the frequencies are sufficiently incommensurable then the phase flow induces a quasiperiodic map on a cross-section transversal to the vector field. Although there are related results for quasiperiodically time-dependent Hamiltonian vector fields, see *e.g.* [1, 8], they are not directly applicable to the maps. The latter have to be dealt with in stability problems for certain systems of billiard type like Fermi-Ulam accelerator. This model, introduced by Fermi in order to explain the origin of the high-energy cosmic radiation [4], consists of a particle bouncing elastically between two parallel walls undergoing periodic motions. The Hamiltonian of such system is not smooth due to the collisions with the walls and one is forced to investigate stability of the corresponding Poincaré map.

The obtained map, however, is not close to an integrable one and a number of canonical transformations have to be carried out to bring the map to such form. For the "periodic" Fermi-Ulam accelerator this has been done in [3, 6, 11]. The reduced periodic map satisfies the conditions of Moser's twist theorem [10] and therefore possesses invariant curves which prevent unbounded motion.

See also [12] for investigation of stability in the quantum Fermi-Ulam accelerator.

More recently, Levi and Zehnder established boundedness of quasiperiodically forced motions in a large class of one dimensional potentials with superquadratic growth at infinity [8]. They also asked if a similar result can be obtained for the Fermi-Ulam system with quasiperiodically moving walls.

In this article, we answer this question positively. We first bring the vector field to a near inegrable form by applying transformations stopping the walls. Then we integrate the vector field using a proper Poincaré section so that to obtain a smooth quasiperiodic twist map. Finally we apply a quasiperiodic version of Moser's twist theorem, which is proven in the first part of the paper.

The proof of the twist theorem uses a "Lagrangian" approach introduced by Moser in the proof of an analogue theorem for elliptic partial differential equations [9] and used in [7] to present a simpler proof of the twist theorem. In contrast to the "Hamiltonian" approach, where one looks for the invariant curves in the phase space, the "Lagrangian" approach is based on the search for the solution of a second order difference equation in the configuration space.

In the subsequent sections, we will derive and solve the second order difference equation. Our proof is different from the one in [7] only in section 2, where the difference equation is introduced. The proof of the boundedness of motion in quasiperiodic Fermi-Ulam problem is presented in the Appendix.

2 Notation and Definitions

2.1 The space of real analytic quasiperiodic functions

We define the space of real analytic quasiperiodic functions $Q(\omega)$ as in [13]

Definition 2.1 *A function $f : \mathbf{R^1} \to \mathbf{R^1}$ is real analytic quasiperiodic $f \in Q(\omega)$ if it can be represented by Fourier series with exponentially decaying coefficients*

$$f(x) = \sum_k f_k e^{i\langle k,\omega\rangle x},$$

where $k = (k_1, k_2, ..., k_n)$, $\omega = (\omega_1, \omega_2, ..., \omega_n)$, $\langle k, \omega\rangle = k_1\omega_1 + ... + k_n\omega_n \neq 0$ if $k \neq 0$.

Generally speaking, an integral of a mean-zero function $f \in Q(\omega)$ may be an unbounded function, see [13] for an example. This might happen because of the small denominators $\langle k, \omega\rangle$ which appear after integrating Fourier series. However, if the frequencies satisfy the following Diophantine inequality:

there exist $K > 0$ and $\sigma > 0$ such that for all $k \neq 0$

$$|\omega_1 k_1 + \omega_2 k_2 + ... + \omega_n k_n| \geq \frac{K}{|k|^\sigma}, \tag{1}$$

where $k = (k_1, ..., k_n)$ and $|k| = |k_1| + ... + |k_n|$, then for any mean-zero $f \in Q(\omega)$ its integral is also in $Q(\omega)$.

To each $f \in Q(\omega)$ there corresponds a multiply periodic analytic function in n variables

$$F(\theta) = \sum_k f_k e^{i\langle k, \theta \rangle},$$

where $\theta = (\theta_1, \theta_2, ..., \theta_n)$, $2\pi-$ periodic in each variable and bounded in a complex neighborhood of $\mathbf{R^n}$: $|\mathrm{Im}\theta_k| \leq r$ for some $r > 0$.

Definition 2.2 *Let $Q_r(\omega) \subset Q(\omega)$ be the set of real analytic functions bounded on the subset $\Pi_r = \{\theta_1, ..., \theta_n : |\mathrm{Im}\theta_k| \leq r\}$, with the supremum norm*

$$|f|_r = \sup_{\theta \in \Pi_r} |f(\theta)|.$$

2.2 Generating functions

In the proof of the invariant curve theorem we will use the generating function of the quasiperiodic twist map. Here, we describe its properties.

Consider an area preserving quasiperiodic monotone twist map defined in a horizontal strip $\mathbf{R} \times I \subset \mathbf{R^2}$ as follows

$$\phi : \mathbf{R} \times I \to \mathbf{R}^2 : \qquad \begin{aligned} x_2 &= x_1 + \phi_1(x_1, y_1) \\ y_2 &= \phi_2(x_1, y_1), \end{aligned} \tag{2}$$

where $\phi_1(x_1, y_1), \phi_2(x_1, y_1) \in Q(\omega)$ as functions of x_1. Using the monotone twist condition $\frac{\partial \phi_1}{\partial y_1}(x_1, y_1) > 0$, we invert the first equation and substitute it in the second equation

$$\begin{aligned} y_1 &= \psi_1(x_2 - x_1, x_1) \\ y_2 &= \psi_2(x_2 - x_1, x_1), \end{aligned} \tag{3}$$

where ψ_1 is the inverse of ϕ_1 and $\psi_2 = \phi_2(x_1, \psi_1(x_2 - x_1, x_1))$. It is easy to check that

$$\frac{\partial \psi_1}{\partial x_2} + \frac{\partial \psi_2}{\partial x_1} = 1 - \left(\frac{\partial \phi_1}{\partial x_1} \frac{\partial \phi_2}{\partial x_2} - \frac{\partial \phi_1}{\partial x_2} \frac{\partial \phi_2}{\partial x_1} \right) = 0$$

using area preservation property of the map. Therefore, there exists a generating function $h(x_1, x_2)$ defining the map implicitly

$$\begin{aligned} y_1 &= -h_{x_1}(x_1, x_2) \\ y_2 &= h_{x_2}(x_1, x_2) \end{aligned} \tag{4}$$

with $h_{12} < 0$, defined in

$$\Omega = \Big\{(x_1, x_2) \in \mathbf{R^2} : a(x_1) < x_2 < b(x_1)\Big\}$$

corresponding to the domain of definition of the map. It is true conversely that the generating function $h(x_1, x_2)$ satisfying the condition $h_{12} < 0$ defines an area preserving monotone twist map.

Integrating the vector field (ψ_1, ψ_2) along a path from some point $(x_{01}, x_{02}) \in \Omega$ to $(x_1, x_2) \in \Omega$ and using Diophantine inequality (1), we obtain the generating function in the form

$$h(x_1, x_2) = \alpha x_1 + H(x_2 - x_1, x_1), \tag{5}$$

where H is quasiperiodic in the second variable. In many applications, not only is this map area-preserving but it also satisfies the exactness condition

Definition 2.3 *The above map ϕ is exact if for any $f(x) \in Q(\omega)$ its average value $\langle f \rangle$ does not change under the mapping*

$$\lim_{X\to\infty} \frac{1}{X} \int_0^X f(x)dx = \lim_{X\to\infty} \frac{1}{X} \int_0^X \phi \circ f(x)dx.$$

Intuitively, non exactness corresponds to the drift in vertical direction and α is the rate of the drift. We now prove

Lemma 2.1 *The map ϕ is exact iff the function $h(x_1 + t, x_2 + t)$ is quasiperiodic in t.*

Proof: Consider the derivative

$$\frac{d}{dt} h(x_1 + t, x_2 + t) = -\psi_1(x_1 + t, x_2 + t) + \psi_2(x_1 + t, x_2 + t) \tag{6}$$

which is a quasiperiodic function. On the other hand using (5) we have

$$h(x_1 + t, x_2 + t) = \alpha t + H_2(x_2 - x_1, x_1 + t), \tag{7}$$

where $f(t) = H_2(x_2 - x_1, x_1 + t) \in Q(\omega)$ as a function of t. Integrating (6) from 0 to T, using (7), dividing by T and taking the limit $T \to \infty$ we obtain the direct relation between exactness of the map and quasiperiodicity of the generating function

$$\begin{aligned} \alpha = \lim_{T\to\infty} \frac{1}{T} \int_0^T \psi_2(x_1 + t, x_2 + t)dt &- \lim_{T\to\infty} \frac{1}{T} \int_0^T \psi_1(x_1 + t, x_2 + t)dt = \\ \lim_{X\to\infty} \frac{1}{X} \int_0^X \phi \circ f(x)dx &- \lim_{X\to\infty} \frac{1}{X} \int_0^X f(x)dx = 0. \end{aligned}$$

QED.

From now on, we assume that the generating function $h(x_1 + t, x_2 + t)$ is quasiperiodic in t. Then we will show that there exists an invariant curve parametrically defined $w(t) = (u(t), v(t))$ so that $u(t) - t$ and $v(t)$ are quasiperiodic. The map restricted to this curve will be a rigid translation $\phi(w(t)) = w(t + \mu)$, with a prescribed rotation number μ.

Lemma 2.2 *The curve $w(t) = (u(t), v(t))$ satisfies the invariance condition $\phi(w(t)) = w(t + \mu)$ iff $u(t)$ satisfies the second order difference equation*

$$E(u)(t) = h_1(u(t), u(t + \mu)) + h_2(u(t - \mu), u(t)) = 0.$$

Proof: The second order difference equation is obtained by shifting t by $-\mu$ in the second equation in (4) and adding the two equations. QED.

Lemma 2.3 *The product $u_t E(u)$ is in $Q(\omega)$ and its mean value is zero*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T u_t E(u)dt = 0.$$

Proof: First we show that $u_t E(u)$ is quasiperiodic. Quasiperiodicity of u_t follows from the definition of $u(t)$ ($u(t) - t \in Q(\omega)$). To show quasiperiodicity of $E(u)$ it suffices to establish that of $h_i(u(t), u(t+\mu))$. The latter can be represented by

$$h_i(u(t), u(t + \mu)) = G(u(t + \mu) - u(t), u(t))$$

according to (5), where G is quasiperiodic in the second variable. Therefore, it remains to show that the function $G(f(t), t+g(t))$ is in $Q(\omega)$ if $f, g \in Q(\omega)$. This follows from representation of quasiperiodic functions by multiply periodic ones, see [13].

To show that $\langle u_t E(u) \rangle = 0$ we use the identity

$$u_t E(u) = \frac{d}{dt} h(u(t), u(t+\mu)) - \Delta^+ [h_2(u(t-\mu), u(t)) u_t(t)],$$

where $\Delta^+ f(t) = f(t+\mu) - f(t)$. The right hand-side is a difference of two mean-zero quasiperiodic functions for one of them is a derivative and the other one is a finite difference of a quasiperiodic function.

QED.

3 Main theorem

The generating function $h(x_1, x_2)$ obtained in the last section is defined up to multiplication by a constant. Indeed, it simply corresponds to scaling y-variable, see (4), and it does not change the second order difference equation. Therefore, for convenience we introduce the normalized generating function

$$h_{nr} = h / \inf_{\Omega} |h_{12}|.$$

Below we will denote the normalized generating function by h.

Theorem 3.1 *Assume that $\omega_1, \omega_2, ..., \omega_n, \mu$ satisfy a Diophantine inequality: there exist $K > 0$ and $\sigma > n$ such that for all $k \neq 0$*

$$|\omega_1 k_1 + \omega_2 k_2 + ... + \omega_n k_n + \mu^{-1} k_{n+1}| \geq \frac{K}{|k|^\sigma}, \tag{8}$$

where $k = (k_1, ..., k_n)$ and $|k| = |k_1| + ... + |k_n| + |k_{n+1}|$. Assume that h is real analytic in

$$\Omega_\alpha = \{(x_1, x_2) \in \mathbf{C^2} : (\mathrm{Re} x_1, \mathrm{Re} x_2) \in \Omega, |\mathrm{Im} x_1|, |\mathrm{Im} x_2| < \alpha\},$$

has bounded derivatives up to the second order in Ω_α $\alpha > 0$, and satisfies the normalized monotone twist condition

$$\inf_{\Omega} h_{12} = -1.$$

Then for any $\epsilon, M, K > 0, \sigma > n \in \mathbf{N}$ there exists $\delta, r > 0$, such that if there exists $u_0(t)$ with

1. $u_0(t) - t \in Q_r(\omega)$
2. $|u_{0t}|_r$, $|(u_{0t})^{-1}|_r < M$

3. $O_\epsilon(u_0(t), u_0(t+\mu)) \in \Omega$

4. $|E(u_0)|_r < \delta$

then there exists a unique solution u *of* $E(u) = 0$ *with* $u(t) - t \in Q_{r/2}(\omega)$, $(u(t), u(t+\mu)) \in \Omega$ *and it is unique up to the translation* $t \to t +$ constant.

Remark 3.1 *It easy to show, using simple measure theoretical argument [13] (pg. 191), that the subset of* ω, μ *which do not satisfy the above Diophantine condition for any* $K > 0, \sigma > n$ *has zero Lebesgue measure. It can be also shown that for a given* $\omega = (\omega_1, \omega_2, ..., \omega_n)$ *satisfying the Diophantine condition*

$$|\omega_1 k_1 + \omega_2 k_2 + ... + \omega_n k_n| \geq \frac{K}{|k|^\sigma} \tag{9}$$

with sufficiently small K *and* $\sigma > \sigma_0 > n$ *there can be found* μ *on a given interval satisfying (8) with the same* K *and* σ*. Besides that the relative measure of* μ *violating (8) decays to zero with* K*.*

Remark 3.2 *The condition* 4 *can be satisfied, in particular, if the map is a sufficiently small perturbation of the integrable map*

$$x_2 = x_1 + y_1 + O(\epsilon)$$
$$y_2 = y_1 + O(\epsilon)$$

by choosing $u_0(t) = t$ *and decreasing* $\epsilon \to 0$*.*

4 The homological equation

4.1 The second order homological equation

Following [7], we solve the equation $E(u) = 0$ using modification of Newton's method. Starting with u_0 such that $E(u_0)$ is sufficiently small, we seek an improvement $u_0 + v$ so that

$$E(u + v) = E(u) + E'(u)v + ...$$

become quadratically small compared to $E(u)$. The first idea would be to kill linear order terms by solving the linear equation

$$E(u) + E'(u)v = 0 \tag{10}$$

as in Newton's method. This equation is, however, difficult to solve because $E'(u)$ is not easy to invert. The way around this difficulty is to modify (10) so that v can be easily found and $E(u+v)$ would be quadratically small. Multiplying (10) by u_t and subtracting the quadratically small term $v\partial_t E(u)$ from the left hand-side we obtain the new equation

$$u_t E'(u)v - vE'(u)u_t + u_t E(u) = 0. \tag{11}$$

Although it is no longer equivalent to (10), nevertheless it still produces v, which makes $E(u+v)$ quadratically small, as we will show later.

Evaluating the terms with $E'(u)$

$$h_{12}^+(u_t v^+ - u_t^+ v) + h_{12}^-(u_t v^- - u_t^- v) + u_t E(u) = 0$$

and choosing the new variable $v = u_t w$ we obtain the equation equivalent to (11)

$$h_{12}^+ u_t u_t^+ (w^+ - w) - h_{12}^- u_t u_t^- (w - w^-) + u_t E(u) = 0.$$

Introducing finite difference operators $\Delta^+ u(t) = u(t+\mu) - u(t)$ and $\Delta^- u(t) = u(t) - u(t-\mu)$ we transform the equation to the second order difference equation

$$\Delta^-(h_{12}^+ u_t u_t^+ \Delta^+ w) + u_t E(u) = 0.$$

4.2 Lemma on the first order homological equation

The second order difference equation obtained at the end of the last section is equivalent to the system of two first order difference equations

$$\begin{cases} \Delta^- \psi = g \\ p^{-1}\Delta^+ w = \psi + \psi_0, \end{cases} \tag{12}$$

where $g = -u_t E(u)$ and $p^{-1} = h_{12}^+ u_t u_t^+$. The solution of each equation is provided by

Lemma 4.1 *Consider the first order difference equation*

$$\Delta\psi = g,$$

where $g \in Q_r(\omega)$, $\langle g \rangle = 0$, and ω satisfies the above Diophantine condition. Then there exists a unique solution $\psi \in Q_{r'}(\omega)$ with $\langle \psi \rangle = 0$ satisfying the estimate

$$|\psi|_{r'} < C(K,\sigma)\frac{|g|_r}{(r-r')^{\sigma+n}}.$$

Proof: We will solve the equation

$$\psi(t+\mu) - \psi(t) = g(t)$$

using Fourier series representation $g(t) = \sum_k g_k e^{i\langle k,\omega\rangle t}$ $\psi(t) = \sum_k \psi_k e^{i\langle k,\omega\rangle t}$. After straightforward calculations we obtain the relation between Fourier coefficients

$$\psi_k = \frac{g_k}{e^{i\langle k,\omega\rangle\mu} - 1}.$$

Using the Diophantine condition (8) we estimate the small denominators

$$|e^{i\langle k,\omega\rangle} - 1| \geq \frac{C(K,\sigma)}{|k|^\sigma}.$$

Since $g \in Q_r(\omega)$ then $|g_k| \leq |g|_r e^{-|k|r}$ and we obtain the estimate on the Fourier coefficients of ψ

$$|\psi_k| \leq C^{-1}(K,\sigma)\,|g|_r\,|k|^\sigma e^{-|k|r} = C^{-1}(K,\sigma)\,|g|_r\,|k|^\sigma e^{-|k|s} e^{-|k|(r-s)}.$$

Using the inequality $x^\sigma e^{-x} \leq c(\sigma)$ we obtain

$$|\psi_k| \leq C(K,\sigma)\,|g|_r\,\frac{e^{-|k|s}}{(r-s)^\sigma}.$$

Now, we estimate

$$|\psi|_{r'} \leq \sum_k |\psi_k| e^{|k|r'} \leq \sum_{m=0}^{\infty} c(n) m^{n-1} |\psi_m| e^{mr'} \leq \frac{C(K,\sigma,n)|g|_r}{(r-s)^\sigma} \sum_{m=0}^{\infty} m^{n-1} e^{-m(s-r')} =$$
$$\frac{C(K,\sigma,n)|g|_r}{(r-s)^\sigma} \sum_{m=0}^{\infty} m^{n-1} e^{-m(s-s')} e^{-m(s'-r')}.$$

Using $x^\sigma e^{-x} \leq c(\sigma)$ and comparing with power series we obtain

$$|\psi|'_r \leq \frac{C(K,\sigma,n)|g|_r}{(r-s)^\sigma (s-s')^{n-1}(1-e^{-(s'-r')})} \leq \frac{C(K,\sigma,n)|g|_r}{(r-s)^\sigma (s-s')^{n-1}(s'-r')}.$$

Fixing $s, s' : r - s = s - s' = s' - r'$ we obtain the result.

QED.

Now, we apply this lemma to find and estimate the solution of the second order difference equation (12):

Lemma 4.2 *Let $r > 0$ and $M' > 0$ and assume that $|E(u)|_r < \infty$, $u(t) - t \in Q_r(\omega)$ and $|u_t|_r, |(u_t)^{-1}|_r, |h_{12}|_r, |(h_{12})^{-1}|_r < M'$, then the second order difference equation (12) has a unique*

solution $w \in Q_\rho(\omega)$ *with* $\langle w \rangle = 0$ *for any* $\rho : 0 < \rho < r$, *so that the correction* $v = u_t w$ *satisfies the estimates*

$$|v|_\rho \leq \frac{C(K,\sigma,n,M')}{(r-\rho)^{2\sigma+2n}}|E(u)|_r, \;\; |v_t|_\rho \leq \frac{C(K,\sigma,n,M')}{(r-\rho)^{2\sigma+2n+1}}|E(u)|_r.$$

Proof: The first equation can be immediately solved since Lemma 4.1 directly applies. The solution satisfies the estimate

$$|\psi|_{r'} < C(K,\sigma,n)\frac{|g|_r}{(r-r')^{\sigma+n}}.$$

To find w we first specify ψ_0 so that the equation

$$\Delta^+ w = p(\psi + \psi_0)$$

can be solved. This requires the right hand-side to be mean-zero

$$\langle p(\psi+\psi_0)\rangle = 0 \Leftrightarrow \psi_0 = -\frac{\langle \psi p\rangle}{\langle p\rangle}$$

for the left hand-side is automatically mean-zero.

Applying Lemma 4.1 again we obtain the solution

$$|w|_\rho \leq C(K,\sigma,n)\frac{|p(\psi+\psi_0)|_{r'}}{(r'-\rho)^{\sigma+n}} \leq C(K,\sigma,n,M')\frac{|g|_r}{(r'-\rho)^{\sigma+n}(r-r')^{\sigma+n}}.$$

Taking $r' = \frac{r+\rho}{2}$ and recalling that $g = -u_t E(u)$ we obtain the first estimate. Using the Cauchy estimate $|v_t|_\rho \leq \frac{|v|_r}{(r-\rho)}$ we obtain the second inequality. **QED.**

Corollary 4.1 *Under the conditions of the Lemma we have*

$$|(u_t+v_t)|_\rho \leq M' + |v_t|_\rho$$
$$|(u_t+v_t)^{-1}|_\rho \leq \frac{M'}{1-M'|v_t|_\rho},$$

provided $|v_t|_\rho M' < 1$.

5 Quadratic decay of error

We now estimate the mismatch for the iteration

$$|E(u+v)|_\rho = |E(u)+E'(u)v+Q|_\rho \le |E(u)+E'(u)v|_\rho + |Q|_\rho.$$

To estimate the first term on the right hand-side, we use (11)

$$|E(u)+E'(u)v|_\rho \le |w\frac{d}{dt}E(u)|_\rho \le C\frac{|E(u)|_r}{(r-\rho)^{2\sigma+2n}}\frac{|E(u)|_r}{(r-\rho)} = C\frac{|E(u)|_r^2}{(r-\rho)^{2\sigma+2n+1}},$$

where we have used the Cauchy estimate $|\frac{d}{dt}E(u)|_\rho \le c\frac{|E(u)|_r}{r-\rho}$.

Now, we estimate the reminder, using Taylor's formula

$$Q = \frac{1}{2}\frac{d^2}{d\lambda^2}E(u+\lambda v),$$

where $\lambda \in (0,1)$, and using estimates on v and h we obtain

$$|Q|_\rho \le c|v|_\rho^2 \le C\frac{|E(u)|_r^2}{(r-\rho)^{4\sigma+4n}}.$$

Therefore the mismatch is indeed quadratically small as in Newton's method

$$|E(u+v)|_\rho \le C\frac{|E(u)|_r^2}{(r-\rho)^{4\sigma+4n+1}}.$$

6 The limiting process

We first choose the sequence of the analyticity domains: $r = r_0 > r_1 > ...$ with $r_m \to r_\infty$, according to $r_m = r_\infty + 2^{-m}(r_0 - r_\infty)$. Using Lemma 4.2 we will construct a sequence $u_0, u_1, ..., u_m, ...$ analytic in the corresponding domains $Q_{r_m}(\omega)$ with the corrections satisfying the inequalities

$$|v_m|_{r_{m+1}} \le \frac{C}{(r_m - r_{m+1})^{2\tau}}|E(u_m)|_{r_m}$$
$$|\frac{d}{dt}v_m|_{r_{m+1}} \le \frac{C}{(r_m - r_{m+1})^{2\tau+1}}|E(u_m)|_{r_m}$$

where $\tau = \sigma + n$. The mismatches $\epsilon_m := |E(u_m)|_{r_m}$ satisfy the estimate

$$\epsilon_{m+1} \le c\frac{\epsilon_m^2}{(r_m - r_{m+1})^{4\tau+1}} = ca^m\epsilon_m^2,$$

where $a = 2^{4\tau+1}$. Rescaling the sequence $\eta_m = ca^{m+1}\epsilon_m$ we obtain $\eta_{m+1} \le \eta_m^2$. This sequence decays to zero if $\eta_0 = ca\epsilon_0 < 1$. In this case the original sequence decays faster than exponentially

$$\epsilon_m \le \frac{(ca\epsilon_0)^{2^m}}{ca^{m+1}}$$

and so does the sequence $|v_m|_{r_\infty} \to 0$.

Therefore, there exists a limit

$$u_\infty = \lim_{m\to\infty} u_m = u_0 + \sum_{k=0}^{\infty} v_k, \tag{13}$$

which is an analytic function in Q_{r_∞}. Now, we can pass to the limit in the main equation to obtain

$$E(u_\infty) = \lim_{m\to\infty} E(u_m) = 0.$$

Therefore, u_∞ is indeed a solution of $E(u) = 0$.

In order to finish this construction, we have to justify application of Lemma 4.2 at each step of the iteration. It suffices to check that $|\frac{d}{dt}u_m|_{r_m}$ and $|\frac{d}{dt}\frac{1}{u_m}|_{r_m}$ stay bounded uniformly in m and ensure that the iterates u_m do not leave the analyticity domain of h.

We will do this by specifying

$$M' = M_\infty = 2M_0 = 2\max(M, \sup_{(x_1,x_2)\in\Omega_\alpha} |h_{12}|, |h_{12}^{-1}|)$$

in Lemma 4.2 and showing that $|\frac{d}{dt}u_m|_{r_m}$ and $|\frac{d}{dt}\frac{1}{u_m}|_{r_m}$ stay below M_∞. Using Corollary 4.1, we obtain inequalities on the iterates of the bounds $M_0, M_1, ...$

$$M_{m+1} \le \max\left(M_m + ca^m\epsilon_m,\ \ \frac{M_m}{1 - ca^m\epsilon_m M_m}\right).$$

We will consider the iterations while $M_m < M_\infty$ and show that by decreasing ϵ_0 we will achieve $M_m \le M_\infty$ for all m. Since $ca^m\epsilon_m \le C(ca\epsilon_0)^{2^m}$ then we can decrease ϵ_0 further so that $ca^m\epsilon_m \le \frac{1}{2M_\infty}$ and we can rewrite the ratio in the brackets

$$\frac{M_m}{1 - ca^m\epsilon_m M_m} \le M_m(1 + 2ca^m\epsilon_m M_m) \le M_m + C(ca\epsilon_0)^{2^m} 2M_\infty^2.$$

Thus, we obtain the estimate

$$M_m \le M_0 + C\sum_{m=1}^{\infty} (ca\epsilon_0)^{2^m}$$

and by decreasing ϵ_0 we obtain the desired inequality $M_m \le M_\infty$.

Finally, using (13) and estimates on $|v_m|$ we obtain that $|u_m - u_0|_{r_m}$ can be made as small as we like (uniformly in m) by further decreasing ϵ_0. Therefore $u_m(t), u_m(t+\mu)$ will not leave Ω_α the domain of analyticity of h, provided $O_\epsilon(u_0(t), u_0(t+\mu)) \in \Omega_\alpha$. This follows from the condition of the theorem and from the proper choice of r: $rM < \alpha - \epsilon$.

Appendix

A Small twist

In many applications the monotone twist decays to zero with the perturbation. Therefore, certain estimates which we used in the proof are not uniformly bounded, *e.g.* h_{12}. This situation arises, for example, in the stability problem of elliptic fixed points of an area-preserving mapping and in the stability problem for Fermi-Ulam system, which is considered in the next section. In this section we show how the above approach applies to the small twist problem.

Consider an area-preserving analytic map ϕ given by

$$\begin{aligned} x_2 &= x_1 + \gamma y_1 + f(x_1, y_1, \gamma) \\ y_2 &= y_1 + g(x_1, y_1, \gamma), \end{aligned} \tag{14}$$

where $\gamma \in (0, \gamma_0)$, $f, g = o(\gamma)$ are real analytic in x_1, y_1 and quasiperiodic in x_1. The generating function for this mapping in the lowest order takes the form

$$h_0(x_1, x_2, \gamma) = \frac{(x_2 - x_1)^2}{2\gamma}$$

and the corresponding term of the second difference equation is given by

$$E_0(u(t)) = \frac{\Delta^2(u)}{\gamma} = \frac{u(t+\mu) - 2u(t) + u(t-\mu)}{\gamma}.$$

Since the twist is of order γ we will keep the rotational number on the same scale $c\gamma < \mu < C\gamma$, using a modified Diophantine inequality. Therefore, the mismatch in the above equation will be of order $\frac{\Delta^2(u)}{\gamma} \sim \frac{\gamma^2}{\gamma} = \gamma$.

Thus, it is natural to consider a rescaled difference equation $E_\gamma(u) = \gamma^{-1} E(u) = 0$, so that if $E_\gamma(u) \sim \delta$ then $E_\gamma(u + v) \sim \delta^2$, where v is a correction found by Newton's method satisfying the estimates uniformly in γ.

Below we show that with this modification the above approach provides the existence of invariant curves in the small twist case. Since the interval, where μ has to be chosen shrinks to zero we use a modified Diophantine condition

$$|(\omega_1 k_1 + \omega_2 k_2 + ... + \omega_n k_n)\gamma + \mu^{-1}\gamma k_{n+1}| \geq \frac{K\gamma}{|k|^\sigma}, \tag{15}$$

where as before for given ω satisfying (9) with sufficiently small K one can find $\mu \in (c\gamma, C\gamma)$ satisfying (15) and the relative measure of μ violating (15) tends to zero with K.

Estimating small denominators

$$|e^{i\langle k,\omega\rangle\mu} - 1| \geq \frac{K\gamma}{|k|^\sigma},$$

we can estimate the solution of the first order homological equation (12). Dividing the first equation by γ we obtain

$$\Delta_\gamma \psi = -u_t E_\gamma(u),$$

where $\Delta_\gamma = \gamma^{-1}\Delta$. Proceeding as in the proof of lemma 4.1 and using that γ in Diophantine condition is cancelled by γ in Δ_γ we obtain the same estimate as in Lemma 4.2. The second equation divided by γ takes the form

$$\Delta_\gamma^+ w = \frac{\psi + \psi_0}{\gamma^{-1} h_{12}^+ u_t u_t^+}$$

and as before γ in Diophantine condition is cancelled by γ in Δ_γ. A uniform estimate is also obtained for $\gamma^{-1}h_{12}$. Therefore we obtain estimates on v which are uniform in γ. The rest of the proof: quadratic decay of errors and the limiting process is straightforward. As a result we can state

Theorem A.1 *Let μ satisfy the Diophantine inequality (15). Then for any $K > 0, \sigma > n, \epsilon > 0, M > 0$, there exist $r > 0$, $\delta > 0$ such that if for some u_0 $(u_0(t) - t \in Q(\omega))$, $O_\epsilon(u_0(t)u_0(t+\mu))$ is in the analyticity domain of h with $|E_\gamma(u_0)| < \delta$ then there exists a solution $u(t)$ of $E(u) = 0$ with $(u(t) - t \in Q(\omega))$. The solution is unique up to the translation $t \to t +$ constant.*

B Boundedness of motion in "quasiperiodic" Fermi-Ulam problem

We consider a classical particle moving between two parallel walls undergoing quasiperiodic motions in the presence of a potential field which is also assumed to be quasiperiodic in time. The motion between the walls is described by

$$\ddot{x} + V'(x,t) = 0,$$

while the motion of the left and right walls is given by $x = b(t)$ and $x = a(t)$, respectively. We will assume that all functions are smooth in x, t and analytic quasiperiodic in t, i. e. $V'(x,t), r(t), l(t) \in Q_r(\omega)$.

First, we apply two transformations to stop the walls. The left wall is stopped by the change of variable $x = y + a(t)$ so that now the particle moves between the walls according to

$$\ddot{y} + V'(y + a(t), t) - \ddot{a}(t) = 0,$$

while the left wall is at rest and the right wall's position is given by $y = b(t) - a(t)$.

We now invoke another transformation, which originated in the theory of heat equation with moving boundary and was later used in quantum version of Fermi-Ulam problem [12]. It is also known as Liouville transformation [2]. We let $y = [b(t) - a(t)]q$ so that the right wall will be also at rest ($q = 1$). The motion between the walls is given by

$$\ddot{c}(t)q + 2\dot{c}(t)\dot{q} + c(t)\ddot{q} + V_1(c(t)q + a(t), t) = 0,$$

where $c(t) = b(t) - a(t)$. In order to get rid of the $\dot{q}$ term we choose the new time $\tau = f(t)$ so that the equation takes the form

$$\ddot{c}q + 2\dot{c}\dot{f}q' + p\ddot{f}q' + \dot{f}^2 q'' + V_1(c(t)q + a(t), t) = 0,$$

where $F' = \frac{d}{d\tau}F$. Requiring that $2\dot{c}\dot{f} + c\ddot{f} = 0 \Leftrightarrow c^2\dot{f} = 1$ we obtain

$$\dot{f}^2 q'' + \ddot{c}q + V_1(c(t)q + a(t), t) = 0 \Leftrightarrow q'' + \ddot{c}(t)c(t)^3 q + c^4(t)V_1(c(t)q + a(t), t) = 0$$
$$\text{with} \quad \tau = f(t) = \int_0^t \frac{ds}{c^2(s)}.$$

The equation takes the form

$$q'' + W_1(q, \tau) = 0,$$

where $W_1(q, \tau) \in Q(\alpha\omega)$ in τ. This follows from the following properties of quasiperiodic functions, see [13].

1. Let $h(t) \in Q(\omega)$ and $\tau = \alpha t + h(t)$ (with $\alpha + \dot{h} > 0$) then the inverse relation is given by $t = \alpha^{-1}t + H(t)$ and $H(t) \in Q(\alpha\omega)$.

2. Let $g(t), f(t) \in Q(\omega)$ then $g(t + f(t)) \in Q(\omega)$.

Indeed, using the first property, for any $F(t) \in Q(\omega)$ we have $F(t(\tau)) = F(\alpha^{-1}\tau + H(\tau))$, where $H(\tau) \in Q(\omega)$. Introducing $G(t) = F(\alpha^{-1}t)$, which is clearly in $Q(\alpha\omega)$, we have $F(t(\tau)) = F(\alpha^{-1}\tau + H(\tau)) = G(\tau + \alpha H(\tau)) \in Q(\alpha\omega)$ by the second property.

Since both walls are at rest we can include collisions in the description of the system using the reflection principle. In the equivalent system the particle moves on the circle $q \in (-1,1)$ in the potential field $W(|q|,\tau)$. The Hamiltonian of this system is given by

$$H = \frac{p^2}{2} + W(|q|,\tau).$$

This system becomes near integrable for large velocities, as we will show by proper rescaling and reduction to the twist map. Let $p = \epsilon^{-1}P, t = \epsilon^{-1}T, H = \epsilon^2 F, q = Q$, then the new Hamiltonian takes the form

$$F = \frac{P^2}{2} + \epsilon^2 W(|Q|,\epsilon T).$$

The rescaled Hamiltonian is formally in near integrable form, however, non smoothness in Q variable makes it necessary to integrate the vector field in Q so that to obtain a smooth map, which can be analyzed for stability. As was recently observed in [5] this can be done by using transformation exchanging time and energy with position and momentum, so that Q become a new time variable, and integrating the obtained equations of motion.

Using invariance of the 1-form: $PdQ - FdT = -(Fd(\epsilon T) - \epsilon P dQ)$ we choose $H = \epsilon P$ as a new Hamiltonian, $t = Q$ as a new time, $\tau = \epsilon T$ as a new position, and $h = F$ as a new momentum

$$H = \epsilon\sqrt{2h} + \epsilon^3 H_1(h,\tau,|t|).$$

The equations of motion are given by

$$\begin{cases} \frac{d\tau}{dt} = \epsilon\frac{1}{\sqrt{2h}} + \epsilon^3\frac{\partial H_1}{\partial h}(h,\tau,|t|) \\ \frac{dh}{dt} = -\epsilon^3\frac{\partial H_1}{\partial \tau}(h,\tau,|t|). \end{cases}$$

Integrating these equations on $t \in (-1,1)$ we obtain the monotone twist map

$$\begin{cases} \tau_1 = \tau_0 + \epsilon\frac{2}{\sqrt{h_0}} + O(\epsilon^3) \\ h_1 = h_0 + O(\epsilon^3). \end{cases} \tag{16}$$

This mapping is analytic in both variables (since H_1 is continuous in t on $(-1,0)$ and $(0,1)$) and quasiperiodic in τ. Now, we will verify the exactness condition. Consider a subset V in the extended phase space (τ,h,t) bounded by by four vertical planes $\tau = 0, \tau = \tau^+, t = -1, t = 1$ and by two 2-dimensional surfaces: one $h_f(t,\tau)$ given by the solution generated by the curve of initial conditions $h = f(\tau)$ (with $f \in Q(\omega)$), the other $h_H(\tau,t)$ given by $H = 0$.

By the Stokes theorem the integral of the symplectic form $d\omega^1 = dh \wedge d\tau + dH \wedge dt$ over a closed two dimensional surface vanishes. Therefore we have

$$\int_{\partial V} d\omega^1 = \int_0^{\tau^+} h_f(1,\tau)d\tau - \int_0^{\tau^+} h_f(-1,\tau)d\tau +$$
$$\int_{-1}^{1}[h_f(t,\tau=0) - h_H(t,\tau=0)]dt - \int_{-1}^{1}[h_f(t,\tau^+) - h_H(t,\tau^+)]dt + \int_{H=0} dh \wedge d\tau = 0.$$

Dividing by τ^+ and taking the limit $\tau^+ \to \infty$ we obtain

$$\lim_{\tau^+\to\infty} \frac{1}{\tau^+}\int_0^{\tau^+} h_f(1,\tau)d\tau - \frac{1}{\tau^+}\int_0^{\tau^+} h_f(-1,\tau)d\tau = -\lim_{\tau^+\to\infty}\frac{1}{\tau^+}\int_{H=0} dh \wedge d\tau,$$

but

$$\int_{H=0} dh \wedge d\tau = -\int_{-1}^{1}\int_0^{\tau^+} \partial_t h_H dt d\tau = -\int_0^{\frac{\tau^+}{\epsilon}}\int_{-1}^{1} W_Q(|Q|,T)dQdT = 0$$

since W is periodic in Q. Thus, we have verified the exactness condition.

Therefore the mapping (16) satisfies the conditions of the monotone small twist theorem for sufficiently small ϵ, which corresponds to large energy solutions. Thus, we obtain invariant manifolds carrying quasiperiodic motions and separating the extended phase space into invariant layers. As $\epsilon \to 0$ (energy grows to infinity) the relative measure of the subset free of the above invariant manifolds decays to zero.

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